Realizations of Real Low-Dimensional Lie Algebras

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Abstract

Using a new powerful technique based on the notion of megaideal, we construct a complete set of inequivalent realizations of real Lie algebras of dimension no greater than four in vector fields on a space of an arbitrary (finite) number of variables. Our classification amends and essentially generalizes earlier works on the subject.

Known results on classification of low-dimensional real Lie algebras, their automorphisms, differentiations, ideals, subalgebras and realizations are reviewed.

1 Introduction

The description of Lie algebra representations by vector fields was first considered by S. Lie. However, this problem is still of great interest and widely applicable e.g., in particular, to integrating of ordinary differential equations [35, 36] (see also some new results and trends in this area, e.g., in [1, 10, 15, 43, 44, 71, 72, 86, 87, 88]), group classification of partial differential equations [6, 24, 30, 92], classification of gravity fields of a general form under the motion groups or groups of conformal transformations [27, 28, 29, 46, 62]. (See, e.g. [21, 24] for other physical applications of realizations of Lie algebras.) Thus, without exaggeration this problem has a major place in modern group analysis of differential equations.

In spite of its importance for applications, the problem of complete description of realizations has not been solved even for the cases when either the dimension of algebras or the dimension of realization space is a fixed small integer. An exception is Lie's classification of all possible Lie groups of point transformations acting on the two-dimensional complex or real space without fixed points [34, 37], which is equivalent to classification of all possible realizations of Lie algebras in vector fields on the two-dimensional complex (real) space (see also [21]).

In this paper we construct a complete set of inequivalent realizations of real Lie algebras of dimension no greater than four in vector fields on a space of an arbitrary (finite) number of variables. For solving this problem, we propose a new powerful technique based on the notion of megaideal.

The plan of the paper is as follows. Results on classifications of abstract Lie algebras are reviewed in Section 2. In Section 3 we give necessary definitions and statements on megaideals and realizations of Lie algebras, which form the theoretical basis of our technique. Previous results on classifications of realizations are reviewed in Section 4. In this section we also explain used notations, abbreviations and conventions and describe the classification technique. The results of our classification are formulated in the form of Tables 2–6. An example of classification of realizations for a four-dimensional algebra is discussed in detail in Section 5. In Section 6 we compare our results with those of [87]. Section 7 contains discussion. In appendix A different results on low-dimensional real Lie algebras (classifications, automorphisms, subalgebras etc.) are collected.

2 On classification of Lie algebras

The necessary step to classify realizations of Lie algebras is classification of these algebras, i.e. classification of possible commutative relations between basis elements. By the Levi–Maltsev theorem any finite-dimensional Lie algebra over a field of characteristic 0 is a semi-direct sum (the Levi–Maltsev decomposition) of the radical (its maximal solvable ideal) and a semi-simple subalgebra (called the Levi factor) (see, e.g., [25]). This result reduces the task of classifying all Lie algebras to the following problems:

1) classification of all semi-simple Lie algebras;

2) classification of all solvable Lie algebras;

3) classification of all algebras that are semi-direct sums of semi-simple Lie algebras and solvable Lie algebras.

Of the problems listed above, only that of classifying all semi-simple Lie algebras is completely solved in the well-known Cartan theorem: any semi-simple complex or real Lie algebra can be decomposed into a direct sum of ideals which are simple subalgebras being mutually orthogonal with respect to the Cartan–Killing form. Thus, the problem of classifying semi-simple Lie algebras is equivalent to that of classifying all non-isomorphic simple Lie algebras. This classification is known (see, e.g., [14, 5]).

At the best of our knowledge, the problem of classifying solvable Lie algebras is completely solved only for Lie algebras of dimension up to and including six (see, for example, [48, 49, 50, 51, 80, 81]). Below we shortly list some results on classifying of low-dimensional Lie algebras.

All the possible complex Lie algebras of dimension ≤ 4 were listed by S. Lie himself [37]. In 1918 L. Bianchi investigated three-dimensional real Lie algebras [7]. Considerably later this problem was again considered by H.C. Lee [32] and G. Vranceanu [85], and their classifications are equivalent to Bianchi's one. Using Lie's results on complex structures, G.I. Kruchkovich [27, 28, 29] classified four-dimensional real Lie algebras which do not contain three-dimensional abelian subalgebras.

Complete, correct and easy to use classification of real Lie algebras of dimension ≤ 4 was first carried out by G.M. Mubarakzyanov [49] (see also citation of these results as well as description of subalgebras and invariants of real low-dimensional Lie algebras in [58, 59]). At the same year a variant of such classification was obtained by J. Dozias [13] and then adduced in [84]. Analogous results are given in [62]. Namely, after citing classifications of L. Bianchi [7] and G.I. Kruchkovich [27], A.Z. Petrov classified four-dimensional real Lie algebras containing three-dimensional abelian ideals.

In [42] M.A.H. MacCallum proposed an alternative scheme of classification and numeration for four-dimensional real Lie algebras. He also reviewed and compared different approaches and results concerning this problem, which were adduced in the previous papers of G.M. Mubarakzyanov [49], F. Bratzlavsky [8], G.I. Kruchkovich and A.Z. Petrov [27, 62], Patera et al [58]. In preprint [2] authors reproduced the classification of four-dimensional real Lie algebras and compared their results with ones by G.M. Mubarakzyanov [49], J. Dozias [13] and Patera et al [58].

In the series of papers [50, 51, 52] G.M. Mubarakzyanov continued his investigations of low-dimensional Lie algebras. He classified five-dimensional real Lie algebras as well as six-dimensional solvable ones with one linearly independent non-nilpotent element. Let us note that for six-dimensional solvable real Lie algebras dimension m of the nilradical is greater than or equal to 3. In the case $m = 3$ such algebras are decomposable. Classification of six-dimensional nilpotent Lie algebras ($m = 6$) was obtained by K.A. Umlauf [83] over complex field and generalized by V.V. Morozov [48] to the case of arbitrary field of characteristic 0.

J. Patera, R.T. Sharp, P. Winternitz and H. Zassenhaus in [58] revised the classification of four- and five-dimensional real Lie algebras by G.M. Mubarakzyanov and nilpotent sixth-dimensional Lie algebras by V.V. Morozov.

Using the proposed in [60] notion of nilpotent frame of a solvable algebra, J. Patera and H. Zassenhaus [61] classified solvable Lie algebras of dimension ≤ 4 over any perfect field. These results were generalized by W.A. de Graaf [11] to arbitrary fields using the computer algebra system MAGMA.

In [80, 82] P. Turkowski classified all real Lie algebras of dimension up to 9, which admit non-trivial Levi decomposition. P. Turkowski [81] also completed Mubarakzyanov's classification of six-dimensional solvable Lie algebras over $\mathbb{R}$, by classifying real Lie algebras of dimension 6 that contain four-dimensional nilradical ($m = 4$).

The recent results and references on seven-dimensional nilpotent Lie algebras can be found in [73].

In the case when the dimension of algebra is not fixed sufficiently general results were obtained only in classification of algebras with nilradicals having special structures (e.g., abelian [54], Heisenberg [69], or triangular algebras [76]) as their nilradical. Invariants of these algebras, i.e. their generalized Casimir operators, were investigated in [53, 55, 76, 77].

In Table 1 we summarize known results about classification Lie algebras in form of the table (A is Lie algebra over the field P, $\dim A$ is the dimension of A, $\dim N$ is the dimension of the nilradical of A).

3 Megaideals and realizations of Lie algebras

Now we define the notion of megaideal that is useful for constructing realizations and proving their inequivalence in a simpler way. Let A be an m-dimensional (real or complex) Lie algebra ($m \in \mathbb{N}$) and let $\mathrm{Aut}(A)$ and $\mathrm{Int}(A)$ denote the groups of all the automorphisms of A and of its inner automorphisms respectively. The Lie algebra of the group $\mathrm{Aut}(A)$ coincides with the Lie algebra $\mathrm{Der}(A)$ of all the derivations of the algebra A. (A derivation D of A is called a linear mapping from A into itself which satisfy the condition $D[u, v] = [Du, v] + [u, Dv]$ for all $u, v \in A$.) $\mathrm{Der}(A)$ contains as an ideal the algebra $\mathrm{Ad}(A)$ of inner derivations of A, which is the Lie algebra of $\mathrm{Int}(A)$. (The inner derivation corresponding to $u \in A$ is the mapping $\mathrm{ad}\, u\colon v \to [v, u]$.) Fixing a basis $\{e_\mu, \mu = \overline{1, m}\}$ in A, we associate an arbitrary linear mapping $l : A \to A$ (e.g., an automorphism or a derivation of A) with a matrix $\alpha = (\alpha_{\nu\mu})_{\mu,\nu=1}^m$ by means of the expanding $l(e_\mu) = \alpha_{\nu\mu} e_\nu$. Then each group of automorphisms of A is associated with a subgroup of the general linear group $GL(m)$ of all the non-degenerated $m \times m$ matrices (over $\mathbb{R}$ or $\mathbb{C}$) as well as each algebra of derivations of A is associated with a subalgebra of the general linear algebra $gl(m)$ of all the $m \times m$ matrices.

Definition. We call a vector subspace of A, which is invariant under any transformation from $\mathrm{Aut}(A)$, a *megaideal* of A.

Since $\mathrm{Int}(A)$ is a normal subgroup of $\mathrm{Aut}(A)$, it is clear that any megaideal of A is a subalgebra and, moreover, an ideal in A. But when $\mathrm{Int}(A) \neq \mathrm{Aut}(A)$ (e.g., for nilpotent algebras) there exist ideals in A, which are not megaideals. Moreover, any megaideal I of A is invariant with respect to all the derivations of A: $\mathrm{Der}(A)I \subset I$, i.e. it is a characteristic subalgebra. Characteristic subalgebras which are not megaideals can exist only if $\mathrm{Aut}(A)$ is a disconnected Lie group.

Both improper subsets of A (the empty set and A itself) are always megaideals in A. The following lemmas are obvious.

Lemma 1. If I_1 and I_2 are megaideals of A then so are $I_1 + I_2$, $I_1 \cap I_2$ and $[I_1, I_2]$, i.e. sums, intersections and Lie products of megaideals are also megaideals.

Corollary 1. All the members of the commutator (derived) and the lower central series of A, i.e. all the derivatives $A^{(n)}$ and all the Lie powers A^n ($A^{(n)} = [A^{(n-1)}, A^{(n-1)}]$, $A^n = [A, A^{n-1}]$, $A^{(0)} = A^0 = A$) are megaideals in A.

This corollary follows from Lemma 1 by induction since A is a megaideal in A.

Corollary 2. The center $A_{(1)}$ and all the other members of the upper central series $\{A_{(n)}, n \geq 0\}$ of A are megaideals in A.

Let us remind that $A_{(0)} = \{0\}$ and $A_{(n+1)}/A_{(n)}$ is the center of $A/A_{(n)}$.

Lemma 2. The radical (i.e. the maximal solvable ideal) and the nil-radical (i.e. the maximal nilpotent ideal) of A are its megaideals.

The above lemmas give a number of invariant subspaces of all the automorphisms in A and, therefore, simplify calculating $\mathrm{Aut}(A)$.

Table 1. Short review of results on classification of low-dimensional algebras

		P	$\dim A$	$\dim N$	Remarks
Lie [37]	1893	$\mathbb{C}$	≤ 4		
Bianchi [7]	1918	$\mathbb{R}$	3		
Lee [32]	1947	$\mathbb{R}$	3		Bianchi results
Vranceanu [85]	1947	$\mathbb{R}$	3		Bianchi results
Dobresku [12]	1953	$\mathbb{R}$	4		according to [42]
Kruchkovich [27, 28]	1954, 1957	$\mathbb{R}$	4		without 3-dim. Abelian ideals
Bratzlavsky [8]	1959	$\mathbb{R}$	4		according to [42]
Mubarakzyanov [49]	1963	$\mathbb{R}, \mathbb{C}$	4		
Dozias [13]	1963	$\mathbb{R}$	4		see [84]
Petrov [62]	1966	$\mathbb{R}$	4		completed Kruchkovich's classification
Ellis, Sciama [16]	1966	$\mathbb{R}$	4		according to [42]
MacCallum [42]	1979, 1999	$\mathbb{R}$	3, 4		
Patera, Zassenhaus [61]	1990	perfect	≤ 4		
Andrada et al [2]	2004	$\mathbb{R}$	4		
de Graaf [11]	2004	arbitrary	4		
Mubarakzyanov [50]	1963	$\mathbb{R}, \mathbb{C}$	5		
Umlauf [83]	1891	$\mathbb{C}$	6	6	nilpotent
Morozov [48]	1958	char $= 0$	6	6	nilpotent
Mubarakzyanov [51]	1963	$\mathbb{R}$	6	3	solvable ($\Rightarrow$decomposable)
Mubarakzyanov [51]	1963	$\mathbb{R}$	6	5	($\Rightarrow$solvable)
Turkowski [81]	1990	$\mathbb{R}$	6	4	($\Rightarrow$solvable)
Turkowski [80]	1988	$\mathbb{R}$	≤ 8		admit non-trivial Levi decomposition
Turkowski [82]	1992	$\mathbb{R}$	9		admit non-trivial Levi decomposition
Patera et al [58]	1976	$\mathbb{R}$	≤ 6		only nilpotent for $\dim A = 6$
Safiullina [70]	1964	$\mathbb{R}$	7	7	nilpotent, according to [73]
Magnin [45]	1986	$\mathbb{R}$	≤ 7	$\dim A$	nilpotent, according to [42]
Seeley [73]	1993	$\mathbb{R}$	7	7	nilpotent
Tsagas [78]	1999	$\mathbb{R}$	8	8	nilpotent
Tsagas et al [79]	2000	$\mathbb{R}$	9	9	nilpotent

Remark. In the above table we emphasize consequent results of Bianchi, Mubarakzyanov, Morozov and Turkowski, which together form classification of real Lie algebras of dimensions no greater than 6.

Example 1. Let mA_1 denote the m-dimensional abelian algebra. $\mathrm{Aut}(mA_1)$ coincides with the group of all the non-degenerated linear transformations of the m-dimensional linear space ($\sim GL(m)$) and $\mathrm{Int}(mA_1)$ contains only the identical transformation. Any vector subspace in the abelian algebra mA_1 is a subalgebra and an ideal in mA_1 and is not a characteristic subalgebra or a megaideal. Therefore, the abelian algebra mA_1 do not contain proper megaideals.

Example 2. Let us fix the canonical basis $\{e_1, e_2, e_3\}$ in the algebra $A = A_{2.1} \oplus A_1$ [49], in which only two first elements has the non-zero commutator $[e_1, e_2] = e_1$. In this basis

$$\mathrm{Aut}(A) \sim \left\{ \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ 0 & 1 & 0 \\ 0 & \alpha_{32} & \alpha_{33} \end{pmatrix} \middle| \ \alpha_{11}\alpha_{33} \neq 0 \right\}, \quad \mathrm{Int}(A) \sim \left\{ \begin{pmatrix} e^{\varepsilon_1} & \varepsilon_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \ \varepsilon_1, \varepsilon_2 \in \mathbb{R} \right\},$$

$$\mathrm{Der}(A) \sim \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle,$$

$$\mathrm{Ad}(A) \sim \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle.$$

A complete set of $\mathrm{Int}(A)$-inequivalent proper subalgebras of A is exhausted by the following ones [59]:

one-dimensional: $\langle pe_2 + qe_3 \rangle$, $\langle e_1 \pm e_3 \rangle$, $\langle e_1 \rangle$, $p^2 + q^2 = 1$;
two-dimensional: $\langle e_1, e_2 + \varkappa e_3 \rangle$, $\langle e_1, e_3 \rangle$, $\langle e_2, e_3 \rangle$.

Among them only $\langle e_1 \rangle$, $\langle e_3 \rangle$, $\langle e_1, e_3 \rangle$ are megaideals, $\langle e_1, e_2 + \varkappa e_3 \rangle$ is an ideal and is not a characteristic subalgebra (and, therefore, a megaideal).

Example 3. Consider the algebra $A = A^{-1}_{3.4}$ from the series $A^a_{3.4}$, $-1 \leq a < 1$, $a \neq 0$ [49]. The non-zero commutators of its canonical basis elements are $[e_1, e_3] = e_1$ and $[e_2, e_3] = -e_2$. $\mathrm{Aut}(A)$ is not connected for this algebra:

$$\mathrm{Aut}(A) \sim \left\{ \begin{pmatrix} \alpha_{11} & 0 & \alpha_{13} \\ 0 & \alpha_{22} & \alpha_{23} \\ 0 & 0 & 1 \end{pmatrix} \middle| \ \alpha_{11}\alpha_{22} \neq 0 \right\} \bigcup \left\{ \begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & 0 & \alpha_{23} \\ 0 & 0 & -1 \end{pmatrix} \middle| \ \alpha_{12}\alpha_{21} \neq 0 \right\},$$

$$\mathrm{Int}(A) \sim \left\{ \begin{pmatrix} e^{\varepsilon_1} & 0 & \varepsilon_2 \\ 0 & e^{-\varepsilon_1} & \varepsilon_3 \\ 0 & 0 & 1 \end{pmatrix} \middle| \ \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R} \right\},$$

$$\mathrm{Der}(A) \sim \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle,$$

$$\mathrm{Ad}(A) \sim \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle.$$

A complete set of $\mathrm{Int}(A)$-inequivalent proper subalgebras of A is exhausted by the following ones [59]:

one-dimensional: $\langle e_1 \rangle$, $\langle e_2 \rangle$, $\langle e_3 \rangle$, $\langle e_1 \pm e_2 \rangle$;
two-dimensional: $\langle e_1, e_2 \rangle$, $\langle e_1, e_3 \rangle$, $\langle e_2, e_3 \rangle$.

Among them only $\langle e_1, e_2 \rangle$ is a megaideal, $\langle e_1 \rangle$ and $\langle e_2 \rangle$ are characteristic subalgebras and are not megaideals.

Let M denote a n-dimensional smooth manifold and $\mathrm{Vect}(M)$ denote the Lie algebra of smooth vector fields (i.e. first-order linear differential operators) on M with the Lie bracket of vector fields as a commutator. Here and below smoothness means analyticity.

Definition. *A realization of a Lie algebra A in vector fields on M* is called a homomorphism $R\colon A \to \mathrm{Vect}(M)$. The realization is said *faithful* if $\ker R = \{0\}$ and *unfaithful* otherwise. Let G be a subgroup of $\mathrm{Aut}(A)$. The realizations $R_1\colon A \to \mathrm{Vect}(M_1)$ and $R_2\colon A \to \mathrm{Vect}(M_2)$ are called *G-equivalent* if there exist $\varphi \in G$ and a diffeomorphism f from M_1 to M_2 such that $R_2(v) = f_* R_1(\varphi(v))$ for all $v \in A$. Here f_* is the isomorphism from $\mathrm{Vect}(M_1)$ to $\mathrm{Vect}(M_2)$ induced by f. If G contains only the identical transformation, the realizations are called *strongly equivalent.* The realizations are *weakly equivalent* if $G = \mathrm{Aut}(A)$. A restriction of the realization R on a subalgebra A_0 of the algebra A is called *a realization induced by* R and is denoted as $R\big|_{A_0}$.

Within the framework of local approach that we use M can be considered as an open subset of $\mathbb{R}^n$ and all the diffeomorphisms are local.

Usually realizations of a Lie algebra have been classified with respect to the weak equivalence. This it is reasonable although the equivalence used in the representation theory is similar to the strong one. The strong equivalence suits better for construction of realizations of algebras using realizations of their subalgebras and is verified in a simpler way than the weak equivalence. It is not specified in some papers what equivalence has been used, and this results in classification mistakes.

To classify realizations of a m-dimensional Lie algebra A in the most direct way, we have to take m linearly independent vector fields of the general form $e_i = \xi^{ia}(x)\partial_a$, where $\partial_a = \partial/\partial x_a$, $x = (x_1, x_2, \ldots, x_n) \in M$, and require them to satisfy the commutation relations of A. As a result, we obtain a system of first-order PDEs for the coefficients ξ^{ia} and integrate it, considering all the possible cases. For each case we transform the solution into the simplest form, using either local diffeomorphisms of the space of x and automorphisms of A if the weak equivalence is meant or only local diffeomorphisms of the space of x for the strong equivalence. A drawback of this method is the necessity to solve a complicated nonlinear system of PDEs. Another way is to classify sequentially realizations of a series of nested subalgebras of A, starting with a one-dimensional subalgebra and ending up with A.

Let V be a subset of $\mathrm{Vect}(M)$ and $r(x) = \dim\langle V(x)\rangle$, $x \in M$. $0 \le r(x) \le n$. The general value of $r(x)$ on M is called the *rank* of V and is denoted as $\operatorname{rank} V$.

Lemma 3. Let B be a subset and R_1 and R_2 be realizations of the algebra A. If $R_1(B)$ and $R_2(B)$ are inequivalent with respect to endomorphisms of $\mathrm{Vect}(M)$ generated by diffeomorphisms on M. Then R_1 and R_2 are strongly inequivalent.

Corollary 2. If there exists a subset B of A such that $\operatorname{rank} R_1(B) \ne \operatorname{rank} R_2(B)$ then the realizations R_1 and R_2 are strongly inequivalent.

Lemma 4. Let I be a megaideal and R_1 and R_2 be realizations of the algebra A. If $R_1\big|_I$ and $R_2\big|_I$ are $\mathrm{Aut}(A)|_I$-inequivalent then R_1 and R_2 are weakly inequivalent.

Corollary 3. If $R_1\big|_I$ and $R_2\big|_I$ are weakly inequivalent then R_1 and R_2 also are weakly inequivalent.

Corollary 4. If there exists a megaideal I of A such that $\operatorname{rank} R_1(I) \ne \operatorname{rank} R_2(I)$ then the realizations R_1 and R_2 are weakly inequivalent.

Remark. In this paper we consider faithful realizations only. If the faithful realizations of Lie algebras of dimensions less than m are known, the unfaithful realizations of m-dimensional algebras can be constructed in an easy way. Indeed, each unfaithful realization of m-dimensional algebra A, having the kernel I, yields a faithful realization of the quotient algebra A/I of dimension less than m. This correspondence is well-defined since the kernel of any homomorphism from an algebra A to an algebra A' is an ideal in A.

4 Realizations of low-dimensional real Lie algebras

The most important and elegant results on realizations of Lie algebras were obtained by S. Lie himself. He classified non-singular Lie algebras of vector fields in one real variable, one complex variable and two complex variables [33, 34]. Using an ingenious geometric argument, Lie also listed the Lie algebras of vector fields in two real variables [37, Vol.3] (a more complete discussion can be found in [21]). Finally, in [37, Vol.3] he claimed to have completely classified all Lie algebras of vector field in three complex variables (in fact he gives details in the case of primitives algebras, and divides the imprimitive cases into three classes, of which only the first two are treated [21]).

Using Lie's classification of Lie algebras of vector fields in two complex variables, A. González-López, N. Kamran and P. Olver [22] studied finite-dimensional Lie algebras of first order differential operators $Q = \xi^i(x)\partial_{x_i} + f(x)$ and classified all of such algebras with two complex variables.

In [43] F.M. Mahomed and P.G.L. Leach investigated realizations of three-dimensional real Lie algebras in terms of Lie vector fields in two variables and used them for treating third order ODEs. Analogous realizations for four-dimensional real Lie algebras without commutative three-dimensional subalgebras were considered by A. Schmucker and G. Czichowski [71].

All the possible realizations of algebra $so(3)$ in the real vectors fields were first classified in [31, 94]. Covariant realizations of physical algebras (Galilei, Poincaré and Euclid ones) were constructed in [17, 18, 19, 20, 31, 38, 39, 90, 93, 94]. Complete description of realizations of the Galilei algebra in the space of two dependent and two independent variables was obtained in [68, 91]. In [26] I.L. Kantor and J. Patera described finite-dimensional Lie algebras of polynomial (degree ≤ 3) vector fields in n real variables that contain the vector fields ∂_{x_i} $(i = \overline{1,n})$. In [64, 65, 66] G. Post studied finite-dimensional Lie algebras of polynomial vector fields of n variables that contain the vector fields ∂_{x_i} $(i = \overline{1,n})$ and $x_i\partial_{x_i}$.

C. Wafo Soh and F.M. Mahomed [87] used Mubarakzyanov's results [49] in order to classify realizations of three- and four-dimensional real Lie algebras in the space of three variables and to describe systems of two second-order ODEs admitting real four-dimensional symmetry Lie algebras, but unfortunately their paper contains some misprints and incorrect statements (see Section 6 of our paper). Therefore, this classification cannot be regarded as complete. The results of [87] are used in [88] to solve the problem of linearization of systems of second-order ordinary differential equations, so some results from [88] also are not completely correct.

A preliminary classification of realizations of solvable three-dimensional Lie algebras in the space of any (finite) number of variables was given in [40]. Analogous results on a complete set of inequivalent realizations for real four-dimensional solvable Lie algebras were announced at the Fourth International Conference "Symmetry in Nonlinear Mathematical Physics" (9–15 July, 2001, Kyiv) and were published in the proceedings of this conference [41, 56].

In this paper we present final results of our classifications of realizations of all the Lie algebras of dimension up to 4. On account of them being cumbersome we adduce only classification of realizations with respect to weak equivalence because it is more complicated to obtain, is more suitable for applications and can be presented in a more compact form. The results are formulated in the form of Tables 2–6. Below equivalence indicates weak equivalence.

Remarks for Tables 2–6. We use the following notation, contractions and agreements.

- We treat Mubarakzyanov's classification of abstract Lie algebras and follow, in general, his numeration of Lie algebras. For each algebra we write down only non-zero commutators between the basis elements. ∂_i is a shorthand for $\partial/\partial x_i$. $R(A,N)$ denotes the N-th realization of the algebra A corresponding to position in the table, and the algebra symbol A can be omitted if is clear what algebra is meant. If it is necessary we also point out parameter symbol $\alpha_1,\ldots,\alpha_k$ in the designation $R(A,N,(\alpha_1,\ldots,\alpha_k))$ of series of realizations.
- The constant parameters of series of solvable Lie algebras (e.g., $A^b_{4.2}$) are denoted as a, b or c. All the other constants as well as the functions in Tables 2–6 are parameters of realization series. The functions are arbitrary differentiable real-valued functions of their arguments, satisfying

only the conditions given in remarks after the respective table. The presence of such remark for a realization is marked in the last column of the table. All the constants are real. The constant ε takes only two values 0 or 1, i.e. $\varepsilon \in \{0; 1\}$. The conditions for the other constant parameters of realization series are given in remarks after the corresponding table.

- For each series of solvable Lie algebras we list, at first, the "common" inequivalent realizations (more precisely, the inequivalent realizations series parametrized with the parameters of algebra series) existing for all the allowed values of the parameters of algebra series. Then, we list the "specific" realizations which exist or are inequivalent to "common" realizations only for some "specific" sets of values of the parameters. Numeration of "specific" realizations for each "specific" set of values of the parameters is continuation of that for "common" realizations.
- In all the conditions of algebra equivalence, which are given in remarks after tables, $(\alpha_{\mu\nu})$ is a non-degenerate $(r \times r)$-matrix, where r is the dimension of the algebra under consideration.
- The summation over repeated indices is implied unless stated otherwise.

Remarks on the series $A_{4,5}$ and $A_{4,6}$. Consider the algebra series $\{A_{4,5}^{a_1,a_2,a_3} \mid a_1a_2a_3 \neq 0\}$ generated by the algebras for which the non-zero commutation relations have the form $[e_1, e_4] = a_1e_1$, $[e_2, e_4] = a_2e_2$, $[e_3, e_4] = a_3e_3$. Two algebras from this series, with the parameters (a_1, a_2, a_3) and $(\tilde a_1, \tilde a_2, \tilde a_3)$ are equivalent iff there exist a real $\lambda \neq 0$ and a permutation (j_1, j_2, j_3) of the set $\{1; 2; 3\}$ such that the condition $\tilde a_i = \lambda a_{j_i}$ $(i = \overline{1,3})$ is satisfied. For the algebras under consideration to be inequivalent, one has to constrain the set of parameter values. There are different ways to do this. A traditional way [10, 49, 58, 59, 87] is to apply the condition $-1 \leq a_2 \leq a_3 \leq a_1 = 1$. But this condition is not sufficient to select inequivalent algebras since the algebras $A_{4,5}^{1,-1,b}$ and $A_{4,5}^{1,-1,-b}$ are equivalent in spite of their parameters satisfying the above constraining condition if $|b| \leq 1$. The additional condition $a_3 \geq 0$ if $a_2 = -1$ guarantees for the algebras with constrained parameters to be inequivalent.

Moreover, it is convenient for us to break the parameter set into three disjoint subsets depending on the number of equal parameters. Each from these subsets is normalized individually. As a result we obtain three inequivalent cases:

$$a_1 = a_2 = a_3 = 1; \quad a_1 = a_2 = 1,\ a_3 \neq 1, 0; \quad -1 \leq a_1 < a_2 < a_3 = 1,\ a_2 > 0 \text{ if } a_1 = -1.$$

An analogous remark is true also for the algebra series $\{A_{4,6}^{a,b} \mid a \neq 0\}$ generated by the algebras for which the non-zero commutation relations have the form $[e_1, e_4] = ae_1$, $[e_2, e_4] = be_2 - e_3$, $[e_3, e_4] = e_2 + be_3$. Two algebras from this series with the different parameters (a, b) and $(\tilde a, \tilde b)$ are equivalent iff $\tilde a = -a$, $\tilde b = -b$. A traditional way of constraining the set of parameter values is to apply the condition $b \geq 0$ that does not exclude the equivalent algebras of the form $A_{4,6}^{a,0}$ and $A_{4,6}^{-a,0}$ from consideration. That is why it is more correct to use the condition $a > 0$ as a constraining relation for the parameters of this series.

The technique of classification is the following.

- For each algebra A from Mubarakzyanov's classification [49] of abstract Lie algebras of dimension $m \leq 4$ we find the automorphism group $\mathrm{Aut}(A)$ and the set of megaideals of A. (Our notions of low-dimensional algebras, choice of their basis elements, and, consequently, the form of commutative relations coincide with Mubarakzyanov's ones.) Calculations of this step is quite simple due to low dimensions and simplicity of the canonical commutation relations. Lemmas 1 and 2, Corollary 1 and other similar statements are useful for such calculations. See also the remarks below.

- We choose a maximal proper subalgebra B of A. As rule, dimension of B is equal to $m - 1$. So, if A is solvable, it necessarily contains a $(m - 1)$-dimensional ideal. The simple algebra $sl(2, \mathbb{R})$ has a two-dimensional subalgebra. The Levi factors of unsolvable four-dimensional algebras ($sl(2, \mathbb{R}) \oplus A_1$ and $so(3) \oplus A_1$) are three-dimensional ideals of these algebras. Only $so(3)$ does not contain a subalgebra of dimension $m - 1 = 2$ that is a reason of difficulties in constructing realizations for this algebra. Moreover, the algebras $sl(2,\mathbb{R})$, $so(3)$, mA_1, $A_{3.1}$, $A_{3.1}\oplus A_1$ and $2A_{2.1}$ exhaust the list of algebras under consideration that do not contain $(m-1)$–dimensional megaideals.

- Let us suppose that a complete list of strongly inequivalent realizations of B has been already constructed. (If B is a megaideal of A and realizations of A are classified only with respect to the weak equivalence, it is sufficient to use only $\mathrm{Aut}(A)|_B$-inequivalent realizations of B.) For each realization $R(B)$ from this list we make the following procedure. We find the set $\mathrm{Diff}^{R(B)}$ of local diffeomorphisms of the space of x, which preserve $R(B)$. Then, we realize the basis vector e_i (or the basis vectors in the case of $so(3)$) from $A\backslash B$ in the most general form $e_i=\xi^{ia}(x)\partial_a$, where $\partial_a=\partial/\partial x_a$, and require that it satisfied the commutation relations of A with the basis vectors from $R(B)$. As a result, we obtain a system of first-order PDEs for the coefficients ξ^{ia} and integrate it, considering all possible cases. For each case we reduce the found solution to the simplest form, using either diffeomorphisms from $\mathrm{Diff}^{R(B)}$ and automorphisms of A if the weak equivalence is meant or only diffeomorphisms from $\mathrm{Diff}^{R(B)}$ for the strong equivalence.
- The last step is to test inequivalence of the constructed realizations. We associate the N-th one of them with the ordered collection of integers (r_{Nj}), where r_{Nj} is equal to the rank of the elements of S_j in the realization $R(A,N)$. Here S_j is either the j-th subset of basis of A with $|S_j|>1$ in the case of strong equivalence or the basis of the j-th megaideals I_j of A with $\dim I_j>1$ in the case of weak equivalence. Inequivalence of realizations with different associated collection of integers immediately follows from Corollary 2 or Corollary 4 respectively. Inequivalence of realizations in the pairs with identical collections of ranks is proved using another method, e.g. Casimir operators (for simple algebras), Lemmas 2 and 3, Corollary 3 and the rule of constraries (see the following section).

We rigorously proved inequivalence of all the constructed realizations. Moreover, we compared our classification with results of the papers, cited in the beginning of the section (see Section 6 for details of comparison with results of one of them).

Remark. Another interesting method to construct realizations of Lie algebras in vector fields was proposed by I. Shirokov [74, 75, 3]. This method is also simple to use and based on classification of subalgebras of Lie algebras.

Remark. The automorphisms of semi-simple algebras are well-known [25]. The automorphisms of three-dimensional algebras were considered in [23]. Let us note that in this paper only connected components of the unity of the automorphism groups were constructed really, i.e. the discrete transformations were missed. The automorphism groups of four-dimensional algebras were published in [9] (with a few misprints, which were corrected in Corrigendum). Namely, for the algebra $A^{\alpha}_{4.2}$ the automorphism groups are to have the following form (we preserve the notations of [9]):

$$\mathrm{Aut}(A^{\alpha}_{4.2})=\left\{\left.\begin{pmatrix} a_1 & 0 & 0 & a_4\\ 0 & a_6 & a_7 & a_8\\ 0 & 0 & a_6 & a_{12}\\ 0 & 0 & 0 & 1\end{pmatrix}\right|\ a_1a_6\neq 0\right\},\quad \alpha\neq 0,1;$$

$$\mathrm{Aut}(A^{1}_{4.2})=\left\{\left.\begin{pmatrix} a_1 & 0 & a_3 & a_4\\ a_5 & a_6 & a_7 & a_8\\ 0 & 0 & a_6 & a_{12}\\ 0 & 0 & 0 & 1\end{pmatrix}\right|\ a_1a_6\neq 0\right\}.$$

Remark. As for any classification which are performed up to an equivalence relation, there exist the problem of choice of canonical forms of realizations for an arbitrary fixed Lie algebra. Such choice can be made in a number of ways and depends, in particular, on the choice of canonical form of commutation relations (i.e. structure constants) of the algebra and the dimension of realization manifold (see e.g. Remark after Table 6).

The closed problem on reduction of realizations to the linear form (i.e. on search of realizations in vector fields having linear coefficients with respect to x), including questions of existence of such forms and a minimum of possible dimensions of the realization manifold, is also opened. This problem is equivalent to construction of finite-dimensional representations of the algebras under consideration.

5 Example: realizations of $A_{4.10}$

We consider in detail constructing of a list of inequivalent realizations for the algebra $A_{4.10}$. The non-zero commutators between the basis elements of $A_{4.10}$ are as follows:

$$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_2, \quad [e_1, e_4] = -e_2, \quad [e_2, e_4] = e_1.$$

The automorphism group $\mathrm{Aut}(A_{4.10})$ is generated by the basis transformations of the form $\tilde{e}_\mu = \alpha_{\nu\mu} e_\nu$, where $\mu, \nu = \overline{1,4}$,

$$(\alpha_{\nu\mu}) = \begin{pmatrix} \pm\alpha_{22} & \alpha_{12} & \alpha_{13} & \pm\alpha_{23} \\ \mp\alpha_{12} & \alpha_{22} & \alpha_{23} & \mp\alpha_{13} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}. \tag{1}$$

The algebra $A_{4.10}$ contains four non-zero megaideals:

$$I_1 = \langle e_1, e_2 \rangle \sim 2A_1, \quad I_2 = \langle e_1, e_2, e_3 \rangle \sim A_{3.3}, \quad I_3 = \langle e_1, e_2, e_4 \rangle \sim A^0_{3.5},$$
$$I_4 = \langle e_1, e_2, e_3, e_4 \rangle \sim A_{4.10}.$$

Realizations of two three-dimensional megaideals I_2 and I_3 can be extended by means of one additional basis element to realizations of $A_{4.10}$. To this end we use I_2. This megaideal has four inequivalent realizations $R(A_{3.3}, N)$ ($N = \overline{1,4}$) in Lie vector fields (see Table 3). Let us emphasize that it is inessential for the algebra $A_{3.3}$ which equivalence (strong or weak) has been used for classifying realizations. For each of these realizations we perform the following procedure. Presenting the fourth basis element in the most general form $e_4 = \xi^a(x)\partial_a$ and commuting e_4 with the other basis elements, we obtain a linear overdetermined system of first-order PDEs for the functions ξ^a. Then we solve this system and simplify its general solution by means of transformations $\tilde{x}_a = f^a(x)$ ($a = \overline{1,n}$) which preserve the forms of e_1, e_2, and e_3 in the considered realization of $A_{3.3}$. To find the appropriate functions $f^a(x)$, we are to solve one more system of PDEs which results from the conditions $e_i\big|_{x_a \to \tilde{x}_a} = (e_i f^a)(x)\partial_{\tilde{x}_a}$ if $\tilde{x}_a = f^a(x)$, $i = \overline{1,3}$. The last step is to prove inequivalence of all the constructed realizations.

So, for the realization $R(A_{3.3}, 1)$ we have $e_1 = \partial_1$, $e_2 = \partial_2$, $e_3 = x_1\partial_1 + x_2\partial_2 + \partial_3$, and the commutation relations imply the following system on the functions ξ^a:

$$\begin{aligned} &[e_1, e_4] = -e_2 \quad \Rightarrow \quad \xi^1_1 = 0, \qquad \xi^2_1 = -1, \qquad \xi^k_1 = 0, \\ &[e_2, e_4] = e_1 \quad \Rightarrow \quad \xi^1_2 = 1, \qquad \xi^2_2 = 0, \qquad \xi^k_2 = 0, \qquad k = \overline{3,n}, \\ &[e_3, e_4] = 0 \quad \Rightarrow \quad \xi^1_3 = \xi^1 - x_2, \quad \xi^2_3 = \xi^2 + x_1, \quad \xi^k_3 = 0, \end{aligned}$$

the general solution of which can be easy found:

$$\xi^1 = x_2 + \theta^1(\hat{x})e^{x_3}, \quad \xi^2 = -x_1 + \theta^2(\hat{x})e^{x_3}, \quad \xi^k = \theta^k(\hat{x}), \quad k = \overline{3,n},$$

where θ^a ($a = \overline{1,n}$) are arbitrary smooth functions of $\hat{x} = (x_4, \ldots, x_n)$. The form of e_1, e_2, and e_3 are preserved only by the transformations

$$\tilde{x}_1 = x_1 + f^1(\hat{x})e^{x_3}, \quad \tilde{x}_2 = x_2 + f^2(\hat{x})e^{x_3}, \quad \tilde{x}_3 = x_3 + f^3(\hat{x}), \quad \tilde{x}_\alpha = f^\alpha(\hat{x}), \quad \alpha = \overline{4,n},$$

where f^a ($a = \overline{1,n}$) are arbitrary smooth functions of $\hat{x}$, and f^α ($\alpha = \overline{4,n}$) are functionally independent. Depending on values of the parameter-functions θ^k ($k = \overline{3,n}$) there exist three cases of possible reduction of e_4 to canonical forms by means of these transformations, namely,

$$\exists\,\alpha:\theta^\alpha\neq 0 \quad\Rightarrow\quad e_4=x_2\partial_1-x_1\partial_2+\partial_4 \quad \text{(the realization } R(A_{4.10},1));$$
$$\theta^\alpha=0,\ \theta^3\neq\text{const} \quad\Rightarrow\quad e_4=x_2\partial_1-x_1\partial_2+x_4\partial_3 \quad \text{(the realization } R(A_{4.10},2));$$
$$\theta^\alpha=0,\ \theta^3=\text{const} \quad\Rightarrow\quad e_4=x_2\partial_1-x_1\partial_2+C\partial_3 \quad \text{(the realization } R(A_{4.10},3,C)).$$

Here C ia an arbitrary constant.

The calculations for other realizations of $A_{3.3}$ are easier than for the first one. Below for each from these realizations we adduce brief only the general solution of the system of PDEs for the coefficients ξ^a, the transformations which preserve the forms of e_1, e_2, and e_3 in the considered realization of $A_{3.3}$, and the respective realizations of $A_{4.10}$.

$R(A_{3.3},2)$: $\xi^1=x_2,\ \xi^2=-x_1,\ \xi^k=\theta^k(\bar{x}),\quad k=\overline{3,n},\quad \bar{x}=(x_3,\dots,x_n);$
$\tilde{x}_1=x_1,\quad \tilde{x}_2=x_2,\quad \tilde{x}_k=f^k(\bar{x});$
$R(A_{4.10},5)$ if $\exists\,k\colon \theta^k\neq 0$ and $R(A_{4.10},6)$ if $\theta^k=0$.

$R(A_{3.3},3)$: $\xi^1=-x_1x_2+\theta^1(x')e^{x_3},\ \xi^2=-(1+x_2^2),\ \xi^k=\theta^k(x');$
$\tilde{x}_1=x_1+f(x')e^{x_3},\quad \tilde{x}_2=x_2,\quad \tilde{x}_3=x_3+f^3(x'),\quad \tilde{x}_\alpha=f^\alpha(x');$
$R(A_{4.10},4);\quad k=\overline{3,n},\quad \alpha=\overline{4,n},\quad x'=(x_2,x_4,x_5,\dots,x_n).$

$R(A_{3.3},4)$: $\xi^1=-x_1x_2,\ \xi^2=-(1+x_2^2),\ \xi^k=\theta^k(\check{x}),\quad k=\overline{3,n},\quad \check{x}=(x_2,\dots,x_n);$
$\tilde{x}_1=x_1,\quad \tilde{x}_2=x_2,\quad \tilde{x}_k=f^k(\check{x});$
$R(A_{4.10},7)$.

Here θ^a $(a=\overline{1,n})$ are arbitrary smooth functions of their arguments, and f^a $(a=\overline{1,n})$ are such smooth functions of their arguments that the respective transformation of x is not singular.

To prove inequivalence of the constructed realizations, we associate the N-th of them with the ordered collection of integers $(r_{N1},r_{N2},r_{N3},r_{N4})$, where $r_{Nj}=\operatorname{rank}R(A_{4.10},N)\big|_{I_j}$, i.e. r_{Nj} is equal to the rank of basis elements of the megaideals I_j in the realization $R(A_{4.10},N)$, $(N=\overline{1,7},\ j=\overline{1,4})$:

$$R(A_{4.10},1)\longrightarrow(2,3,3,4);\quad R(A_{4.10},2)\longrightarrow(2,3,3,3);$$
$$R(A_{4.10},3,C)\longrightarrow(2,3,3,3)\ \text{ if }\ C\neq 0\ \text{ and }\ R(A_{4.10},3,0)\longrightarrow(2,3,2,3);$$
$$R(A_{4.10},4)\longrightarrow(1,2,2,3);\quad R(A_{4.10},5)\longrightarrow(2,2,3,3);$$
$$R(A_{4.10},6)\longrightarrow(2,2,2,2);\quad R(A_{4.10},7)\longrightarrow(1,1,2,2).$$

Inequivalence of realizations with different associated collections of integers follows immediately from Corollary 4. The collections of ranks of megaideals coincide only for the pairs of realizations of two forms

$$\{R(A_{4.10},2),R(A_{4.10},3,C)\}\quad\text{and}\quad\{R(A_{4.10},3,C),R(A_{4.10},3,\tilde{C})\},$$

where $C,\tilde{C}\neq 0$. Inequivalence of realizations in these pairs is to be proved using another method, e.g. the rule of contraries.

Let us suppose that the realizations $R(A_{4.10},2)$ and $R(A_{4.10},3,C)$ are equivalent and let us fix their bases given in Table 5. Then, by the definition of equivalence there exists an automorphism of $A_{4.10}$ $\tilde{e}_\mu=\alpha_{\nu\mu}e_\nu$ and a change of variables $\tilde{x}_a=g^a(x)$ which transform the basis of $R(A_{4.10},2)$ into the basis of $R(A_{4.10},3,C)$. (Here $\mu,\nu=\overline{1,4}$, $a=\overline{1,n}$, and the matrix $(\alpha_{\nu\mu})$ has the form (1).) For this condition to hold true, the function g^3 is to satisfy the following system of PDEs:

$$g^3_1=0,\quad g^3_2=0,\quad g^3_3=1,\quad x_4g^3_3=C$$

which implies the contradictory equality $x_4=C$. Therefore, the considered realizations are inequivalent.

In an analogous way we obtain that the realizations $R(A_{4.10},3,C)$ and $R(A_{4.10},3,\tilde{C})$ are equivalent iff $C=\tilde{C}$.

Table 2. Realizations of one and two-dimensional real Lie algebras

Algebra	N	Realization	$(*)$
A_1	1	∂_1	
$2A_1$	1 2	$\partial_1,\, \partial_2$ $\partial_1,\, x_2\partial_1$	
$A_{2.1}$ $[e_1,e_2]=e_1$	1 2	$\partial_1,\, x_1\partial_1+\partial_2$ $\partial_1,\, x_1\partial_1$	

Table 3. Realizations of three-dimensional solvable Lie algebras

Algebra	N	Realization	$(*)$
$3A_1$	1 2 3 4 5	$\partial_1,\, \partial_2,\, \partial_3$ $\partial_1,\, \partial_2,\, x_3\partial_1+x_4\partial_2$ $\partial_1,\, \partial_2,\, x_3\partial_1+\varphi(x_3)\partial_2$ $\partial_1,\, x_2\partial_1,\, x_3\partial_1$ $\partial_1,\, x_2\partial_1,\, \varphi(x_2)\,\partial_1$	 $(*)$ $(*)$
$A_{2.1}\oplus A_1$ $[e_1,e_2]=e_1$	1 2 3 4	$\partial_1,\, x_1\partial_1+\partial_3,\, \partial_2$ $\partial_1,\, x_1\partial_1+x_3\partial_2,\, \partial_2$ $\partial_1, x_1\partial_1,\, \partial_2$ $\partial_1,\, x_1\partial_1+x_2\partial_2,\, x_2\partial_1$	
$A_{3.1}$ $[e_2,e_3]=e_1$	1 2 3	$\partial_1,\, \partial_2,\, x_2\partial_1+\partial_3$ $\partial_1,\, \partial_2,\, x_2\partial_1+x_3\partial_2$ $\partial_1,\, \partial_2,\, x_2\partial_1$	
$A_{3.2}$ $[e_1,e_3]=e_1$ $[e_2,e_3]=e_1+e_2$	1 2 3	$\partial_1,\, \partial_2,\, (x_1+x_2)\,\partial_1+x_2\partial_2+\partial_3$ $\partial_1,\, \partial_2,\, (x_1+x_2)\,\partial_1+x_2\partial_2$ $\partial_1,\, x_2\partial_1,\, x_1\partial_1-\partial_2$	
$A_{3.3}$ $[e_1,e_3]=e_1$ $[e_2,e_3]=e_2$	1 2 3 4	$\partial_1,\, \partial_2,\, x_1\partial_1+x_2\partial_2+\partial_3$ $\partial_1,\, \partial_2,\, x_1\partial_1+x_2\partial_2$ $\partial_1,\, x_2\partial_1,\, x_1\partial_1+\partial_3$ $\partial_1,\, x_2\partial_1,\, x_1\partial_1$	
$A^a_{3.4}$, $\lvert a\rvert\le 1$, $a\ne 0,1$ $[e_1,e_3]=e_1$ $[e_2,e_3]=ae_2$	1 2 3	$\partial_1,\, \partial_2,\, x_1\partial_1+ax_2\partial_2+\partial_3$ $\partial_1,\, \partial_2,\, x_1\partial_1+ax_2\partial_2$ $\partial_1,\, x_2\partial_1,\, x_1\partial_1+(1-a)x_2\partial_2$	
$A^b_{3.5}$, $b\ge 0$ $[e_1,e_3]=be_1-e_2$ $[e_2,e_3]=e_1+be_2$	1 2 3	$\partial_1,\, \partial_2,\, (bx_1+x_2)\partial_1+(-x_1+bx_2)\partial_2+\partial_3$ $\partial_1,\, \partial_2,\, (bx_1+x_2)\partial_1+(-x_1+bx_2)\partial_2$ $\partial_1,\, x_2\partial_1,\, (b-x_2)x_1\partial_1-(1+x_2^2)\partial_2$	

Remarks for Table 3.

$R(3A_1,3,\varphi)$. $\varphi=\varphi(x_3)$. The realizations $R(3A_1,3,\varphi)$ and $R(3A_1,3,\tilde\varphi)$ are equivalent iff

$$\begin{aligned}\tilde x_3 &= -(\alpha_{11}x_3+\alpha_{12}\varphi(x_3)-\alpha_{13})/(\alpha_{31}x_3+\alpha_{32}\varphi(x_3)-\alpha_{33}),\\ \tilde\varphi &= -(\alpha_{21}x_3+\alpha_{22}\varphi(x_3)-\alpha_{23})/(\alpha_{31}x_3+\alpha_{32}\varphi(x_3)-\alpha_{33}).\end{aligned} \tag{2}$$

$R(3A_1,5,\varphi)$. $\varphi=\varphi(x_2)$, $\varphi''\ne 0$. The realizations $R(3A_1,5,\varphi)$ and $R(3A_1,5,\tilde\varphi)$ are equivalent iff

$$\begin{aligned}\tilde x_2 &= -(\alpha_{21}x_2+\alpha_{22}\varphi(x_2)-\alpha_{23})/(\alpha_{11}x_2+\alpha_{12}\varphi(x_2)-\alpha_{13}),\\ \tilde\varphi &= -(\alpha_{31}x_2+\alpha_{32}\varphi(x_2)-\alpha_{33})/(\alpha_{11}x_2+\alpha_{12}\varphi(x_2)-\alpha_{13}).\end{aligned} \tag{3}$$

Table 4. Realizations of real decomposable solvable four-dimensional Lie algebras

Algebra	N	Realization	$(*)$
$4A_1$	1	$\partial_1,\, \partial_2,\, \partial_3,\, \partial_4$	
	2	$\partial_1,\, \partial_2,\, \partial_3,\, x_4\partial_1 + x_5\partial_2 + x_6\partial_3$	
	3	$\partial_1,\, \partial_2,\, \partial_3,\, x_4\partial_1 + x_5\partial_2 + \theta(x_4, x_5)\partial_3$	$(*)$
	4	$\partial_1,\, \partial_2,\, \partial_3,\, x_4\partial_1 + \varphi(x_4)\partial_2 + \psi(x_4)\partial_3$	$(*)$
	5	$\partial_1,\, \partial_2,\, x_3\partial_1 + x_4\partial_2,\, x_5\partial_1 + x_6\partial_2$	
	6	$\partial_1,\, \partial_2,\, x_3\partial_1 + x_4\partial_2,\, x_5\partial_1 + \theta(x_3, x_4, x_5)\partial_2$	$(*)$
	7	$\partial_1,\, \partial_2,\, x_3\partial_1 + \varphi(x_3, x_4)\partial_2,\, x_4\partial_1 + \psi(x_3, x_4)\partial_2$	$(*)$
	8	$\partial_1,\, \partial_2,\, x_3\partial_1 + \varphi(x_3)\partial_2,\, \theta(x_3)\partial_1 + \psi(x_3)\partial_2$	$(*)$
	9	$\partial_1,\, x_2\partial_1,\, x_3\partial_1,\, x_4\partial_1$	
	10	$\partial_1,\, x_2\partial_1,\, x_3\partial_1,\, \theta(x_2, x_3)\partial_1$	$(*)$
	11	$\partial_1,\, x_2\partial_1,\, \varphi(x_2)\partial_1,\, \psi(x_2)\partial_1$	$(*)$
$A_{2.1} \oplus 2A_1$	1	$\partial_1,\, x_1\partial_1 + \partial_4,\, \partial_2,\, \partial_3$	
$[e_1, e_2] = e_1$	2	$\partial_1,\, x_1\partial_1 + x_4\partial_2 + x_5\partial_3,\, \partial_2,\, \partial_3$	
	3	$\partial_1,\, x_1\partial_1 + x_4\partial_2 + \varphi(x_4)\partial_3,\, \partial_2,\, \partial_3$	$(*)$
	4	$\partial_1,\, x_1\partial_1,\, \partial_2,\, \partial_3$	
	5	$\partial_1,\, x_1\partial_1 + x_3\partial_3,\, \partial_2,\, x_3\partial_1 + x_4\partial_2$	
	6	$\partial_1,\, x_1\partial_1 + x_3\partial_3,\, \partial_2,\, x_3\partial_1$	
	7	$\partial_1,\, x_1\partial_1 + \partial_4,\, \partial_2,\, x_3\partial_2$	
	8	$\partial_1,\, x_1\partial_1 + x_4\partial_2,\, \partial_2,\, x_3\partial_2$	
	9	$\partial_1,\, x_1\partial_1 + \varphi(x_3)\partial_2,\, \partial_2,\, x_3\partial_2$	$(*)$
	10	$\partial_1,\, x_1\partial_1 + x_2\partial_2 + x_3\partial_3,\, x_2\partial_1,\, x_3\partial_1$	
$2A_{2.1}$	1	$\partial_1,\, x_1\partial_1 + \partial_3,\, \partial_2,\, x_2\partial_2 + \partial_4$	
$[e_1, e_2] = e_1$	2	$\partial_1,\, x_1\partial_1 + \partial_3,\, \partial_2,\, x_2\partial_2 + x_4\partial_3$	
$[e_3, e_4] = e_3$	3	$\partial_1,\, x_1\partial_1 + \partial_3,\, \partial_2,\, x_2\partial_2 + C\partial_3$	$(*)$
	4	$\partial_1,\, x_1\partial_1 + x_3\partial_2,\, \partial_2,\, x_2\partial_2 + x_3\partial_3$	
	5	$\partial_1,\, x_1\partial_1,\, \partial_2,\, x_2\partial_2$	
	6	$\partial_1,\, x_1\partial_1 + x_2\partial_2,\, x_2\partial_1,\, -x_2\partial_2 + \partial_3$	
	7	$\partial_1,\, x_1\partial_1 + x_2\partial_2,\, x_2\partial_1,\, -x_2\partial_2$	
$A_{3.1} \oplus A_1$	1	$\partial_1,\, \partial_3,\, x_3\partial_1 + \partial_4,\, \partial_2$	
$[e_2, e_3] = e_1$	2	$\partial_1,\, \partial_3,\, x_3\partial_1 + x_4\partial_2 + x_5\partial_3,\, \partial_2$	
	3	$\partial_1,\, \partial_3,\, x_3\partial_1 + \varphi(x_4)\partial_2 + x_4\partial_3,\, \partial_2$	$(*)$
	4	$\partial_1,\, \partial_3,\, x_3\partial_1 + x_4\partial_2,\, \partial_2$	
	5	$\partial_1,\, \partial_3,\, x_3\partial_1,\, \partial_2$	
	6	$\partial_1,\, \partial_3,\, x_3\partial_1 + \partial_4,\, x_2\partial_1$	
	7	$\partial_1,\, \partial_3,\, x_3\partial_1 + x_4\partial_3,\, x_2\partial_1$	
	8	$\partial_1,\, \partial_3,\, x_3\partial_1 + \varphi(x_2)\partial_3,\, x_2\partial_1$	$(*)$
$A_{3.2} \oplus A_1$	1	$\partial_1,\, \partial_2,\, (x_1 + x_2)\partial_1 + x_2\partial_2 + \partial_3,\, \partial_4$	
$[e_1, e_3] = e_1$	2	$\partial_1,\, \partial_2,\, (x_1 + x_2)\partial_1 + x_2\partial_2 + \partial_3,\, x_4\partial_3$	
$[e_2, e_3] = e_1 + e_2$	3	$\partial_1,\, \partial_2,\, (x_1 + x_2)\partial_1 + x_2\partial_2,\, \partial_3$	
	4	$\partial_1,\, \partial_2,\, (x_1 + x_2)\partial_1 + x_2\partial_2 + \partial_3,\, x_4e^{x_3}(x_3\partial_1 + \partial_2)$	
	5	$\partial_1,\, \partial_2,\, (x_1 + x_2)\partial_1 + x_2\partial_2 + \partial_3,\, e^{x_3}(x_3\partial_1 + \partial_2)$	
	6	$\partial_1,\, \partial_2,\, (x_1 + x_2)\partial_1 + x_2\partial_2 + \partial_3,\, e^{x_3}\partial_1$	
	7	$\partial_1,\, x_2\partial_1,\, x_1\partial_1 - \partial_2,\, \partial_3$	
	8	$\partial_1,\, x_2\partial_1,\, x_1\partial_1 - \partial_2,\, x_3e^{-x_2}\partial_1$	
	9	$\partial_1,\, x_2\partial_1,\, x_1\partial_1 - \partial_2,\, e^{-x_2}\partial_1$	
$A_{3.3} \oplus A_1$	1	$\partial_1,\, \partial_2,\, x_1\partial_1 + x_2\partial_2 + \partial_3,\, \partial_4$	
$[e_1, e_3] = e_1$	2	$\partial_1,\, \partial_2,\, x_1\partial_1 + x_2\partial_2 + \partial_3,\, x_4\partial_3$	
$[e_2, e_3] = e_2$	3	$\partial_1,\, \partial_2,\, x_1\partial_1 + x_2\partial_2,\, \partial_3$	
	4	$\partial_1,\, \partial_2,\, x_1\partial_1 + x_2\partial_2 + \partial_3,\, e^{x_3}(\partial_1 + x_4\partial_2)$	
	5	$\partial_1,\, \partial_2,\, x_1\partial_1 + x_2\partial_2 + \partial_3,\, e^{x_3}\partial_1$	
	6	$\partial_1,\, x_2\partial_1,\, x_1\partial_1 + \partial_3,\, \partial_4$	
	7	$\partial_1,\, x_2\partial_1,\, x_1\partial_1 + \partial_3,\, x_4\partial_3$	
	8	$\partial_1,\, x_2\partial_1,\, x_1\partial_1 + \partial_3,\, \varphi(x_2)\partial_3$	$(*)$
	9	$\partial_1,\, x_2\partial_1,\, x_1\partial_1 + \partial_3,\, e^{x_3}\partial_1$	

Continuation of Table 4.

Algebra	N	Realization	$(*)$
$A^a_{3.4}\oplus A_1$	1	$\partial_1,\,\partial_2,\,x_1\partial_1+ax_2\partial_2+\partial_3,\,\partial_4$	
$\lvert a\rvert\le 1,\ \ a\ne 0,1$	2	$\partial_1,\,\partial_2,\,x_1\partial_1+ax_2\partial_2+\partial_3,\,x_4\partial_3$	
$[e_1,e_3]=e_1$	3	$\partial_1,\,\partial_2,\,x_1\partial_1+ax_2\partial_2,\,\partial_3$	
$[e_2,e_3]=ae_2$	4	$\partial_1,\,\partial_2,\,x_1\partial_1+ax_2\partial_2+\partial_3,\,e^{x_3}\partial_1+x_4e^{ax_3}\partial_2$	
	5	$\partial_1,\,\partial_2,\,x_1\partial_1+ax_2\partial_2+\partial_3,\,e^{x_3}\partial_1+e^{ax_3}\partial_2$	
	6	$\partial_1,\,\partial_2,\,x_1\partial_1+ax_2\partial_2+\partial_3,\,e^{x_3}\partial_1$	
	7	$\partial_1,\,x_2\partial_1,\,x_1\partial_1+(1-a)x_2\partial_2,\,\partial_3$	
	8	$\partial_1,\,x_2\partial_1,\,x_1\partial_1+(1-a)x_2\partial_2,\,x_3\lvert x_2\rvert^{\frac{1}{1-a}}\partial_1$	
	9	$\partial_1,\,x_2\partial_1,\,x_1\partial_1+(1-a)x_2\partial_2,\,\lvert x_2\rvert^{\frac{1}{1-a}}\partial_1$	
$a\ne -1$	10	$\partial_1,\,\partial_2,\,x_1\partial_1+ax_2\partial_2+\partial_3,\,e^{ax_3}\partial_2$	
$A^b_{3.5}\oplus A_1,\ \ b\ge 0$	1	$\partial_1,\,\partial_2,\,(bx_1+x_2)\partial_1+(-x_1+bx_2)\partial_2+\partial_3,\,\partial_4$	
$[e_1,e_3]=be_1-e_2$	2	$\partial_1,\,\partial_2,\,(bx_1+x_2)\partial_1+(-x_1+bx_2)\partial_2+\partial_3,\,x_4\partial_3$	
$[e_2,e_3]=e_1+be_2$	3	$\partial_1,\,\partial_2,\,(bx_1+x_2)\partial_1+(-x_1+bx_2)\partial_2,\,\partial_3$	
	4	$\partial_1,\,\partial_2,\,(bx_1+x_2)\partial_1+(-x_1+bx_2)\partial_2+\partial_3,\,x_4e^{bx_3}(\cos x_3\partial_1-\sin x_3\partial_2)$	
	5	$\partial_1,\,\partial_2,\,(bx_1+x_2)\partial_1+(-x_1+bx_2)\partial_2+\partial_3,\,e^{bx_3}(\cos x_3\partial_1-\sin x_3\partial_2)$	
	6	$\partial_1,\,x_2\partial_1,\,(b-x_2)x_1\partial_1-(1+x_2^2)\partial_2,\,\partial_3$	
	7	$\partial_1,\,x_2\partial_1,\,(b-x_2)x_1\partial_1-(1+x_2^2)\partial_2,\,x_3\sqrt{1+x_2^2}\,e^{-b\arctan x_2}\partial_1$	
	8	$\partial_1,\,x_2\partial_1,\,(b-x_2)x_1\partial_1-(1+x_2^2)\partial_2,\,\sqrt{1+x_2^2}\,e^{-b\arctan x_2}\partial_1$	

Remarks for Table 4.

$R(4A_1,3,\theta)$. $\theta=\theta(x_4,x_5)$. The realizations $R(4A_1,3,\theta)$ and $R(4A_1,3,\tilde\theta)$ are equivalent iff

$$\tilde\xi^a=-(\xi^b\alpha_{ba}-\alpha_{4a})/(\xi^c\alpha_{c4}-\alpha_{44}),\tag{4}$$

where $\xi^1=x_4$, $\xi^2=x_5$, $\xi^3=\theta(x_4,x_5)$, $\tilde\xi^1=\tilde x_4$, $\tilde\xi^2=\tilde x_5$, $\tilde\xi^3=\tilde\theta(\tilde x_4,\tilde x_5)$, $a,b,c=\overline{1,3}$.

$R(4A_1,4,(\varphi,\psi))$. $\varphi=\varphi(x_4)$, $\psi=\psi(x_4)$. The realizations $R(4A_1,4,(\varphi,\psi))$ and $R(4A_1,4,(\tilde\varphi,\tilde\psi))$ are equivalent iff condition (4) is satisfied, where $\xi^1=x_4$, $\xi^2=\varphi(x_4)$, $\xi^3=\psi(x_4)$, $\tilde\xi^1=\tilde x_4$, $\tilde\xi^2=\tilde\varphi(\tilde x_4)$, $\tilde\xi^3=\tilde\psi(\tilde x_4)$.

$R(4A_1,6,\theta)$. $\theta=\theta(x_3,x_4,x_5)$. The realizations $R(4A_1,3,\theta)$ and $R(4A_1,3,\tilde\theta)$ are equivalent iff

$$(\xi^{ik}\alpha_{k,2+j}-\alpha_{2+i,2+j})\tilde\xi^{jl}=-(\xi^{ik}\alpha_{kl}-\alpha_{2+i,l}),\tag{5}$$

where $\xi^{11}=x_3$, $\xi^{12}=x_4$, $\xi^{21}=x_5$, $\xi^{22}=\theta(x_3,x_4,x_5)$, $\tilde\xi^{11}=\tilde x_3$, $\tilde\xi^{12}=\tilde x_4$, $\tilde\xi^{21}=\tilde x_5$, $\tilde\xi^{22}=\tilde\theta(\tilde x_3,\tilde x_4,\tilde x_5)$, $i,j,k,l=1,2$.

$R(4A_1,7,(\varphi,\psi))$. $\varphi=\varphi(x_3,x_4)$, $\psi=\psi(x_3,x_4)$. The realizations $R(4A_1,7,(\varphi,\psi))$ and $R(4A_1,7,(\tilde\varphi,\tilde\psi))$ are equivalent iff condition (5) is satisfied, where $\xi^{11}=x_3$, $\xi^{12}=\varphi(x_3,x_4)$, $\xi^{21}=x_4$, $\xi^{22}=\psi(x_3,x_4)$, $\tilde\xi^{11}=\tilde x_3$, $\tilde\xi^{12}=\tilde\varphi(\tilde x_3,\tilde x_4)$, $\tilde\xi^{21}=\tilde x_4$, $\tilde\xi^{22}=\tilde\psi(\tilde x_3,\tilde x_4)$.

$R(4A_1,8,(\varphi,\psi,\theta))$. $\varphi=\varphi(x_3)$, $\psi=\psi(x_3)$, $\theta=\theta(x_3)$, and the vector-functions (x_3,φ) and (θ,ψ) are linearly independent. The realizations $R(4A_1,8,(\varphi,\psi,\theta))$ and $R(4A_1,8,(\tilde\varphi,\tilde\psi,\tilde\theta))$ are equivalent iff condition (5) is satisfied, where $\xi^{11}=x_3$, $\xi^{12}=\varphi(x_3)$, $\xi^{21}=\theta(x_3)$, $\xi^{22}=\psi(x_3)$, $\tilde\xi^{11}=\tilde x_3$, $\tilde\xi^{12}=\tilde\varphi(\tilde x_3)$, $\tilde\xi^{21}=\tilde\theta(\tilde x_3)$, $\tilde\xi^{22}=\tilde\psi(\tilde x_3)$.

$R(4A_1,10,\theta)$. $\theta=\theta(x_2,x_3)$, and the function θ is nonlinear with respect to (x_2,x_3). The realizations $R(4A_1,10,\theta)$ and $R(4A_1,10,\tilde\theta)$ are equivalent iff

$$(\xi^a\alpha_{1,b+1}-\alpha_{a+1,b+1})\tilde\xi^b=-(\xi^a\alpha_{11}-\alpha_{a1}),\tag{6}$$

where $\xi^1=x_2$, $\xi^2=x_3$, $\xi^3=\theta(x_2,x_3)$, $\xi^1=\tilde x_2$, $\xi^2=\tilde x_3$, $\xi^3=\tilde\theta(\tilde x_2,\tilde x_3)$, $a,b=\overline{1,3}$.

$R(4A_1,11,(\varphi,\theta))$. $\varphi=\varphi(x_2)$, $\psi=\psi(x_2)$, and the functions 1, x_2, φ and ψ are linearly independent. The realizations $R(4A_1,11,(\varphi,\theta))$ and $R(4A_1,11,(\tilde\varphi,\tilde\theta))$ are equivalent iff condition (6) is satisfied, where $\xi^1=x_2$, $\xi^2=\varphi(x_2)$, $\xi^3=\psi(x_2)$, $\xi^1=\tilde x_2$, $\xi^2=\tilde\varphi(\tilde x_2)$, $\xi^3=\tilde\psi(\tilde x_2)$.

$R(A_{2.1}\oplus 2A_1,3,\varphi)$. $\varphi=\varphi(x_4)$. The realizations $R(A_{2.1}\oplus 2A_1,3,\varphi)$ and $R(A_{2.1}\oplus 2A_1,3,\tilde\varphi)$ are equivalent iff
$\tilde x_4=-\alpha_{23}+\alpha_{33}x_4+\alpha_{43}\varphi,\quad \tilde\varphi=-\alpha_{24}+\alpha_{34}x_4+\alpha_{44}\varphi$
$(\tilde\varphi=\tilde\varphi(\tilde x_4),\ \alpha_{22}=1,\ \alpha_{12}=\alpha_{13}=\alpha_{14}=\alpha_{31}=\alpha_{32}=\alpha_{41}=\alpha_{42}=0)$.

$R(A_{2.1}\oplus 2A_1,9,\varphi)$. $\varphi=\varphi(x_3)$. The realizations $R(A_{2.1}\oplus 2A_1,9,\varphi)$ and $R(A_{2.1}\oplus 2A_1,9,\tilde\varphi)$ are equivalent iff
$\tilde x_3=-(\alpha_{33}x_3-\alpha_{43})/(\alpha_{34}x_3-\alpha_{44}),\quad \tilde\varphi=(\alpha_{33}+\alpha_{34}\tilde x_3)\varphi-(\alpha_{23}+\alpha_{24}\tilde x_3)$
$(\tilde\varphi=\tilde\varphi(\tilde x_3),\ \alpha_{22}=1,\ \alpha_{12}=\alpha_{13}=\alpha_{14}=\alpha_{31}=\alpha_{32}=\alpha_{41}=\alpha_{42}=0)$.

$R(2A_{2.1},3,C)$. $|C|\le 1$. If $C\ne\tilde C$ ($|C|\le 1$, $|\tilde C|\le 1$), the realizations $R(2A_{2.1},3,C)$ and $R(2A_{2.1},3,\tilde C)$ are inequivalent.

$R(A_{3.1}\oplus A_1,3,\varphi)$. $\varphi=\varphi(x_4)$. The realizations $R(A_{3.1}\oplus A_1,3,\varphi)$ and $R(A_{3.1}\oplus A_1,3,\tilde\varphi)$ are equivalent iff
$\tilde x_4=-(\alpha_{22}x_4-\alpha_{32})/(\alpha_{23}x_4-\alpha_{33}),\quad \tilde\varphi=-(\alpha_{44}\varphi+\alpha_{24}x_4-\alpha_{34})/(\alpha_{23}x_4-\alpha_{33})$
$(\tilde\varphi=\tilde\varphi(\tilde x_4),\ \alpha_{11}=\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32},\ \alpha_{12}=\alpha_{13}=\alpha_{14}=\alpha_{42}=\alpha_{43}=0)$.

$R(A_{3.1}\oplus A_1,8,\varphi)$. $\varphi=\varphi(x_2)$. The realizations $R(A_{3.1}\oplus A_1,8,\varphi)$ and $R(A_{3.1}\oplus A_1,8,\tilde\varphi)$ are equivalent iff
$\tilde x_2=(\alpha_{11}x_2-\alpha_{41})/\alpha_{44},\quad \tilde\varphi=-(\alpha_{22}\varphi-\alpha_{32})/(\alpha_{23}\varphi-\alpha_{33})$
$(\tilde\varphi=\tilde\varphi(\tilde x_2),\ \alpha_{11}=\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32},\ \alpha_{12}=\alpha_{13}=\alpha_{14}=\alpha_{42}=\alpha_{43}=0)$.

$R(A_{3.3}\oplus A_1,8,\varphi)$. $\varphi=\varphi(x_2)\ne 0$. The realizations $R(A_{3.3}\oplus A_1,8,\varphi)$ and $R(A_{3.3}\oplus A_1,8,\tilde\varphi)$ are equivalent iff
$\tilde x_2=-(\alpha_{11}x_2-\alpha_{21})/(\alpha_{12}x_2-\alpha_{22}),\quad \tilde\varphi=-\varphi/(\alpha_{34}\varphi-\alpha_{44})$
$(\tilde\varphi=\tilde\varphi(\tilde x_2),\ \alpha_{13}=\alpha_{14}=\alpha_{23}=\alpha_{24}=\alpha_{41}=\alpha_{42}=\alpha_{43}=0,\ \alpha_{33}=1)$.

Table 5. Realizations of real indecomposable solvable four-dimensional Lie algebras

Algebra	N	Realization	$(*)$
$A_{4.1}$	1	$\partial_1,\ \partial_2,\ \partial_3,\ x_2\partial_1+x_3\partial_2+\partial_4$	
$[e_2,e_4]=e_1$	2	$\partial_1,\ \partial_2,\ \partial_3,\ x_2\partial_1+x_3\partial_2+x_4\partial_3$	
$[e_3,e_4]=e_2$	3	$\partial_1,\ \partial_2,\ \partial_3,\ x_2\partial_1+x_3\partial_2$	
	4	$\partial_1,\ \partial_2,\ x_3\partial_1+x_4\partial_2,\ x_2\partial_1+x_4\partial_3-\partial_4$	
	5	$\partial_1,\ \partial_2,\ -\frac{1}{2}x_3^2\partial_1+x_3\partial_2,\ x_2\partial_1-\partial_3$	
	6	$\partial_1,\ x_2\partial_1,\ \partial_3,\ x_2x_3\partial_1-\partial_2$	
	7	$\partial_1,\ x_2\partial_1,\ x_3\partial_1,\ -\partial_2-x_2\partial_3$	
	8	$\partial_1,\ x_2\partial_1,\ \frac{1}{2}x_2^2\partial_1,\ -\partial_2$	
$A^b_{4.2},\ \ b\neq 0$	1	$\partial_1,\ \partial_2,\ \partial_3,\ bx_1\partial_1+(x_2+x_3)\partial_2+x_3\partial_3+\partial_4$	
$[e_1,e_4]=be_1$	2	$\partial_1,\ \partial_2,\ \partial_3,\ bx_1\partial_1+(x_2+x_3)\partial_2+x_3\partial_3$	
$[e_2,e_4]=e_2$	3	$\partial_1,\ \partial_2,\ x_4\partial_1+x_3\partial_2,\ bx_1\partial_1+x_2\partial_2+(b-1)x_4\partial_4-\partial_3$	
$[e_3,e_4]=e_2+e_3$	4	$\partial_1,\ \partial_2,\ x_3\partial_2,\ bx_1\partial_1+x_2\partial_2-\partial_3$	
	5	$\partial_1,\ x_2\partial_1,\ \partial_3,\ (bx_1+x_2x_3)\partial_1+(b-1)x_2\partial_2+x_3\partial_3$	
	6	$\partial_1,\ x_2\partial_1,\ x_3\partial_1,\ bx_1\partial_1+(b-1)x_2\partial_2+((b-1)x_3-x_2)\partial_3$	
$b\neq 1$	7	$\partial_1,\ \partial_2,\ e^{(1-b)x_3}\partial_1+x_3\partial_2,\ bx_1\partial_1+x_2\partial_2-\partial_3$	
	8	$\partial_1,\ x_2\partial_1,\ \frac{x_2}{1-b}\ln\lvert x_2\rvert\partial_1,\ bx_1\partial_1+(b-1)x_2\partial_2$	
$b=1$	7	$\partial_1,\ x_2\partial_1,\ \partial_3,\ (x_1+x_2x_3)\partial_1+x_3\partial_3+\partial_4$	
$A_{4.3}$	1	$\partial_1,\ \partial_2,\ \partial_3,\ x_1\partial_1+x_3\partial_2+\partial_4$	
$[e_1,e_4]=e_1$	2	$\partial_1,\ \partial_2,\ \partial_3,\ x_1\partial_1+x_3\partial_2+x_4\partial_3$	
$[e_3,e_4]=e_2$	3	$\partial_1,\ \partial_2,\ \partial_3,\ x_1\partial_1+x_3\partial_2$	
	4	$\partial_1,\partial_2,\ x_3\partial_1+x_4\partial_2,\ x_1\partial_1+x_3\partial_3-\partial_4$	
	5	$\partial_1,\ \partial_2,\ \varepsilon e^{-x_3}\partial_1+x_3\partial_2,\ x_1\partial_1-\partial_3$	
	6	$\partial_1,\ x_2\partial_1,\ \partial_3,\ (x_1+x_2x_3)\partial_1+x_2\partial_2$	
	7	$\partial_1,\ x_2\partial_1,\ x_3\partial_1,\ x_1\partial_1+x_2\partial_2+(x_3-x_2)\partial_3$	
	8	$\partial_1,\ x_2\partial_1,\ -x_2\ln\lvert x_2\rvert\partial_1,\ x_1\partial_1+x_2\partial_2$	
$A_{4.4}$	1	$\partial_1,\ \partial_2,\ \partial_3,\ (x_1+x_2)\partial_1+(x_2+x_3)\partial_2+x_3\partial_3+\partial_4$	
$[e_1,e_4]=e_1$	2	$\partial_1,\ \partial_2,\ \partial_3,\ (x_1+x_2)\partial_1+(x_2+x_3)\partial_2+x_3\partial_3$	
$[e_2,e_4]=e_1+e_2$	3	$\partial_1,\ \partial_2,\ x_3\partial_1+x_4\partial_2,\ (x_1+x_2)\partial_1+x_2\partial_2+x_4\partial_3-\partial_4$	
$[e_3,e_4]=e_2+e_3$	4	$\partial_1,\ \partial_2,\ -\frac{1}{2}x_3^2\partial_1+x_3\partial_2,\ (x_1+x_2)\partial_1+x_2\partial_2-\partial_3$	
	5	$\partial_1,\ x_2\partial_1,\ \partial_3,\ (x_1+x_2x_3)\partial_1-\partial_2+x_3\partial_3$	
	6	$\partial_1,\ x_2\partial_1,\ x_3\partial_1,\ x_1\partial_1-\partial_2-x_2\partial_3$	
	7	$\partial_1,\ x_2\partial_1,\ \frac{1}{2}x_2^2\partial_1,\ x_1\partial_1-\partial_2$	
$A^{a,b,c}_{4.5}$, $abc\neq 0$	1	$\partial_1,\ \partial_2,\ \partial_3,\ ax_1\partial_1+bx_2\partial_2+cx_3\partial_3+\partial_4$	
$[e_1,e_4]=ae_1$	2	$\partial_1,\ \partial_2,\ \partial_3,\ ax_1\partial_1+bx_2\partial_2+cx_3\partial_3$	
$[e_2,e_4]=be_2$	3	$\partial_1,\ \partial_2,\ x_3\partial_1+x_4\partial_2,\ ax_1\partial_1+bx_2\partial_2+(a-c)x_3\partial_3+(b-c)x_4\partial_4$	
$[e_3,e_4]=ce_3$	4	$\partial_1,\ x_2\partial_1,\ x_3\partial_1,\ ax_1\partial_1+(a-b)x_2\partial_2+(a-c)x_3\partial_3$	
$a=b=c=1$	5	$\partial_1,\ \partial_2,\ x_3\partial_1+x_4\partial_2,\ x_1\partial_1+x_2\partial_2+\partial_5$	
	6	$\partial_1,\ \partial_2,\ x_3\partial_1+\varphi(x_3)\partial_2,\ x_1\partial_1+x_2\partial_2+\partial_4$	$(*)$
	7	$\partial_1,\ \partial_2,\ x_3\partial_1+\varphi(x_3)\partial_2,\ x_1\partial_1+x_2\partial_2$	$(*)$
	8	$\partial_1,\ x_2\partial_1,\ x_3\partial_1, x_1\partial_1+\partial_4$	
	9	$\partial_1,\ x_2\partial_1,\ \varphi(x_2)\partial_1,\ x_1\partial_1+\partial_3$	$(*)$
	10	$\partial_1,\ x_2\partial_1,\ \varphi(x_2)\partial_1,\ x_1\partial_1$	$(*)$
$a=b=1,\ c\neq 1$	5	$\partial_1,\ x_2\partial_1,\ \partial_3,\ x_1\partial_1+cx_3\partial_3+\partial_4$	
	6	$\partial_1,\ x_2\partial_1,\ \partial_3,\ x_1\partial_1+cx_3\partial_3$	
	7	$\partial_1,\ \partial_2,\ e^{(1-c)x_3}\partial_1,\ x_1\partial_1+x_2\partial_2+\partial_3$	
$-1\leq a<b<c=1$	5	$\partial_1,\ \partial_2,\ \varepsilon_1e^{(a-1)x_3}\partial_1+\varepsilon_2e^{(b-1)x_3}\partial_2,\ ax_1\partial_1+bx_2\partial_2+\partial_3$	$(*)$
$b>0$ if $a=-1$	6	$\partial_1,\ x_2\partial_1,\ \partial_3,\ ax_1\partial_1+(a-b)x_2\partial_2+x_3\partial_3$	
	7	$\partial_1,\ e^{(a-b)x_2}\partial_1,\ e^{(a-1)x_2}\partial_1,\ ax_1\partial_1+\partial_2$	

Continuation of Table 5.

Algebra	N	Realization	$(*)$
$A^{a,b}_{4.6}$, $a>0$	1	$\partial_1,\ \partial_2,\ \partial_3,\ ax_1\partial_1+(bx_2+x_3)\partial_2+(-x_2+bx_3)\partial_3+\partial_4$	
$[e_1,e_4]=ae_1$	2	$\partial_1,\ \partial_2,\ \partial_3,\ ax_1\partial_1+(bx_2+x_3)\partial_2+(-x_2+bx_3)\partial_3$	
$[e_2,e_4]=be_2-e_3$	3	$\partial_1,\ \partial_2,\ x_3\partial_1+x_4\partial_2,\ (ax_1-x_2x_3)\partial_1+(b-x_4)x_2\partial_2+(a-b-x_4)x_3\partial_3-(1+x_4^2)\partial_4$	
$[e_3,e_4]=e_2+be_3$	4	$\partial_1,\ \partial_2,\ \varepsilon e^{(b-a)\arctan x_3}\sqrt{1+x_3^2}\partial_1+x_3\partial_2,$	
		$(ax_1-\varepsilon x_2e^{(b-a)\arctan x_3}\sqrt{1+x_3^2}\,)\partial_1+(b-x_3)x_2\partial_2-(1+x_3^2)\partial_3$	
	5	$\partial_1,\ x_2\partial_1,\ x_3\partial_1,\ ax_1\partial_1+((a-b)x_2+x_3)\partial_2+(-x_2+(a-b)x_3)\partial_3$	
	6	$\partial_1,\ e^{(a-b)x_2}\cos x_2\partial_1,\ -e^{(a-b)x_2}\sin x_2\partial_1,\ ax_1\partial_1+\partial_2$	
$A_{4.7}$	1	$\partial_1,\ \partial_2,\ x_2\partial_1+\partial_3,\ (2x_1+\frac{1}{2}x_3^2)\partial_1+(x_2+x_3)\partial_2+x_3\partial_3+\partial_4$	
$[e_2,e_3]=e_1$	2	$\partial_1,\ \partial_2,\ x_2\partial_1+\partial_3,\ (2x_1+\frac{1}{2}x_3^2)\partial_1+(x_2+x_3)\partial_2+x_3\partial_3$	
$[e_1,e_4]=2e_1$	3	$\partial_1,\ \partial_2,\ x_2\partial_1+x_3\partial_2,\ 2x_1\partial_1+x_2\partial_2-\partial_3$	
$[e_2,e_4]=e_2$	4	$\partial_1,\ x_2\partial_1,\ -\partial_2,\ (2x_1-\frac{1}{2}x_2^2)\partial_1+x_2\partial_2+\partial_3$	
$[e_3,e_4]=e_2+e_3$	5	$\partial_1,\ x_2\partial_1,\ -\partial_2,\ (2x_1-\frac{1}{2}x_2^2)\partial_1+x_2\partial_2$	
$A^b_{4.8}$, $\lvert b\rvert\le 1$	1	$\partial_1,\ \partial_2,\ x_2\partial_1+\partial_3,\ (1+b)x_1\partial_1+x_2\partial_2+bx_3\partial_3+\partial_4$	
$[e_2,e_3]=e_1$	2	$\partial_1,\ \partial_2,\ x_2\partial_1+\partial_3,\ (1+b)x_1\partial_1+x_2\partial_2+bx_3\partial_3$	
$[e_1,e_4]=(1+b)e_1$	3	$\partial_1,\ \partial_2,\ x_2\partial_1+x_3\partial_2,\ (1+b)x_1\partial_1+x_2\partial_2+(1-b)x_3\partial_3$	
$[e_2,e_4]=e_2$	4	$\partial_1,\ \partial_2,\ x_2\partial_1,\ (1+b)x_1\partial_1+x_2\partial_2+\partial_3$	
$[e_3,e_4]=be_3$	5	$\partial_1,\ \partial_2,\ x_2\partial_1,\ (1+b)x_1\partial_1+x_2\partial_2$	
$b=1$	6	$\partial_1,\ \partial_2,\ x_2\partial_1+x_3\partial_2,\ 2x_1\partial_1+x_2\partial_2+\partial_4$	
$b=-1$	6	$\partial_1,\ \partial_2,\ x_2\partial_1+\partial_3,\ x_4\partial_1+x_2\partial_2-x_3\partial_3$	
	7	$\partial_1,\ \partial_2,\ x_2\partial_1,\ x_3\partial_1+x_2\partial_2$	
$b\ne\pm1$	6	$\partial_1,\ x_2\partial_1,\ -\partial_2,\ (1+b)x_1\partial_1+bx_2\partial_2+\partial_3$	
	7	$\partial_1,\ x_2\partial_1,\ -\partial_2,\ (1+b)x_1\partial_1+bx_2\partial_2$	
$b=0$	8	$\partial_1,\ \partial_2,\ x_2\partial_1+\partial_3,\ x_1\partial_1+x_2\partial_2+x_4\partial_3$	
	9	$\partial_1,\ \partial_2,\ x_2\partial_1+\partial_3,\ x_1\partial_1+x_2\partial_2+C\partial_3$	$(*)$
$A^a_{4.9}$, $a\ge 0$	1	$\partial_1,\ \partial_2,\ x_2\partial_1+\partial_3,\ \frac{1}{2}(4ax_1+x_3^2-x_2^2)\partial_1+(ax_2+x_3)\partial_2+(-x_2+ax_3)\partial_3+\partial_4$	
$[e_2,e_3]=e_1$			
$[e_1,e_4]=2ae_1$	2	$\partial_1,\ \partial_2,\ x_2\partial_1+\partial_3,\ \frac{1}{2}(4ax_1+x_3^2-x_2^2)\partial_1+(ax_2+x_3)\partial_2+(-x_2+ax_3)\partial_3$	
$[e_2,e_4]=ae_2-e_3$			
$[e_3,e_4]=e_2+ae_3$	3	$\partial_1,\ \partial_2,\ x_2\partial_1+x_3\partial_2,\ (2ax_1-\frac{1}{2}x_2^2)\partial_1+(a-x_3)x_2\partial_2-(1+x_3^2)\partial_3$	
$a=0$	4	$\partial_1,\ \partial_2,\ x_2\partial_1+\partial_3,\ \frac{1}{2}(x_3^2-x_2^2+2x_4)\partial_1+x_3\partial_2-x_2\partial_3$	
$A_{4.10}$	1	$\partial_1,\ \partial_2,\ x_1\partial_1+x_2\partial_2+\partial_3,\ x_2\partial_1-x_1\partial_2+\partial_4$	
$[e_1,e_3]=e_1$	2	$\partial_1,\ \partial_2,\ x_1\partial_1+x_2\partial_2+\partial_3,\ x_2\partial_1-x_1\partial_2+x_4\partial_3$	
$[e_2,e_3]=e_2$	3	$\partial_1,\ \partial_2,\ x_1\partial_1+x_2\partial_2+\partial_3,\ x_2\partial_1-x_1\partial_2+C\partial_3$	$(*)$
$[e_1,e_4]=-e_2$	4	$\partial_1,\ x_2\partial_1,\ x_1\partial_1+\partial_3,\ -x_1x_2\partial_1-(1+x_2^2)\partial_2$	
$[e_2,e_4]=e_1$	5	$\partial_1,\ \partial_2,\ x_1\partial_1+x_2\partial_2,\ x_2\partial_1-x_1\partial_2+\partial_3$	
	6	$\partial_1,\ \partial_2,\ x_1\partial_1+x_2\partial_2,\ x_2\partial_1-x_1\partial_2$	
	7	$\partial_1,\ x_2\partial_1,\ x_1\partial_1,\ -x_1x_2\partial_1-(1+x_2^2)\partial_2$	

Remarks for Table 5.

$R(A^{1,1,1}_{4.5},N,\varphi)$, $N=6,7$. $\varphi=\varphi(x_3)$. The realizations $R(A^{1,1,1}_{4.5},N,\varphi)$ and $R(A^{1,1,1}_{4.5},N,\tilde\varphi)$ are equivalent iff condition (2) is satisfied ($\tilde\varphi=\tilde\varphi(\tilde x_3)$, $\alpha_{41}=\alpha_{42}=\alpha_{43}=0$).

$R(A^{1,1,1}_{4.5},N,\varphi)$, $N=9,10$. $\varphi=\varphi(x_2)$, $\varphi''\ne 0$. The realizations $R(A^{1,1,1}_{4.5},N,\varphi)$ and $R(A^{1,1,1}_{4.5},N,\tilde\varphi)$ are equivalent iff condition (3) is satisfied ($\tilde\varphi=\tilde\varphi(\tilde x_2)$, $\alpha_{41}=\alpha_{42}=\alpha_{43}=0$).

$R(A^{a,b,c}_{4.5},5,(\varepsilon_1,\varepsilon_2))$, where $-1\le a<b<c=1$, $b>0$ if $a=-1$. $\varepsilon_i\in\{0;1\}$, $(\varepsilon_1,\varepsilon_2)\ne(0,0)$ (three different variants are possible). All the variants are inequivalent.

$R(A^0_{4.8},9,C)$. $C\ne 0$ (since $R(A^0_{4.8},9,0)=R(A^0_{4.8},2)$).

$R(A_{4.10},3,C)$. C is an arbitrary constant.

Table 6. Realizations of real unsolvable three- and four-dimensional Lie algebras

Algebra	N	Realization	$(*)$
$sl(2,\mathbb{R})$	1	$\partial_1,\ x_1\partial_1+x_2\partial_2,\ x_1^2\partial_1+2x_1x_2\partial_2+x_2\partial_3$	
$[e_1,e_2]=e_1$	2	$\partial_1,\ x_1\partial_1+x_2\partial_2,\ (x_1^2-x_2^2)\partial_1+2x_1x_2\partial_2$	
$[e_2,e_3]=e_3$	3	$\partial_1,\ x_1\partial_1+x_2\partial_2,\ (x_1^2+x_2^2)\partial_1+2x_1x_2\partial_2$	
$[e_1,e_3]=2e_2$	4	$\partial_1,\ x_1\partial_1+x_2\partial_2,\ x_1^2\partial_1+2x_1x_2\partial_2$	
	5	$\partial_1,\ x_1\partial_1,\ x_1^2\partial_1$	
$sl(2,\mathbb{R})\oplus A_1$	1	$\partial_1,\ x_1\partial_1+x_2\partial_2,\ x_1^2\partial_1+2x_1x_2\partial_2+x_2\partial_3,\ \partial_4$	
$[e_1,e_2]=e_1$	2	$\partial_1,\ x_1\partial_1+x_2\partial_2,\ x_1^2\partial_1+2x_1x_2\partial_2+x_2\partial_3,\ x_2\partial_1+2x_2x_3\partial_2+(x_3^2+x_4)\partial_3$	
$[e_2,e_3]=e_3$	3	$\partial_1,\ x_1\partial_1+x_2\partial_2,\ x_1^2\partial_1+2x_1x_2\partial_2+x_2\partial_3,\ x_2\partial_1+2x_2x_3\partial_2+(x_3^2+c)\partial_3,$ $c\in\{-1;0;1\}$	
$[e_1,e_3]=2e_2$	4	$\partial_1,\ x_1\partial_1+x_2\partial_2,\ (x_1^2+x_2^2)\partial_1+2x_1x_2\partial_2,\ \partial_3$	
	5	$\partial_1,\ x_1\partial_1+x_2\partial_2,\ (x_1^2-x_2^2)\partial_1+2x_1x_2\partial_2,\ \partial_3$	
	6	$\partial_1,\ x_1\partial_1+x_2\partial_2,\ x_1^2\partial_1+2x_1x_2\partial_2,\ \partial_3$	
	7	$\partial_1,\ x_1\partial_1+x_2\partial_2,\ x_1^2\partial_1+2x_1x_2\partial_2,\ x_2x_3\partial_2$	
	8	$\partial_1,\ x_1\partial_1+x_2\partial_2,\ x_1^2\partial_1+2x_1x_2\partial_2,\ x_2\partial_2$	
	9	$\partial_1,\ x_1\partial_1,\ x_1^2\partial_1,\ \partial_2$	
$so(3)$	1	$-\sin x_1\tan x_2\partial_1-\cos x_1\partial_2,\ -\cos x_1\tan x_2\partial_1+\sin x_1\partial_2,\ \partial_1$	
$[e_2,e_3]=e_1$	2	$-\sin x_1\tan x_2\partial_1-\cos x_1\partial_2+\sin x_1\sec x_2\partial_3,$	
$[e_3,e_1]=e_2$		$-\cos x_1\tan x_2\partial_1+\sin x_1\partial_2+\cos x_1\sec x_2\partial_3,\ \partial_1$	
$[e_1,e_2]=e_3$			
$so(3)\oplus A_1$	1	$-\sin x_1\tan x_2\partial_1-\cos x_1\partial_2,\ -\cos x_1\tan x_2\partial_1+\sin x_1\partial_2,\ \partial_1,\ \partial_3$	
$[e_2,e_3]=e_1$	2	$-\sin x_1\tan x_2\partial_1-\cos x_1\partial_2+\sin x_1\sec x_2\partial_3,$	
$[e_3,e_1]=e_2$		$-\cos x_1\tan x_2\partial_1+\sin x_1\partial_2+\cos x_1\sec x_2\partial_3,\ \partial_1,\ \partial_3$	
$[e_1,e_2]=e_3$	3	$-\sin x_1\tan x_2\partial_1-\cos x_1\partial_2+\sin x_1\sec x_2\partial_3,$	
		$-\cos x_1\tan x_2\partial_1+\sin x_1\partial_2+\cos x_1\sec x_2\partial_3,\ \partial_1,\ x_4\partial_3$	
	4	$-\sin x_1\tan x_2\partial_1-\cos x_1\partial_2+\sin x_1\sec x_2\partial_3,$	
		$-\cos x_1\tan x_2\partial_1+\sin x_1\partial_2+\cos x_1\sec x_2\partial_3,\ \partial_1,\ \partial_4$	

Remark. The realizations $R(so(3),1)$ and $R(so(3),2)$ are well-known. At the best of our knowledge, completeness of the list of these realizations was first proved in [94]. We do not assert that the adduced forms of realizations are optimal for all applications and that the classification from Table 6 is canonical.

Consider the realization $R(so(3),1)$ of rank 2 in more details. The corresponding transformation group acts transitively on the manifold S^2. With the stereographic projection $\tan x_1 = t/x$, $\mathrm{cotan}\, x_2 = \sqrt{x^2+t^2}$ it can be reduced to the well known realization on the plane [21]:

$$(1+t^2)\partial_t + xt\partial_x, \quad x\partial_t - t\partial_x, \quad -xt\partial_t - (1+x^2)\partial_x.$$

If dimension of the x-space is not smaller than 3, the variables x_1, x_2 and the implicit variable x_3 in $R(so(3),1)$ can be interpreted as the angles and the radius of the spherical coordinates that is equivalent to imbedding S^2 in $\mathbb{R}^3$. Then, in the corresponding Cartesian coordinates the realization $R(so(3),1)$ has the well-known form:

$$x_2\partial_3 - x_3\partial_2, \qquad x_3\partial_1 - x_1\partial_3, \qquad x_1\partial_2 - x_2\partial_1,$$

which is generated by the standard representation of $SO(3)$ as the rotation group in $\mathbb{R}^3$.

6 Comparison of our results and those of [87]

The results of this paper include, as a particular case, realizations in three variables $x = (x_1, x_2, x_3)$, which were considered in [87]. That is why it is interesting for us to compare the lists of realizations.

In general, a result of classification may contain errors of two types:

- missing some inequivalent cases and
- including mutually equivalent cases.

Summarizing the comparison given below, we can state that errors of both types are in [87]. Namely, for three-dimensional algebras 3 cases are missing, 1 case is equivalent to other case, and 1 case can be reduced to 3 essentially simpler cases. For four-dimensional algebras 34 cases are missing, 13 cases are equivalent to other cases. Such errors are usually caused by incorrect performing of changes of variables and also shortcomings in the algorithms employed. See other errors in the comparison list.

Below we keep notations of [87] ($\mathcal{L}_{\dots}$, $\mathcal{L}^{\dots}_{\dots}$, $X_{\dots}$) and our notations ($A_{\dots}$, $R(A_{\dots}, \dots)$, $e_{\dots}$) for algebras, realizations and their basis elements. We list the pairs of equivalent realizations $\mathcal{L}^{k_1}_{r.m_1}$ and $R(A_{r.m_2}, k_2)$ using the shorthand notation $k_1 \sim k_2$ as well as all the differences of classifications. In the cases when equivalence of realizations is not obvious we give the necessary transformations of variables and basis changes.

Three-dimensional algebras

$\mathcal{L}_{3.1} \sim 3A_1$. $1 \sim 1$; $2 \sim 3$ (one of the parameter-functions of $\mathcal{L}_{3.1}$ can be made equal to t); $3 \sim 4$; the realization $R(3A_1, 5)$ is missing in [87].

$\mathcal{L}_{3.2} \sim A_{2.1} \oplus A_1$ ($X_1 = -e_2$, $X_2 = e_1$, $X_3 = e_3$). $1 \sim 3$; $2 \sim 1$; the series of realizations $\mathcal{L}^3_{3.2}$ with two parameter-functions f and g can be reduced to three realizations:
$R(A_{2.1} \oplus A_1, 2)$ if $f' \neq 0$ ($x_1 = y - xg(t)/f(t)$, $x_2 = \ln|x|/f(t)$, $x_3 = 1/f(t)$),
$R(A_{2.1} \oplus A_1, 3)$ if $f' = 0$ and $f \neq 0$ ($x_1 = y - xg(t)/f$, $x_2 = \ln|x|/f$, $x_3 = t$, $X_1 = -e_2 - (1/f)e_3$, $X_2 = e_1$, $X_3 = e_3$), which coincides with $\mathcal{L}^1_{3.2}$,
$R(A_{2.1} \oplus A_1, 4)$ if $f = 0$ and, therefore, $g \neq 0$ ($x_1 = g(t)x$, $x_2 = y$, $x_3 = t$).

$\mathcal{L}_{3.3} \sim A_{3.1}$. $1 \sim 3$; $2 \sim 2$; $3 \sim 1$; $4 \sim \mathcal{L}^1_{3.3}$.

$\mathcal{L}_{3.4} \sim A_{3.2}$. $1 \sim 2$; $2 \sim 1$; $3 \sim 3$.

$\mathcal{L}_{3.5} \sim A_{3.3}$. $1 \sim 2$; $2 \sim 1$; $3 \sim 4$; $4 \sim 3$.

$\mathcal{L}^a_{3.6} \sim A^a_{3.4}$. $1 \sim 2$; $2 \sim 1$; $3 \sim 3$.

$\mathcal{L}^a_{3.7} \sim A^a_{3.5}$. $1 \sim 2$; $2 \sim 1$; $3 \sim 3$.

$\mathcal{L}_{3.8} \sim sl(2, \mathbb{R})$. $1 \sim 5$; $2 \sim 1$; $3 \sim 3$ ($x_1 = (x+t)/2$, $x_2 = (x-t)/2$); $4 \sim 4$ ($x_1 = -x/t$, $x_2 = 1/t^2$, $x_3 = y$); the realization $R(sl(2, \mathbb{R}), 2)$ is missing in [87].

$\mathcal{L}_{3.9} \sim so(3)$. $1 \sim 1$ ($x_1 = \arctan t/x$, $x_2 = \operatorname{arccot}\sqrt{x^2+t^2}$, $e_1 = X_3$, $e_2 = -X_1$, $e_3 = X_2$); the realization $R(so(3), 2)$ is missing in [87].

Four-dimensional algebras

$\mathcal{L}_{4.1} \sim 4A_1$. $1 \sim 8$ (one of the parameter-functions of $\mathcal{L}_{3.1}$ can be made equal to t); $2 \sim 10$; the realization $R(4A_1, 11)$ is missing in [87].

$\mathcal{L}_{4.2} \sim A_{2.1} \oplus 2A_1$ ($X_1 = -e_2$, $X_2 = e_1$, $X_3 = e_3$, $X_4 = e_4$). $1 \sim 10$; $2 \sim 4$; $3 \sim \mathcal{L}^2_{4.2}$ ($\tilde{x} = \ln|t|$, $\tilde{y} = y$, $\tilde{t} = x/t$); $4 \sim \mathcal{L}^5_{4.2}$ ($\tilde{x} = x/t$, $\tilde{y} = y$, $\tilde{t} = 1/t$); $5 \sim 6$; $6 \sim 9$; $7 \sim \mathcal{L}^1_{4.2}$ if $f = 0$ ($\tilde{x} = ye^{-x}/g(t)$, $\tilde{y} = e^{-x}/g(t)$, $\tilde{t} = te^{-x}/g(t)$) or $7 \sim \mathcal{L}^6_{4.2}$ if $f \neq 0$ ($\tilde{x} = -e^{-x}/f(t)$, $\tilde{y} = y - xg(t)/f(t)$, $\tilde{t} = t$).

$\mathcal{L}_{4.3} \sim 2A_{2.1}$ ($X_1 = -e_2$, $X_2 = e_1$, $X_3 = -e_4$, $X_4 = e_3$). $1 \sim \mathcal{L}^3_{4.3}$ ($\tilde{x} = t$, $\tilde{y} = x$, $\tilde{t} = y$; $\tilde{X}_1 = X_3$, $\tilde{X}_2 = X_4$, $\tilde{X}_3 = X_1$, $\tilde{X}_4 = X_2$); $2 \sim 7$; $3 \sim 3^{C=0}$; $4 \sim 6$ ($x_1 = y$, $x_2 = t$, $x_3 = \ln|x/t|$); $5 \sim \mathcal{L}^3_{4.3}$ ($\tilde{x} = 1/x$, $\tilde{y} = y/x$, $\tilde{t} = t$); $6 \sim 3^{C=1}$ ($x_1 = y$, $x_2 = x/t$, $x_3 = \ln|t|$); $7 \sim 4$ ($x_1 = y$, $x_2 = x/t$, $x_3 = 1/t$); $8 \sim 5$; the realization $R(2A_{2.1}, 3, C)$ ($C \neq 0, 1$) is missing in [87].

$\mathcal{L}_{4.4} \sim A_{3.1} \oplus A_1$. The realization $\mathcal{L}^1_{4.4}$ is a particular case of $\mathcal{L}^4_{4.4}$; $2 \sim 5$; the basis operators of $\mathcal{L}^3_{4.4}$ do not satisfy the commutative relations of $\mathcal{L}_{4.4}$; $4 \sim 8$.

$\mathcal{L}_{4.5} \sim A_{3.2} \oplus A_1$. $1 \sim 8$ ($x_1 = x$, $x_2 = t$, $x_3 = ye^t$); $2 \sim 6$ ($x_1 = x$, $x_2 = y$, $x_3 = \ln|t|$); $3 \sim 5$ ($x_1 = x - tye^{-t}$, $x_2 = ye^{-t}$, $x_3 = -t$); $4 \sim 3$ ($x_1 = t$, $x_2 = x$, $x_3 = y$); the realizations $R(A_{3.2} \oplus A_1, 7)$ and $R(A_{3.2} \oplus A_1, 9)$ are missing in [87].

$\mathcal{L}^1_{4.6} \sim A_{3.3} \oplus A_1$. $1 \sim 9$ ($x_1 = x$, $x_2 = t$, $x_3 = \ln|y|$); $2 \sim 5$ ($x_1 = x$, $x_2 = y$, $x_3 = \ln|t|$); $3 \sim \mathcal{L}^{1,2}_{4.6}$ ($\tilde{x} = y$, $\tilde{y} = x$, $\tilde{t} = t$; $\tilde{X}_1 = X_2$, $\tilde{X}_2 = X_1$, $\tilde{X}_3 = -X_3$, $\tilde{X}_4 = X_4$); $4 \sim 3$; $5 \sim \mathcal{L}^{1,2}_{4.6}$ ($\tilde{x} = x$, $\tilde{y} = ty$, $\tilde{t} = t$; $\tilde{X}_1 = X_1 + X_2$, $\tilde{X}_2 = X_2$, $\tilde{X}_3 = X_3$, $\tilde{X}_4 = X_4$); the series of realizations $R(A_{3.3} \oplus A_1, 8)$ are missing in [87].

$\mathcal{L}^a_{4.6} \sim A^a_{3.4} \oplus A_1$ ($-1 \leq a < 1$, $a \neq 0$). $1 \sim 8$ ($x_1 = x$, $x_2 = t$, $x_3 = y|t|^{-\frac{1}{1-a}}$); $2 \sim 6$ ($x_1 = x$, $x_2 = y$, $x_3 = \ln|t|$); $3 \sim 10$ if $a \neq -1$ ($x_1 = x$, $x_2 = y$, $x_3 = \frac{1}{a}\ln|t|$), $3 \sim \mathcal{L}^{-1,2}_{4.6}$ for $a = -1$ ($\tilde{x} = y$, $\tilde{y} = x$, $\tilde{t} = t$; $\tilde{X}_1 = X_2$, $\tilde{X}_2 = X_1$, $\tilde{X}_3 = X_3$, $\tilde{X}_4 = X_4$); $4 \sim 3$; $5 \sim 5$ ($x_1 = x$, $x_2 = yt^a + xt^{a-1}$, $x_3 = \ln|t|$); the realizations $R(A^a_{3.4} \oplus A_1, 7)$ and $R(A^a_{3.4} \oplus A_1, 9)$ are missing in [87].

$\mathcal{L}_{4.7} \sim A^0_{3.5} \oplus A_1$. The basis operators of $\mathcal{L}^1_{4.7}$ do not satisfy the commutative relations of $\mathcal{L}_{4.7}$; $2 \sim 3$; $3 \sim 5$; the realizations $R(A^0_{3.5} \oplus A_1, 6)$, $R(A^0_{3.5} \oplus A_1, 7)$, and $R(A^0_{3.5} \oplus A_1, 8)$ are missing in [87]; the zero value of the parameter of algebra series $A^a_{3.5} \oplus A_1$ is not special with respect to constructing of inequivalent realizations.

$\mathcal{L}^a_{4.8} \sim A^a_{3.5} \oplus A_1$ ($a > 0$). $1 \sim 3$; $2 \sim 5$ (the notation of X_4 contains some misprints); the realizations $R(A^a_{3.5} \oplus A_1, 6)$, $R(A^a_{3.5} \oplus A_1, 7)$, and $R(A^a_{3.5} \oplus A_1, 8)$ are missing in [87].

$\mathcal{L}_{4.9} \sim sl(2,\mathbb{R}) \oplus A_1$ ($e_1 = X_1$, $e_2 = X_2$, $e_3 = -X_3$, $e_4 = X_4$). $1 \sim 3^{c=0}$ ($x_1 = t + x/(1+y)$, $x_2 = t/(1+y)$, $x_3 = -y(1+y)$); $2 \sim 4$ ($x_1 = (t+x)/2$, $x_2 = (t-x)/2$, $x_3 = y$); $3 \sim 6$ ($x_1 = -x/t$, $x_2 = -1/t^2$, $x_3 = y$); $4 \sim 9$; the realizations $R(sl(2,\mathbb{R}) \oplus A_1, 3, c)$ ($c = \pm 1$), $R(sl(2,\mathbb{R}) \oplus A_1, 5)$, $R(sl(2,\mathbb{R}) \oplus A_1, 7)$, and $R(sl(2,\mathbb{R}) \oplus A_1, 8)$ are missing in [87].

$\mathcal{L}_{4.10} \sim so(3) \oplus A_1$. $1 \sim 1$ ($x_1 = \arctan t/x$, $x_2 = \operatorname{arccot}\sqrt{x^2+t^2}$, $x_3 = y$, $e_1 = X_3$, $e_2 = -X_1$, $e_3 = X_2$, $e_4 = X_4$); the realization $R(so(3) \oplus A_1, 2)$ is missing in [87].

$\mathcal{L}_{4.11} \sim A_{4.1}$. $1 \sim 7$; $2 \sim 5$; $3 \sim 3$; the realizations $R(A_{4.1}, 6)$ and $R(A_{4.1}, 8)$ are missing in [87].

$\mathcal{L}^a_{4.12} \sim A^a_{4.2}$ ($a \neq 0$). $1 \sim 6$; $2 \sim 4$; $3 \sim 2$; $4 \sim 7$ for $a \neq 1$ and $4 \sim \mathcal{L}^{a,2}_{4.12}$ for $a = 1$; the realizations $R(A^a_{4.2}, 5)$, $R(A^a_{4.2}, 7)$ ($a = 1$), and $R(A^a_{4.2}, 8)$ ($a \neq 1$) are missing in [87].

$\mathcal{L}_{4.13} \sim A_{4.3}$. $1 \sim 7$; $2 \sim 3$; $3 \sim 5$; the realizations $R(A_{4.3}, 6)$ and $R(A_{4.3}, 8)$ are missing in [87].

$\mathcal{L}_{4.14} \sim A_{4.4}$. $1 \sim 6$; $2 \sim 2$; $3 \sim 4$; the realizations $R(A_{4.4}, 5)$ and $R(A_{4.4}, 7)$ are missing in [87].

$\mathcal{L}^{a,b}_{4.15} \sim A^{a,b,1}_{4.5}$ ($-1 \leq a < b < 1$, $ab \neq 0$, $e_1 = -X_2$, $e_2 = X_3$, $e_3 = X_1$, $e_4 = X_4$). $1 \sim 4$ ($x_1 = -x/t$, $x_2 = -y/t$, $x_3 = -1/t$); $2 \sim 2$; $3 \sim 5^{\varepsilon_1=0}$ ($x_1 = -y$, $x_2 = x/t$, $x_3 = (1-b)^{-1}\ln|t|$); $4 \sim 6$ ($x_1 = -y$, $x_2 = t$, $x_3 = x$); $5 \sim 5^{\varepsilon_1=\varepsilon_2=1}$ ($x_1 = -y + e^{(a-1)t}x$, $x_2 = e^{(b-1)t}x$, $x_3 = t$); the realizations $R(A^{a,b,1}_{4.5}, 5^{\varepsilon_2=0})$ and $R(A^{a,b,1}_{4.5}, 7)$ are missing in [87].

$\mathcal{L}^{a,a}_{4.15} \sim A^{1,1,a^{-1}}_{4.5}$ ($-1 < a < 1$, $a \neq 0$, $e_1 = X_3$, $e_2 = X_2$, $e_3 = X_1$, $e_4 = X_4$, $\mathcal{L}^{-1,-1}_{4.15} \sim \mathcal{L}^{-1,1}_{4.15}$). $1 \sim 4$ ($x_1 = x/y$, $x_2 = t/y$, $x_3 = 1/y$); $2 \sim 2$; $3 \sim 7$ ($x_1 = x/t$, $x_2 = y$, $x_3 = a(a-1)^{-1}\ln|t|$); $4 \sim 6$ ($x_1 = y/t$, $x_2 = 1/t$, $x_3 = x$).

$\mathcal{L}^{a,1}_{4.15} \sim A^{1,1,a}_{4.5}$ $(-1 \le a < 1,\ a \ne 0,\ e_1 = X_1,\ e_2 = X_3,\ e_3 = X_2,\ e_4 = X_4)$. $1 \sim 4$; $2 \sim 2$; $3 \sim 6$; $4 \sim 7$ $(x_1 = x,\ x_2 = y/t,\ x_3 = (a-1)^{-1} \ln |t|)$.

$\mathcal{L}_{4.16} \sim A^{1,1,1}_{4.5}$. $1 \sim 2$; $2 \sim 4$; $3 \sim 7$ (the function $f(t)$ can be made equal to t); the realizations $R(A^{1,1,1}_{4.5}, 9)$ and $R(A^{1,1,1}_{4.5}, 10)$ are missing in [87].

$\mathcal{L}^{a,b}_{4.17} \sim A^{a,b}_{4.6}$. $1 \sim 5$; $2 \sim 2$; $3 \sim 4$; the realizations $R(A^{a,b}_{4.6}, 6)$ is missing in [87].

$\mathcal{L}_{4.18} \sim A_{4.7}$. $1 \sim 5$ $(x_1 = x/2,\ x_2 = t,\ x_3 = y)$; $2 \sim 4$ $(x_1 = x/2,\ x_2 = t,\ x_3 = y)$; $3 \sim 2$ $(x_1 = x/2,\ x_2 = t,\ x_3 = y)$; $4 \sim 3$ $(x_1 = y,\ x_2 = x,\ x_3 = -t)$.

$\mathcal{L}_{4.19} \sim A^{-1}_{4.8}$. $1 \sim 7$; $2 \sim \mathcal{L}^{8}_{4.19}$ and $3 \sim \mathcal{L}^{1}_{4.19}$ $(\tilde{x} = t,\ \tilde{y} = x,\ \tilde{t} = -y,\ \tilde{X}_1 = X_1,\ \tilde{X}_2 = -X_3,\ \tilde{X}_3 = X_2,\ \tilde{X}_4 = -X_4)$; $4 \sim \mathcal{L}^{5}_{4.19}$ $(\tilde{x} = t,\ \tilde{y} = x,\ \tilde{t} = e^{-2y},\ \tilde{X}_1 = X_1,\ \tilde{X}_2 = -X_3,\ \tilde{X}_3 = X_2,\ \tilde{X}_4 = -X_4)$; $5 \sim 4$ $(x_1 = y,\ x_2 = x,\ x_3 = \frac{1}{2} \ln |t|)$; $6 \sim 3$; $7 \sim 2$; $8 \sim 5$.

$\mathcal{L}^{b}_{4.20} \sim A^{b}_{4.8}$ $(-1 < b \le 1)$. $1 \sim 5$; $2 \sim 7$ and $3 \sim 6$ (these realizations can be inscribed in the list of inequivalent realizations iff $b \ne \pm 1$); $4 \sim 4$ for $b \ne 1$ $(x_1 = y,\ x_2 = x,\ x_3 = (1-b)^{-1} \ln |t|)$ and $4 \sim \mathcal{L}^{b,1}_{4.20}$ if $b = 1$; $5 \sim 3$; $6 \sim 2$; the realizations $R(A^{0}_{4.8}, 9, C)$ is missing in [87].

$\mathcal{L}^{a}_{4.21} \sim A^{a}_{4.9}$ $(a \ge 0)$. $1 \sim 2$; $2 \sim 3$; the realizations $R(A^{0}_{4.9}, 4)$ is missing in [87].

$\mathcal{L}_{4.22} \sim A_{4.10}$. $1 \sim 7$; $2 \sim 6$; $3 \sim 4$; $4 \sim 5$; $5 \sim 3$.

7 Conclusion

We plan to extend this study by including the results of classifying realizations with respect to the strong equivalence and more detailed description of algebraic properties of low-dimensional Lie algebras and the classification technique. We have also begun the investigations on complete description of differential invariants and operators of invariant differentiation for all the constructed realizations, as well as ones on applications of the obtained results. (Let us note that the complete system of differential invariants for all the Lie groups, from Lie's classification, of point and contact transformations acting on a two-dimensional complex space was determined in [57]. The differential invariants of one-parameter groups of local transformations were exhaustively described in [63] in the case of arbitrary number of independent and dependent variables.) Using the above classification of inequivalent realizations of real Lie algebras of dimension no greater than four, one can solve, in a quite clear way, the group classification problems for the following classes of differential equations with real variables:

- ODEs of order up to four;
- systems of two second-order ODEs;
- systems of two, three and four first-order ODEs;
- general systems of two hydrodynamic-type equations with two independent variables;
- first-order PDEs with two independent variables;
- second-order evolutionary PDEs.

All the above classes of differential equations occur frequently in applications (classical, fluid and quantum mechanics, general relativity, mathematical biology, etc). Third- and fourth-order ODEs and the second class were investigated, in some way, in [10, 43, 71, 87]. Now we perform group classification for the third and fourth classes and fourth-order ODEs. Solving the group classification problem for the last class is necessary in order to construct first-order differential constraints being compatible with well-known nonlinear second-order PDEs.

Our results can be also applied to solving the interesting and important problem of studying finite-dimensional Lie algebras of first-order differential operators. (There are considerable number of papers devoted to this problem, see e.g. [3, 22, 47].)

It is obvious that our classification can be transformed to classification of realizations of complex Lie algebras of dimension no greater than four in vector fields on a space of an arbitrary (finite) number of complex variables. We also hope to solve the analogous problem for five-dimensional algebras in the near future.

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A Low-dimensional real Lie algebras

In this appendix different results on low-dimensional real Lie algebras are collected. Below we follow the Mubarakzyanov's classification and numeration of these algebras. For each algebra A we adduce

- only non-zero commutators between basis elements,
- the group Int of inner automorphisms (namely, the general form of matrices that form Int in the chosen basis of A),
- the complete automorphism group Aut (more precisely, the general form of matrices that form Aut in the chosen basis of A, these matrices are always supposed non-singular),
- a basis of the differentiation algebra Der,
- the set M_0 of proper megaideals (excluding improper megaideals $\{0\}$ and the whole algebra A which exist for any algebra),
- the set Ch_0 of proper characteristic ideals which are not megaideals,
- the set I_0 of proper ideals which are not characteristic,
- the complete set S of Int-inequivalent proper subalgebras arranged according to dimensions and structures.

We use following parameters:

$$\varepsilon = \pm 1, \quad \varkappa, \gamma, \nu, \zeta, p, q, \alpha_{ij}, \theta_i \in \mathbb{R}, \quad p^2 + q^2 = 1,$$

and the notations:

$m^{i_1\ldots i_k}_{j_1\ldots j_k}$ (for the fixed values $i_1, \ldots, i_k$ and $j_1, \ldots, j_k$) denotes the determinant of the matrix $(a_{ij})^{i\in\{i_1\ldots i_k\}}_{j\in\{j_1\ldots j_k\}}$;

$s^{i_1\ldots i_k}_{j_1 j_2}$ denotes the scalar product of subcolumns of the j_1-th and j_2-th columns, namely $s^{i_1\ldots i_k}_{j_1 j_2} = a_{i_1 j_1}a_{i_1 j_2} + a_{i_2 j_1}a_{i_2 j_2} + \cdots + a_{i_k j_1}a_{i_k j_2}$;

E_{ij} (for the fixed values i and j) denotes the matrix $(\delta_{ii'}\delta_{jj'})$ with i' and j' running the numbers of rows and column correspondingly, i.e. the matrix with the unit on the cross of the i-th row and the j-th column and the zero otherwise;

indices i, j, k, i', j' and k' run from 1 to $\dim A$; δ_{ik} is the Kronecker delta.

The lists of inequivalent subalgebras of three- and four-dimensional Lie algebras were taken from the Patera's and Winternitz's classification [59], where the basis elements of subalgebras are adduced in such manner that the basis elements of the corresponding derived algebra are written to the right of a semicolon. For Abelian subalgebras the semicolon which should be on the extreme right is omitted.

A.1 Three-dimensional real Lie algebras

$3A_1$ (Abelian, Bianchi I)
Any subspace of $3A_1$ as a usual vector space is a subalgebra and, moreover, an ideal of $3A_1$, and $\mathrm{Aut}(3A_1) = GL(3,\mathbb{R})$.

$$\mathrm{Int}\colon \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{Aut}\colon \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

Der: $E_{11},\ E_{12},\ E_{13},\ E_{21},\ E_{22},\ E_{23},\ E_{31},\ E_{32},\ E_{33}$

M_0: —
Ch_0: —
$\mathrm{I}_0 = S$
S: 1-dim $\sim A_1$: $\langle e_1 + \varkappa e_2 + \gamma e_3\rangle,\ \ \langle e_2 + \varkappa e_3\rangle,\ \ \langle e_3\rangle$
2-dim $\sim 2A_1$: $\langle e_1 + \varkappa e_3, e_2 + \gamma e_3\rangle,\ \ \langle e_1 + \varkappa e_2, e_3\rangle,\ \ \langle e_2, e_3\rangle$

Note, that an another normalization of parameters is possible for the 1-dimensional subalgebras: $\langle \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3\rangle$, $0 < \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \leq 1$.

$A_{2.1} \oplus A_1$: $[e_1, e_2] = e_1$ (decomposable solvable, Bianchi III)

$$\mathrm{Int}\colon \begin{pmatrix} e^{\theta_2} & \theta_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{Aut}\colon \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ 0 & 1 & 0 \\ 0 & \alpha_{32} & \alpha_{33} \end{pmatrix} \qquad \mathrm{Der}\colon E_{11},\ E_{12},\ E_{32},\ E_{33}$$

M_0: $\langle e_1\rangle$, $\langle e_3\rangle$, $\langle e_1, e_3\rangle$
Ch_0: —
I_0: $\langle e_2 + \varkappa e_3; e_1\rangle$
S: 1-dim $\sim A_1$: $\langle pe_3 + qe_2\rangle,\ \ \langle e_1 + \varepsilon e_3\rangle,\ \ \langle e_1\rangle$
2-dim $\sim 2A_1$: $\langle e_2, e_3\rangle,\ \ \langle e_1, e_3\rangle$
$\sim A_{2.1}$: $\langle e_2 + \varkappa e_3; e_1\rangle$

$A_{3.1}$: $[e_2, e_3] = e_1$ (Weyl algebra, nilpotent, Bianchi II)

$$\mathrm{Int}\colon \begin{pmatrix} 1 & \theta_3 & \theta_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{Aut}\colon \begin{pmatrix} m_{23}^{23} & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{22} & \alpha_{23} \\ 0 & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

Der: $E_{11} + E_{22},\ E_{12},\ E_{13},\ E_{23},\ E_{32},\ -E_{22} + E_{33}$

M_0: $\langle e_1\rangle$
Ch_0: —
I_0: $\langle e_1, pe_2 + qe_3\rangle$
S: 1-dim $\sim A_1$: $\langle e_1\rangle,\ \ \langle pe_2 + qe_3\rangle$
2-dim $\sim 2A_1$: $\langle e_1, pe_2 + qe_3\rangle$

$A_{3.2}$: $[e_1, e_3] = e_1,\ [e_2, e_3] = e_1 + e_2$ (solvable, Bianchi IV)

$$\mathrm{Int}\colon \begin{pmatrix} e^{\theta_3} & \theta_3 e^{\theta_3} & \theta_1 + \theta_2 \\ 0 & e^{\theta_3} & \theta_2 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{Aut}\colon \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{11} & \alpha_{23} \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{Der}\colon E_{11} + E_{22},\ E_{12},\ E_{13},\ E_{23}$$

M_0: $\langle e_1\rangle,\ \ \langle e_1, e_2\rangle$
Ch_0: —
I_0: —
S: 1-dim $\sim A_1$: $\langle e_1\rangle,\ \ \langle e_2\rangle,\ \ \langle e_3\rangle$
2-dim $\sim 2A_1$: $\langle e_1, e_2\rangle$
$\sim A_{2.1}$: $\langle e_3; e_1\rangle$

$\boldsymbol{A_{3.3}}$: $[e_1,e_3]=e_1,\ [e_2,e_3]=e_2$ (solvable, Bianchi V)

$$\text{Int}:\ \begin{pmatrix} e^{\theta_3} & 0 & \theta_1 \\ 0 & e^{\theta_3} & \theta_2 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{Aut}:\ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{11} & \alpha_{23} \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{Der}:\ E_{11},\ E_{12},\ E_{13},\ E_{21},\ E_{22},\ E_{23}$$

M_0: $\langle e_1,e_2\rangle$
Ch_0: —
I_0: $\langle pe_1+qe_2\rangle$
S: 1-dim $\sim A_1$: $\langle e_3\rangle$, $\langle pe_1+qe_2\rangle$
2-dim $\sim 2A_1$: $\langle e_1,e_2\rangle$
$\sim A_{2.1}$: $\langle e_3;pe_1+qe_2\rangle$

$\boldsymbol{A^a_{3.4}}$: $[e_1,e_3]=e_1,\ [e_2,e_3]=ae_2,\ -1\le a<1,\ a\ne 0$ (solvable, Bianchi VI)

$$\text{Int}:\ \begin{pmatrix} e^{\theta_3} & 0 & \theta_1 \\ 0 & e^{a\theta_3} & \theta_2 \\ 0 & 0 & 1 \end{pmatrix}$$

$-1<a<1,\ a\ne 0$:

$$\text{Aut}:\ \begin{pmatrix} \alpha_{11} & 0 & \alpha_{13} \\ 0 & \alpha_{22} & \alpha_{23} \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{Der}:\ E_{11},\ E_{13},\ E_{22},\ E_{23}$$

M_0: $\langle e_1\rangle$, $\langle e_2\rangle$, $\langle e_1,e_2\rangle$
Ch_0: —
I_0: —
S: 1-dim $\sim A_1$: $\langle e_1\rangle$, $\langle e_2\rangle$, $\langle e_3\rangle$, $\langle e_1+\varepsilon e_2\rangle$
2-dim $\sim 2A_1$: $\langle e_1,e_2\rangle$
$\sim A_{2.1}$: $\langle e_3;e_1\rangle$, $\langle e_3;e_2\rangle$

$a=-1$ ($\sim p(1,1)$, i.e. the Poincaré algebra in $\mathbb{R}^{1,1}$):

$$\text{Aut}:\ \begin{pmatrix} \alpha_{11} & 0 & \alpha_{13} \\ 0 & \alpha_{22} & \alpha_{23} \\ 0 & 0 & 1 \end{pmatrix},\ \begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & 0 & \alpha_{23} \\ 0 & 0 & -1 \end{pmatrix} \qquad \text{Der}:\ E_{11},\ E_{13},\ E_{22},\ E_{23}$$

M_0: $\langle e_1,e_2\rangle$
Ch_0: $\langle e_1\rangle$, $\langle e_2\rangle$
I_0: —
S: 1-dim $\sim A_1$: $\langle e_1\rangle$, $\langle e_2\rangle$, $\langle e_3\rangle$, $\langle e_1+\varepsilon e_2\rangle$
2-dim $\sim 2A_1$: $\langle e_1,e_2\rangle$
$\sim A_{2.1}$: $\langle e_3;e_1\rangle$, $\langle e_3;e_2\rangle$

$\boldsymbol{A^b_{3.5}}$: $[e_1,e_3]=be_1-e_2,\ [e_2,e_3]=e_1+be_2,\ b\ge 0$ (solvable, Bianchi VII)
The canonical commutation relations of the Lie algebra of Bianchi type VII: $[e_1',e_2']=0$, $[e_1',e_3']=e_2'$, $[e_2',e_3']=-e_1'+he_2'$, $h^2<4$ can be reduced to the canonical commutation relations of the algebra $A^b_{3.5}$ by the basis transformation: $e_1=-\frac{2b}{h}e_1'+be_2'$, $e_2=e_2'$, $e_3=\frac{2b}{h}e_3'$, $b=\frac{h}{\sqrt{4-h^2}}$.

$$\text{Int}:\ \begin{pmatrix} \cos\theta_3 e^{b\theta_3} & \sin\theta_3 e^{b\theta_3} & b\theta_1+\theta_2 \\ -\sin\theta_3 e^{b\theta_3} & \cos\theta_3 e^{b\theta_3} & -\theta_1+b\theta_2 \\ 0 & 0 & 1 \end{pmatrix}$$

$b>0$:

$$\text{Aut}:\ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ -\alpha_{12} & \alpha_{11} & \alpha_{23} \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{Der}:\ E_{11}+E_{22},\ E_{12}-E_{21},\ E_{13},\ E_{23}$$

M_0: $\langle e_1,e_2\rangle$

Ch$_0$: —
I$_0$: —
S: 1-dim $\sim A_1$: $\langle e_1\rangle$, $\langle e_3\rangle$
  2-dim $\sim 2A_1$: $\langle e_1, e_2\rangle$

$b=0$ ($\sim e(2)$, i.e. the Euclidian algebra in $\mathbb{R}^2$):

Aut: $\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ -\alpha_{12} & \alpha_{11} & \alpha_{23} \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & -\alpha_{11} & \alpha_{23} \\ 0 & 0 & -1 \end{pmatrix}$ Der: $E_{11}+E_{22},\ E_{12}-E_{21},\ E_{13},\ E_{23}$

M$_0$: $\langle e_1, e_2\rangle$
Ch$_0$: —
I$_0$: —
S: 1-dim $\sim A_1$: $\langle e_1\rangle$, $\langle e_3\rangle$
  2-dim $\sim 2A_1$: $\langle e_1, e_2\rangle$

$\boldsymbol{sl(2,\mathbb{R})}$: $[e_1,e_2]=e_1,\ [e_2,e_3]=e_3,\ [e_1,e_3]=2e_2$
  ($\sim su(1,1)\sim so(1,2)\sim sp(1,\mathbb{R})$, simple, Bianchi VIII)

The canonical commutation relations of the Lie algebra $sl(2,\mathbb{R})$ can be reduced to the canonical commutation relations $[e'_1,e'_2]=-e'_3,\ [e'_2,e'_3]=e'_1,\ [e'_3,e'_1]=e'_2$ of the algebra $so(1,2)$ by the basis transformation $e'_i=e_j\beta_{ji}$, where

$$(\beta_{ij})=\begin{pmatrix} 0 & \frac12 & \frac12 \\ 1 & 0 & 0 \\ 0 & -\frac12 & \frac12 \end{pmatrix}.$$

In the basis $\{e'_i\}$ Int coincides with the matrix group Lor$(1,2)$ of 3×3 Lorentz matrices, i.e.

Int$(so(1,2))$: $J^1(\varphi_1)J^2(\varphi_2)J^3(\varphi_3)$, where

$$J^1(\varphi_1)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\varphi_1 & \sinh\varphi_1 \\ 0 & \sinh\varphi_1 & \cosh\varphi_1 \end{pmatrix},\qquad J^2(\varphi_2)=\begin{pmatrix} \cosh\varphi_2 & 0 & \sinh\varphi_2 \\ 0 & 1 & 0 \\ \sinh\varphi_2 & 0 & \cosh\varphi_2 \end{pmatrix},$$

$$J^3(\varphi_3)=\begin{pmatrix} \cos\varphi_3 & \sin\varphi_3 & 0 \\ -\sin\varphi_3 & \cos\varphi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, Int$(sl(2,\mathbb{R}))$: $(\beta_{ij})J^1(\varphi_1)J^2(\varphi_2)J^3(\varphi_3)(\beta_{ij})^{-1}$

Aut$(so(1,2))$ coincides with the matrix group $SO(1,2)$ of 3×3 special pseudo-orthogonal matrices, which is generated by the elements of Int$(so(1,2))=$ Lor$(1,2)$ and of the additional discrete transformation:

$$\mathcal{E}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Hence, the automorphism group of the algebra $sl(2,\mathbb{R})$ can be written in the form:

Aut$(sl(2,\mathbb{R}))$: $(\beta_{ij})J^1(\varphi_1)J^2(\varphi_2)J^3(\varphi_3)(\beta_{ij})^{-1},\ (\beta_{ij})\mathcal{E}J^1(\varphi_1)J^2(\varphi_2)J^3(\varphi_3)(\beta_{ij})^{-1}$
Der: $E_{12}+2E_{23},\ 2E_{21}+E_{32},\ E_{11}-E_{33}$
M$_0$: —
Ch$_0$: —
I$_0$: —
S: 1-dim $\sim A_1$: $\langle e_1\rangle$, $\langle e_2\rangle$, $\langle e_1+e_3\rangle$
  2-dim $\sim A_{2.1}$: $\langle e_2; e_1\rangle$

$\boldsymbol{so(3)}$: $[e_2,e_3]=e_1,\ [e_3,e_1]=e_2,\ [e_1,e_2]=e_3$ ($\sim su(2)\sim sp(1)$, simple, Bianchi IX)

For the algebra $so(3)$, Aut coincides with Int and is the matrix group $SO(3)$ of 3×3 special orthogonal matrices, i.e.

$$\mathrm{Aut}=\mathrm{Int}\colon \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}, \quad \text{where } \alpha_{ij}\alpha_{kj}=\delta_{ik} \text{ and } \det(\alpha_{ij})=1.$$

In an explicit form any 3×3 special orthogonal matrix (α_{ij}) can be presented via the Euler angles φ_i: $(\alpha_{ij})=J^1(\varphi_1)J^2(\varphi_2)J^3(\varphi_3)$, where $J^i(\varphi_i)$ is the matrix of rotation on the angle φ_i with respect to the i-th axis:

$$J^1(\varphi_1)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_1 & \sin\varphi_1 \\ 0 & -\sin\varphi_1 & \cos\varphi_1 \end{pmatrix}, \qquad J^2(\varphi_2)=\begin{pmatrix} \cos\varphi_2 & 0 & -\sin\varphi_2 \\ 0 & 1 & 0 \\ \sin\varphi_2 & 0 & \cos\varphi_2 \end{pmatrix},$$

$$J^3(\varphi_3)=\begin{pmatrix} \cos\varphi_3 & \sin\varphi_3 & 0 \\ -\sin\varphi_3 & \cos\varphi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Der: $E_{23}+E_{32},\ E_{31}+E_{13},\ E_{12}-E_{21}$

M_0: —
Ch_0: —
I_0: —
S: 1-dim $\sim A_1$: $\langle e_1\rangle$

A.2 Four-dimensional real Lie algebras

$\boldsymbol{4A_1}$: (Abelian)
Any subspace of $4A_1$ as a usual vector space is a subalgebra and, moreover, an ideal of $4A_1$, and $\mathrm{Aut}(4A_1)=GL(4,\mathbb{R})$.

$$\mathrm{Int}\colon \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{Aut}\colon \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix}$$

Der: $E_{11},\ E_{12},\ E_{13},\ E_{14},\ E_{21},\ E_{22},\ E_{23},\ E_{24},\ E_{31},\ E_{32},\ E_{33},\ E_{34},\ E_{41},\ E_{42},\ E_{43},\ E_{44}$

M_0: —
Ch_0: —
$\mathrm{I}_0=S$
S: 1-dim $\sim A_1$: $\langle e_1+\varkappa e_2+\gamma e_3+\zeta e_4\rangle$, $\langle e_2+\varkappa e_3+\gamma e_4\rangle$, $\langle e_3+\varkappa e_4\rangle$, $\langle e_4\rangle$
2-dim $\sim 2A_1$: $\langle e_1+\varkappa e_3+\gamma e_4, e_2+\zeta e_3+\nu e_4\rangle$, $\langle e_1+\varkappa e_2+\gamma e_4, e_3+\zeta e_4\rangle$, $\langle e_2+\varkappa e_4, e_3+\gamma e_4\rangle$, $\langle e_1+\varkappa e_2+\gamma e_3, e_4\rangle$, $\langle e_2+\varkappa e_3, e_4\rangle$, $\langle e_3, e_4\rangle$.
3-dim $\sim 3A_1$: $\langle e_1+\varkappa e_4, e_2+\gamma e_4, e_3+\zeta e_4\rangle$, $\langle e_1+\varkappa e_2, e_3, e_4\rangle$, $\langle e_2, e_3, e_4\rangle$, $\langle e_1+\varkappa e_3, e_2+\gamma e_3, e_4\rangle$

$\boldsymbol{A_{2.1}\oplus 2A_1}$: $[e_1,e_2]=e_1$ (decomposable solvable)

$$\mathrm{Int}\colon \begin{pmatrix} e^{\theta_2} & \theta_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{Aut}\colon \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ 0 & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix}$$

Der: $E_{11},\ E_{12},\ E_{32},\ E_{33},\ E_{34},\ E_{42},\ E_{43},\ E_{44}$

M_0: $\langle e_1\rangle$, $\langle e_3,e_4\rangle$, $\langle e_1,e_3,e_4\rangle$
Ch_0: —
I_0: $\langle pe_3+qe_4\rangle$, $\langle e_2+\varkappa e_3+\gamma e_4; e_1\rangle$, $\langle e_1, pe_3+qe_4\rangle$, $\langle e_2+\varkappa(qe_3-pe_4), pe_3+qe_4; e_1\rangle$
S: 1-dim $\sim A_1$: $\langle e_1\rangle$, $\langle pe_3+qe_4\rangle$, $\langle e_2+\varkappa e_3+\gamma e_4\rangle$, $\langle e_1+pe_3+qe_4\rangle$
2-dim $\sim 2A_1$: $\langle e_2+\varkappa(qe_3-pe_4), pe_3+qe_4\rangle$, $\langle e_3,e_4\rangle$, $\langle e_1+qe_3-pe_4, pe_3+qe_4\rangle$, $\langle e_1, pe_3+qe_4\rangle$
$\sim A_{2.1}$: $\langle e_2+\varkappa e_3+\gamma e_4; e_1\rangle$
3-dim $\sim 3A_1$: $\langle e_2,e_3,e_4\rangle$, $\langle e_1,e_3,e_4\rangle$
$\sim A_{2.1}\oplus A_1$: $\langle e_2+\varkappa(qe_3-pe_4), pe_3+qe_4; e_1\rangle$

$\boldsymbol{2A_{2.1}}$: $[e_1,e_2]=e_1,\ [e_3,e_4]=e_3$ (decomposable solvable)

$$\text{Int}\colon \begin{pmatrix} e^{\theta_2} & \theta_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{\theta_4} & \theta_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{Aut}\colon \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha_{33} & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \alpha_{13} & \alpha_{14} \\ 0 & 0 & 0 & 1 \\ \alpha_{31} & \alpha_{32} & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Der: $E_{11},\ E_{12},\ E_{33},\ E_{34}$

M_0: $\langle e_1,e_3\rangle,\ \langle e_2+\varepsilon e_4;e_1,e_3\rangle$
Ch_0: $\langle e_1\rangle, \langle e_3\rangle, \langle e_2;e_1\rangle, \langle e_4;e_3\rangle, \langle e_2,e_3;e_1\rangle, \langle e_1,e_4;e_3\rangle,\ \langle e_2+\varkappa e_4;e_1,e_3\rangle,\ \varkappa\neq 0,\pm 1$
I_0: —
S: 1-dim $\sim A_1$: $\langle e_1\rangle,\ \langle e_3\rangle,\ \langle e_4\rangle,\ \langle e_2+\varkappa e_4\rangle,\ \langle e_2+\varepsilon e_3\rangle,\ \langle e_1+\varepsilon e_3\rangle,\ \langle e_1+\varepsilon e_4\rangle$
2-dim $\sim 2A_1$: $\langle e_2,e_4\rangle,\ \langle e_2,e_3\rangle,\ \langle e_1,e_4\rangle,\ \langle e_1,e_3\rangle$
$\sim A_{2.1}$: $\langle e_2+\varkappa e_4;e_1\rangle,\ \langle e_4+\varkappa e_2;e_3\rangle,\ \langle e_2+\varepsilon e_3;e_1\rangle,\ \langle e_4+\varepsilon e_1;e_3\rangle,\ \langle e_2+e_4;e_1+\varepsilon e_3\rangle$
3-dim $\sim A_{2.1}\oplus A_1$: $\langle e_2,e_4;e_1\rangle,\ \langle e_2,e_3;e_1\rangle,\ \langle e_2,e_4;e_3\rangle,\ \langle e_1,e_4;e_3\rangle$
$\sim A_{3.3}$: $\langle e_2+e_4;e_1,e_3\rangle$
$\sim A^a_{3.4}$: $\langle e_2+\varkappa e_4;e_1,e_3\rangle$

The parameter a in the algebra $A^a_{3.4}$ should satisfy condition:

$$a=\begin{cases} \varkappa, & -1\leq\varkappa<1,\ \varkappa\neq 0, \\ \frac{1}{\varkappa}, & |\varkappa|>1 \end{cases}$$

$\boldsymbol{A_{3.1}\oplus A_1}$: $[e_2,e_3]=e_1$ (decomposable nilpotent)

$$\text{Int}\colon \begin{pmatrix} 1 & \theta_3 & \theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{Aut}\colon \begin{pmatrix} m^{23}_{23} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ 0 & \alpha_{22} & \alpha_{23} & 0 \\ 0 & \alpha_{32} & \alpha_{33} & 0 \\ 0 & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix}$$

Der: $E_{12},\ E_{13},\ E_{14},\ E_{11}+E_{22},\ E_{23},\ E_{32},\ E_{11}+E_{33},\ E_{42},\ E_{43},\ E_{44}$

M_0: $\langle e_1\rangle,\ \langle e_1,e_4\rangle$
Ch_0: —
I_0: $\langle e_1+\varkappa e_4\rangle,\ \langle e_4\rangle,\ \langle e_1,e_4+\varkappa(pe_2+qe_3)\rangle,\ \langle e_1,pe_2+qe_3\rangle,\ \langle e_1,pe_2+qe_3,e_4\rangle,$
$\langle e_2+\nu e_4,e_3+\gamma e_4;e_1\rangle,\ \varkappa\neq 0$
S: 1-dim $\sim A_1$: $\langle e_1+\varkappa e_4\rangle,\ \langle e_4\rangle,\ \langle pe_2+qe_3+\varkappa e_4\rangle$
2-dim $\sim 2A_1$: $\langle e_1,e_4+\varkappa(pe_2+qe_3)\rangle,\ \langle e_4,pe_2+qe_3\rangle,\ \langle e_1+\varkappa e_4,pe_2+qe_3\rangle$
3-dim $\sim 3A_1$: $\langle e_1,pe_2+qe_3,e_4\rangle$
$\sim A_{3.1}$: $\langle e_2+\varkappa e_4,e_3+\gamma e_4;e_1\rangle$

$\boldsymbol{A_{3.2}\oplus A_1}$: $[e_1,e_3]=e_1,\ [e_2,e_3]=e_1+e_2$ (decomposable solvable)

$$\text{Int}\colon \begin{pmatrix} e^{\theta_3} & \theta_3 e^{\theta_3} & \theta_1+\theta_2 & 0 \\ 0 & e^{\theta_3} & \theta_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{Aut}\colon \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 \\ 0 & \alpha_{11} & \alpha_{23} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \alpha_{43} & \alpha_{44} \end{pmatrix}$$

Der: $E_{12},\ E_{13},\ E_{11}+E_{22},\ E_{23},\ E_{43},\ E_{44}$

M_0: $\langle e_1\rangle,\ \langle e_4\rangle,\ \langle e_1,e_2\rangle, \langle e_1,e_4\rangle,\ \langle e_1,e_2,e_4\rangle$
Ch_0: —
I_0: $\langle e_3+\varkappa e_4;e_1,e_2\rangle$
S: 1-dim $\sim A_1$: $\langle e_1\rangle,\ \langle e_1+\varepsilon e_4\rangle,\ \langle e_2+\varkappa e_4\rangle,\ \langle e_3+\varkappa e_4\rangle,\ \langle e_4\rangle$
2-dim $\sim 2A_1$: $\langle e_1+\varkappa e_4,e_2\rangle,\ \langle e_1,e_2+\varepsilon e_4\rangle,\ \langle e_1,e_4\rangle,\ \langle e_2,e_4\rangle,\ \langle e_3,e_4\rangle$
$\sim A_{2.1}$: $\langle e_3+\varkappa e_4;e_1\rangle$
3-dim $\sim 3A_1$: $\langle e_1,e_2,e_4\rangle$
$\sim A_{2.1}\oplus A_1$: $\langle e_3,e_4;e_1\rangle$
$\sim A_{3.2}$: $\langle e_3+\varkappa e_4;e_1,e_2\rangle$

$\boldsymbol{A_{3.3} \oplus A_1}$: $[e_1, e_3] = e_1,\ [e_2, e_3] = e_2$ (decomposable solvable)

$$\text{Int}: \begin{pmatrix} e^{\theta_3} & 0 & \theta_1 & 0 \\ 0 & e^{\theta_3} & \theta_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{Aut}: \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \alpha_{43} & \alpha_{44} \end{pmatrix}$$

Der: $E_{11},\ E_{12},\ E_{13},\ E_{21},\ E_{22},\ E_{23},\ E_{43},\ E_{44}$

M$_0$: $\langle e_4\rangle,\ \langle e_1, e_2\rangle,\ \langle e_1, e_2, e_4\rangle$
Ch$_0$: —
I$_0$: $\langle pe_1 + qe_2\rangle,\ \langle pe_1 + qe_2, e_4\rangle,\ \langle e_3 + \varkappa e_4; e_1, e_2\rangle$
S: 1-dim $\sim A_1$: $\langle e_4\rangle,\ \langle pe_1 + qe_2\rangle,\ \langle e_3 + \varkappa e_4\rangle,\ \langle pe_1 + qe_2 + \varepsilon e_4\rangle$
2-dim $\sim 2A_1$: $\langle e_1, e_2\rangle,\ \langle e_3, e_4\rangle,\ \langle pe_1 + qe_2, e_4\rangle,\ \langle e_1, e_2 + \varepsilon e_4\rangle,\ \langle e_1 + \varepsilon e_4, e_2 + \varkappa e_1\rangle$
$\sim A_{2.1}$: $\langle e_3 + \varkappa e_4; pe_1 + qe_2\rangle$
3-dim $\sim 3A_1$: $\langle e_1, e_2, e_4\rangle$
$\sim A_{2.1} \oplus A_1$: $\langle e_3, e_4; pe_1 + qe_2\rangle$
$\sim A_{3.2}$: $\langle e_3 + \varkappa e_4; e_1, e_2\rangle$

$\boldsymbol{A^a_{3.4} \oplus A_1}$: $[e_1, e_3] = e_1,\ [e_2, e_3] = ae_2,\ -1 \le a < 1,\ a \ne 0$ (decomposable solvable)

$$\text{Int}: \begin{pmatrix} e^{\theta_3} & 0 & \theta_1 & 0 \\ 0 & e^{a\theta_3} & \theta_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$-1 < a < 1,\ a \ne 0$:

$$\text{Aut}: \begin{pmatrix} \alpha_{11} & 0 & \alpha_{13} & 0 \\ 0 & \alpha_{22} & \alpha_{23} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \alpha_{43} & \alpha_{44} \end{pmatrix} \qquad \text{Der}: E_{11},\ E_{13},\ E_{22},\ E_{23},\ E_{43},\ E_{44}$$

M$_0$: $\langle e_1\rangle,\ \langle e_2\rangle,\ \langle e_4\rangle,\ \langle e_1, e_2\rangle,\ \langle e_1, e_4\rangle,\ \langle e_2, e_4\rangle,\ \langle e_1, e_2, e_4\rangle$
Ch$_0$: —
I$_0$: $\langle e_3 + \varkappa e_4; e_1, e_2\rangle$
S: 1-dim $\sim A_1$: $\langle e_1\rangle,\ \langle e_2\rangle,\ \langle e_4\rangle,\ \langle e_1 + \varepsilon e_4\rangle,\ \langle e_2 + \varepsilon e_4\rangle,\ \langle e_3 + \varkappa e_4\rangle,\ \langle e_1 + \varepsilon e_2 + \varkappa e_4\rangle$
2-dim $\sim 2A_1$: $\langle e_1, e_2\rangle,\ \langle e_1, e_4\rangle,\ \langle e_2, e_4\rangle,\ \langle e_3, e_4\rangle,\ \langle e_1 + \varepsilon e_2, e_4\rangle,\ \langle e_1, e_2 + \varepsilon e_4\rangle,\ \langle e_1 + \varepsilon e_4, e_2 + \varkappa e_4\rangle$
$\sim A_{2.1}$: $\langle e_3 + \varkappa e_4; e_1\rangle,\ \langle e_3 + \varkappa e_4; e_2\rangle$
3-dim $\sim 3A_1$: $\langle e_1, e_2, e_4\rangle$
$\sim A_{2.1} \oplus A_1$: $\langle e_3, e_4; e_1\rangle,\ \langle e_3, e_4; e_2\rangle$
$\sim A^a_{3.4}$: $\langle e_3 + \varkappa e_4; e_1, e_2\rangle$

$a = -1$:

$$\text{Aut}: \begin{pmatrix} \alpha_{11} & 0 & \alpha_{13} & 0 \\ 0 & \alpha_{22} & \alpha_{23} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \alpha_{43} & \alpha_{44} \end{pmatrix}, \begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} & 0 \\ \alpha_{21} & 0 & \alpha_{23} & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & \alpha_{43} & \alpha_{44} \end{pmatrix} \qquad \text{Der}: E_{11},\ E_{13},\ E_{22},\ E_{23},\ E_{43},\ E_{44}$$

M$_0$: $\langle e_4\rangle,\ \langle e_1, e_2\rangle,\ \langle e_1, e_2, e_4\rangle$
Ch$_0$: $\langle e_1\rangle,\ \langle e_2\rangle,\ \langle e_1, e_4\rangle,\ \langle e_2, e_4\rangle$
I$_0$: $\langle e_3 + \varkappa e_4; e_1, e_2\rangle$
S: 1-dim $\sim A_1$: $\langle e_1\rangle,\ \langle e_2\rangle,\ \langle e_4\rangle,\ \langle e_1 + \varepsilon e_4\rangle,\ \langle e_2 + \varepsilon e_4\rangle,\ \langle e_3 + \varkappa e_4\rangle,\ \langle e_1 + \varepsilon e_2 + \varkappa e_4\rangle$
2-dim $\sim 2A_1$: $\langle e_1, e_2\rangle,\ \langle e_1, e_4\rangle,\ \langle e_2, e_4\rangle,\ \langle e_3, e_4\rangle,\ \langle e_1 + \varepsilon e_2, e_4\rangle,\ \langle e_1, e_2 + \varepsilon e_4\rangle,\ \langle e_1 + \varepsilon e_4, e_2 + \varkappa e_4\rangle$
$\sim A_{2.1}$: $\langle e_3 + \varkappa e_4; e_1\rangle,\ \langle e_3 + \varkappa e_4; e_2\rangle$
3-dim $\sim 3A_1$: $\langle e_1, e_2, e_4\rangle$
$\sim A_{2.1} \oplus A_1$: $\langle e_3, e_4; e_1\rangle,\ \langle e_3, e_4; e_2\rangle$
$\sim A^a_{3.4}$: $\langle e_3 + \varkappa e_4; e_1, e_2\rangle$

$\boldsymbol{A_{3.5}^b \oplus A_1}$: $[e_1,e_3]=be_1-e_2,\ [e_2,e_3]=e_1+be_2,\ b\ge 0$ (decomposable solvable)

$$\text{Int}\colon \begin{pmatrix} \cos\theta_3 e^{b\theta_3} & \sin\theta_3 e^{b\theta_3} & b\theta_1+\theta_2 & 0 \\ -\sin\theta_3 e^{b\theta_3} & \cos\theta_3 e^{b\theta_3} & -\theta_1+b\theta_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$b>0$:

$$\text{Aut}\colon \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 \\ -\alpha_{12} & \alpha_{11} & \alpha_{23} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \alpha_{43} & \alpha_{44} \end{pmatrix} \qquad \text{Der}\colon\ E_{12}-E_{21},\ E_{13},\ E_{11}+E_{22},\ E_{23},\ E_{43},\ E_{44}$$

M_0: $\langle e_4\rangle,\ \langle e_1,e_2\rangle,\ \langle e_1,e_2,e_4\rangle$
Ch_0: —
I_0: $\langle e_3+\varkappa e_4; e_1,e_2\rangle$
S: 1-dim $\sim A_1$: $\langle e_4\rangle,\ \langle e_1+\varkappa e_4\rangle,\ \langle e_3+\gamma e_4\rangle,\ \varkappa\ge 0$
2-dim $\sim 2A_1$: $\langle e_1+\varkappa e_4, e_2\rangle,\ \langle e_1,e_4\rangle,\ \langle e_3,e_4\rangle,\ \varkappa\ge 0$
3-dim $\sim 3A_1$: $\langle e_1,e_2,e_4\rangle$
$\sim A_{3.5}^b$: $\langle e_3+\varkappa e_4; e_1,e_2\rangle$

$b=0$:

$$\text{Aut}\colon \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 \\ -\alpha_{12} & \alpha_{11} & \alpha_{23} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \alpha_{43} & \alpha_{44} \end{pmatrix},\ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 \\ \alpha_{12} & -\alpha_{11} & \alpha_{23} & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & \alpha_{43} & \alpha_{44} \end{pmatrix}$$

Der: $E_{12}-E_{21},\ E_{13},\ E_{11}+E_{22},\ E_{23},\ E_{43},\ E_{44}$

M_0: $\langle e_4\rangle,\ \langle e_1,e_2\rangle,\ \langle e_1,e_2,e_4\rangle$
Ch_0: —
I_0: $\langle e_3+\varkappa e_4; e_1,e_2\rangle$
S: 1-dim $\sim A_1$: $\langle e_4\rangle,\ \langle e_1+\varkappa e_4\rangle,\ \langle e_3+\gamma e_4\rangle,\ \varkappa\ge 0$
2-dim $\sim 2A_1$: $\langle e_1+\varkappa e_4, e_2\rangle,\ \langle e_1,e_4\rangle,\ \langle e_3,e_4\rangle,\ \varkappa\ge 0$
3-dim $\sim 3A_1$: $\langle e_1,e_2,e_4\rangle$
$\sim A_{3.5}^0$: $\langle e_3+\varkappa e_4; e_1,e_2\rangle$

$\boldsymbol{sl(2,\mathbb{R})\oplus A_1}$: $[e_1,e_2]=e_1,\ [e_2,e_3]=e_3,\ [e_1,e_3]=2e_2$ (unsolvable)

$$\text{Int}\colon \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ where the submatrix } (\alpha_{ij})_{i,j=\overline{1,3}} \text{ belongs to } \mathrm{Int}(sl(2,\mathbb{R}))$$

$$\text{Aut}\colon \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 0 \\ 0 & 0 & 0 & \alpha_{44} \end{pmatrix}, \text{ where the submatrix } (\alpha_{ij})_{i,j=\overline{1,3}} \text{ belongs to } \mathrm{Aut}(sl(2,\mathbb{R}))$$

Der: $E_{11}-E_{33},\ E_{12}+2E_{23},\ 2E_{21}+E_{32},\ E_{44}$

M_0: $\langle e_4\rangle,\ \langle ; e_1,e_2,e_3\rangle$
Ch_0: —
I_0: —
S: 1-dim $\sim A_1$: $\langle e_1\rangle,\ \langle e_4\rangle,\ \langle e_2+\varkappa e_4\rangle,\ \langle e_1+\varepsilon e_4\rangle,\ \langle e_1+e_3+\gamma e_4\rangle,\ \varkappa\ge 0$
2-dim $\sim 2A_1$: $\langle e_1,e_4\rangle,\ \langle e_2,e_4\rangle,\ \langle e_1+e_3,e_4\rangle$
$\sim A_{2.1}$: $\langle e_2+\varkappa e_4; e_1\rangle$
3-dim $\sim A_{2.1}\oplus A_1$: $\langle e_2,e_4; e_1\rangle$
$\sim sl(2,\mathbb{R})$: $\langle ; e_1,e_2,e_3\rangle$

$\boldsymbol{so(3)\oplus A_1}$: $[e_1,e_2]=e_3,\ [e_2,e_3]=e_1,\ [e_3,e_1]=e_2$ (unsolvable)

Int: $\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}&0\\ \alpha_{21}&\alpha_{22}&\alpha_{23}&0\\ \alpha_{31}&\alpha_{32}&\alpha_{33}&0\\ 0&0&0&1\end{pmatrix}$, where the submatrix $(\alpha_{ij})_{i,j=\overline{1,3}}$ belongs to $SO(3)$

Aut: $\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}&0\\ \alpha_{21}&\alpha_{22}&\alpha_{23}&0\\ \alpha_{31}&\alpha_{32}&\alpha_{33}&0\\ 0&0&0&\alpha_{44}\end{pmatrix}$, where the submatrix $(\alpha_{ij})_{i,j=\overline{1,3}}$ belongs to $SO(3)$

Der: $E_{12}-E_{21},\ E_{13}-E_{31},\ E_{32}-E_{23},\ E_{44}$

M_0: $\langle e_4\rangle,\ \langle ;e_1,e_2,e_3\rangle$
Ch_0: —
I_0: —
S: 1-dim $\sim A_1$: $\langle e_4\rangle,\ \langle e_1+\varkappa e_4\rangle,\ \ \varkappa\geq 0$
2-dim $\sim 2A_1$: $\langle e_1,e_4\rangle$
3-dim $\sim so(3)$: $\langle ;e_1,e_2,e_3\rangle$

$\boldsymbol{A_{4.1}}$: $[e_2,e_4]=e_1,\ [e_3,e_4]=e_2$ (indecomposable nilpotent)

Int: $\begin{pmatrix}1&\theta_4&\frac12\theta_4^2&\theta_2\\ 0&1&\theta_4&\theta_3\\ 0&0&1&0\\ 0&0&0&1\end{pmatrix}$ Aut: $\begin{pmatrix}\alpha_{33}\alpha_{44}^2&\alpha_{23}\alpha_{44}&\alpha_{13}&\alpha_{14}\\ 0&\alpha_{33}\alpha_{44}&\alpha_{23}&\alpha_{24}\\ 0&0&\alpha_{33}&\alpha_{34}\\ 0&0&0&\alpha_{44}\end{pmatrix}$

Der: $E_{13},\ E_{14},\ E_{12}+E_{23},\ E_{24},\ E_{11}+E_{22}+E_{33},\ E_{34},\ 2E_{11}+E_{22}+E_{44}$

M_0: $\langle e_1\rangle,\ \langle e_1,e_2\rangle,\ \langle e_1,e_2,e_3\rangle$
Ch_0: —
I_0: $\langle e_4+\varkappa e_3,e_2;e_1\rangle$
S: 1-dim $\sim A_1$: $\langle e_1\rangle,\ \langle e_2\rangle,\ \langle e_3+\varkappa e_1\rangle,\ \langle e_4+\varkappa e_3\rangle$
2-dim $\sim 2A_1$: $\langle e_1,e_2\rangle,\ \langle e_1,e_3\rangle,\ \langle e_2,e_3+\varkappa e_1\rangle,\ \langle e_1,e_4+\varkappa e_3\rangle$
3-dim $\sim 3A_1$: $\langle e_1,e_2,e_3\rangle$
$\sim A_{3.1}$: $\langle e_4+\varkappa e_3,e_2;e_1\rangle$

$\boldsymbol{A_{4.2}^b}$: $[e_1,e_4]=be_1,\ [e_2,e_4]=e_2,\ [e_3,e_4]=e_2+e_3,\ b\neq 0$ (indecomposable solvable)

Int: $\begin{pmatrix}e^{b\theta_4}&0&0&\theta_1\\ 0&e^{\theta_4}&\theta_4e^{\theta_4}&\theta_2+\theta_3\\ 0&0&e^{\theta_4}&\theta_3\\ 0&0&0&1\end{pmatrix}$

$b\neq 0,\ b\neq 1$:

Aut: $\begin{pmatrix}\alpha_{11}&0&0&\alpha_{14}\\ 0&\alpha_{22}&\alpha_{23}&\alpha_{24}\\ 0&0&\alpha_{22}&\alpha_{34}\\ 0&0&0&1\end{pmatrix}$

Der: $E_{11},\ E_{14},\ E_{23},\ E_{24},\ E_{22}+E_{33},\ E_{34}$

M_0: $\langle e_1\rangle,\ \langle e_2\rangle,\ \langle e_1,e_2\rangle,\ \langle e_2,e_3\rangle,\ \langle e_1,e_2,e_3\rangle$
Ch_0: —
I_0: —
S: 1-dim $\sim A_1$: $\langle e_1\rangle,\ \langle e_2\rangle,\ \langle e_4\rangle,\ \langle e_3+\varkappa e_1\rangle,\ \langle e_1+\varepsilon e_2\rangle$
2-dim $\sim 2A_1$: $\langle e_1,e_2\rangle,\ \langle e_1+\varkappa e_2,e_3\rangle,\ \langle e_2,e_3\rangle,\ \langle e_1+\varepsilon e_3,e_2\rangle$
$\sim A_{2.1}$: $\langle e_4;e_1\rangle,\ \langle e_4;e_2\rangle$
3-dim $\sim 3A_1$: $\langle e_1,e_2,e_3\rangle$

$\sim A_{3.2}$: $\langle e_4; e_2, e_3\rangle$
$\sim A^a_{3.4}$: $\langle e_4; e_1, e_2\rangle$

The parameter a in the algebra $A^a_{3.4}$ should satisfy condition: $a = \begin{cases} b, & -1 \le b < 1,\ b \ne 0, \\ \frac{1}{b}, & |b| > 1 \end{cases}$

$b = 1$:

Aut: $\begin{pmatrix} \alpha_{11} & 0 & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ 0 & 0 & \alpha_{22} & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Der: $E_{11},\ E_{13},\ E_{14},\ E_{21},\ E_{23},\ E_{24},\ E_{22}+E_{33},\ E_{34}$

M$_0$: $\langle e_2\rangle,\ \langle e_1, e_2\rangle,\ \langle e_1, e_2, e_3\rangle$
Ch$_0$: —
I$_0$: $\langle e_1 + \varkappa e_2\rangle,\ \langle e_2, e_3 + \varkappa e_1\rangle$
S: 1-dim $\sim A_1$: $\langle pe_1 + qe_2\rangle,\ \langle e_3 + \varkappa e_1\rangle,\ \langle e_4\rangle$
2-dim $\sim 2A_1$: $\langle e_1, e_2\rangle,\ \langle e_1 + \varkappa e_2, e_3\rangle,\ \langle e_2, e_3 + \varkappa e_1\rangle$
$\sim A_{2.1}$: $\langle e_4; pe_1 + qe_2\rangle$
3-dim $\sim 3A_1$: $\langle e_1, e_2, e_3\rangle$
$\sim A_{3.2}$: $\langle e_4; e_2, e_3 + \varkappa e_1\rangle$
$\sim A_{3.3}$: $\langle e_4; e_1, e_2\rangle$

***A*$_{4.3}$**: $[e_1, e_4] = e_1,\ [e_3, e_4] = e_2$ (indecomposable solvable)

Int: $\begin{pmatrix} e^{\theta_4} & 0 & 0 & \theta_1 \\ 0 & 1 & \theta_4 & \theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ Aut: $\begin{pmatrix} \alpha_{11} & 0 & 0 & \alpha_{14} \\ 0 & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ 0 & 0 & \alpha_{22} & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Der: $E_{11},\ E_{14},\ E_{23},\ E_{24},\ E_{22}+E_{33},\ E_{34}$

M$_0$: $\langle e_1\rangle,\ \langle e_2\rangle,\ \langle e_1, e_2\rangle,\ \langle e_2, e_3\rangle,\ \langle e_1, e_2, e_3\rangle$
Ch$_0$: —
I$_0$: $\langle e_4 + \varkappa e_3, e_2; e_1\rangle$
S: 1-dim $\sim A_1$: $\langle e_1\rangle,\ \langle e_2\rangle,\ \langle e_1 + \varepsilon e_2\rangle,\ \langle e_3 + \varkappa e_1\rangle,\ \langle e_4 + \varkappa e_3\rangle$
2-dim $\sim 2A_1$: $\langle e_1, e_2\rangle,\ \langle e_1 + \varkappa e_2, e_3\rangle,\ \langle e_2, e_3\rangle,\ \langle e_2, e_3 + \varepsilon e_1\rangle,\ \langle e_2, e_4 + \varkappa e_3\rangle$
$\sim A_{2.1}$: $\langle e_4 + \varkappa e_3; e_1\rangle$
3-dim $\sim 3A_1$: $\langle e_1, e_2, e_3\rangle$
$\sim A_{2.1} \oplus A_1$: $\langle e_4 + \varkappa e_3, e_2; e_1\rangle$
$\sim A_{3.1}$: $\langle e_3, e_4; e_2\rangle$

***A*$_{4.4}$**: $[e_1, e_4] = e_1,\ [e_2, e_4] = e_1 + e_2,\ [e_3, e_4] = e_2 + e_3$ (indecomposable solvable)

Int: $\begin{pmatrix} e^{\theta_4} & \theta_4 e^{\theta_4} & \frac{1}{2}\theta_4^2 e^{\theta_4} & \theta_1 + \theta_2 \\ 0 & e^{\theta_4} & \theta_4 e^{\theta_4} & \theta_2 + \theta_3 \\ 0 & 0 & e^{\theta_4} & \theta_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ Aut: $\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ 0 & \alpha_{11} & \alpha_{12} & \alpha_{24} \\ 0 & 0 & \alpha_{11} & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Der: $E_{11}+E_{22}+E_{33},\ E_{13},\ E_{14},\ E_{24},\ E_{12}+E_{23},\ E_{34}$

M$_0$: $\langle e_1\rangle,\ \langle e_1, e_2\rangle,\ \langle e_1, e_2, e_3\rangle$
Ch$_0$: —
I$_0$: —
S: 1-dim $\sim A_1$: $\langle e_1 + \varkappa e_3\rangle,\ \langle e_2\rangle,\ \langle e_3\rangle,\ \langle e_4\rangle$
2-dim $\sim 2A_1$: $\langle e_1 + \varkappa e_3, e_2\rangle,\ \langle e_1, e_3\rangle,\ \langle e_2, e_3\rangle$
$\sim A_{2.1}$: $\langle e_4; e_1\rangle$
3-dim $\sim 3A_1$: $\langle e_1, e_2, e_3\rangle$
$\sim A_{3.2}$: $\langle e_4; e_1, e_2\rangle$

$\boldsymbol{A^{a,b,c}_{4.5}}$: $[e_1,e_4]=ae_1,\ [e_2,e_4]=be_2,\ [e_3,e_4]=ce_3,\ abc\neq 0$ (indecomposable solvable)

$$\mathrm{Int}\colon \begin{pmatrix} e^{a\theta_4} & 0 & 0 & \theta_1 \\ 0 & e^{b\theta_4} & 0 & \theta_2 \\ 0 & 0 & e^{c\theta_4} & \theta_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$-1\leq a<b<c=1,\ \ b>0$ if $a=-1$:

$$\mathrm{Aut}\colon \begin{pmatrix} \alpha_{11} & 0 & 0 & \alpha_{14} \\ 0 & \alpha_{22} & 0 & \alpha_{24} \\ 0 & 0 & \alpha_{33} & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Der: $E_{11},\ E_{14},\ E_{22},\ E_{24},\ E_{33},\ E_{34}$

M$_0$: $\langle e_1\rangle,\ \langle e_2\rangle,\ \langle e_3\rangle,\ \langle e_1,e_2\rangle,\ \langle e_2,e_3\rangle,\ \langle e_1,e_3\rangle,\ \langle e_1,e_2,e_3\rangle$
Ch$_0$: —
I$_0$: —
S: 1-dim $\sim A_1$: $\langle e_1\rangle,\ \langle e_2\rangle,\ \langle e_3\rangle,\ \langle e_4\rangle,\ \langle e_1+\varepsilon e_3\rangle, \langle e_2+\varepsilon e_3\rangle,\ \langle e_1+\varepsilon e_2+\varkappa e_3\rangle,\ \ \varkappa\neq 0$
2-dim $\sim 2A_1$: $\langle e_1,e_2\rangle,\ \langle e_1,e_3\rangle,\ \langle e_2,e_3\rangle,\ \langle e_1,e_2+\varepsilon e_3\rangle,\ \langle e_2,e_1+\varepsilon e_3\rangle,\ \langle e_3,e_1+\varepsilon e_2\rangle,$
$\langle e_1+\varepsilon e_3,e_2+\varkappa e_3\rangle,\ \ \varkappa\neq 0$
$\sim A_{2.1}$: $\langle e_4;e_1\rangle,\ \langle e_4;e_2\rangle,\ \langle e_4;e_3\rangle$
3-dim $\sim 3A_1$: $\langle e_1,e_2,e_3\rangle$
$\sim A^v_{3.4}$: $\langle e_4;e_1,e_3\rangle^*,\ \ \langle e_4;e_2,e_3\rangle^{**},\ \ \langle e_4;e_1,e_2\rangle^{***}$

The parameter v in the algebra $A^v_{3.4}$ should satisfy the condition $v=a$ in the case *, $v=b$ in the case ** and $v=\begin{cases} a/b, & |a/b|<1, \\ b/a, & |a/b|>1 \end{cases}$ in the case ***.

$a=b=1,\ c\neq 1$:

$$\mathrm{Aut}\colon \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & 0 & \alpha_{24} \\ 0 & 0 & \alpha_{33} & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Der: $E_{11},\ E_{12},\ E_{14},\ E_{21},\ E_{22},\ E_{24},\ E_{33},\ E_{34}$

M$_0$: $\langle e_3\rangle,\ \langle e_1,e_2\rangle,\ \langle e_1,e_2,e_3\rangle$
Ch$_0$: —
I$_0$: $\langle pe_1+qe_2\rangle,\ \langle e_3,pe_1+qe_2\rangle$
S: 1-dim $\sim A_1$: $\langle e_3\rangle,\ \langle pe_1+qe_2\rangle,\ \langle e_4\rangle,\ \langle e_2+\varepsilon e_3\rangle,\ \langle e_1+\varkappa e_2+\varepsilon e_3\rangle$
2-dim $\sim 2A_1$: $\langle e_3,pe_1+qe_2\rangle,\ \langle e_1,e_2\rangle,\ \langle e_2,e_1+\varepsilon e_3\rangle,\ \langle e_1+\varkappa e_2,e_2+\varepsilon e_3\rangle$
$\sim A_{2.1}$: $\langle e_4;e_3\rangle,\ \langle e_4;pe_1+qe_2\rangle$
3-dim $\sim 3A_1$: $\langle e_1,e_2,e_3\rangle$
$\sim A_{3.3}$: $\langle e_4;e_1,e_2\rangle$
$\sim A^c_{3.4}$: $\langle e_4;e_3,pe_1+qe_2\rangle$

$a=b=c=1$:

$$\mathrm{Aut}\colon \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Der: $E_{11},\ E_{12},\ E_{13},\ E_{14},\ E_{21},\ E_{22},\ E_{23},\ E_{24},\ E_{31},\ E_{32},\ E_{33},\ E_{34}$

M$_0$: $\langle e_1,e_2,e_3\rangle$
Ch$_0$: —
I$_0$: $\langle e_1+\varkappa e_2+\gamma e_3\rangle,\ \langle e_2+\varkappa e_3\rangle,\ \langle e_3\rangle,\ \langle e_1+\varkappa e_3,e_2+\gamma e_3\rangle,\ \langle e_1+\varkappa e_2,e_3\rangle,\ \langle e_2,e_3\rangle$
S: 1-dim $\sim A_1$: $\langle e_1+\varkappa e_2+\gamma e_3\rangle,\ \langle e_2+\varkappa e_3\rangle,\ \langle e_3\rangle,\ \langle e_4\rangle$

2-dim $\sim 2A_1$: $\langle e_1+\varkappa e_3, e_2+\gamma e_3\rangle$, $\langle e_1+\varkappa e_2, e_3\rangle$, $\langle e_2, e_3\rangle$
$\sim A_{2.1}$: $\langle e_4; e_1+\varkappa e_2+\gamma e_3\rangle$, $\langle e_4; e_2+\varkappa e_3\rangle$, $\langle e_4; e_3\rangle$
3-dim $\sim 3A_1$: $\langle e_1, e_2, e_3\rangle$
$\sim A_{3.3}$: $\langle e_4; e_1+\varkappa e_3, e_2+\gamma e_3\rangle$, $\langle e_4; e_1+\varkappa e_2, e_3\rangle$, $\langle e_4; e_2, e_3\rangle$

$\boldsymbol{A^{a,b}_{4.6}}$: $[e_1,e_4]=ae_1,\ [e_2,e_4]=be_2-e_3,\ [e_3,e_4]=e_2+be_3,\ a>0$ (indecomposable solvable)

$$\text{Int}:\begin{pmatrix} e^{a\theta_4} & 0 & 0 & a\theta_1 \\ 0 & e^{b\theta_4}\cos\theta_4 & e^{b\theta_4}\sin\theta_4 & b\theta_2+\theta_3 \\ 0 & -e^{b\theta_4}\sin\theta_4 & e^{b\theta_4}\cos\theta_4 & -\theta_2+b\theta_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{Aut}:\begin{pmatrix} \alpha_{11} & 0 & 0 & \alpha_{14} \\ 0 & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ 0 & -\alpha_{23} & \alpha_{22} & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Der: $E_{11},\ E_{14},\ E_{24},\ E_{34},\ E_{22}+E_{33},\ E_{23}-E_{32}$

M$_0$: $\langle e_1\rangle$, $\langle e_2, e_3\rangle$, $\langle e_1, e_2, e_3\rangle$
Ch$_0$: —
I$_0$: —
S: 1-dim $\sim A_1$: $\langle e_1+\varkappa e_3\rangle$, $\langle e_3\rangle$, $\langle e_4\rangle$, $\varkappa\geq 0$
2-dim $\sim 2A_1$: $\langle e_1+\varkappa e_3, e_2\rangle$, $\langle e_2, e_3\rangle$, $\varkappa\geq 0$
$\sim A_{2.1}$: $\langle e_4; e_1\rangle$
3-dim $\sim 3A_1$: $\langle e_1, e_2, e_3\rangle$
$\sim A^b_{3.5}$: $\langle e_4; e_2, e_3\rangle$

$\boldsymbol{A_{4.7}}$: $[e_2,e_3]=e_1,\ [e_1,e_4]=2e_1,\ [e_2,e_4]=e_2,\ [e_3,e_4]=e_2+e_3$ (indecomposable solvable)

$$\text{Int}:\begin{pmatrix} e^{2\theta_4} & -\theta_3 e^{\theta_4} & -\theta_3\theta_4 e^{\theta_4}+\theta_2 e^{\theta_4} & \theta_1+\theta_2\theta_3-\frac{1}{2}\theta_3^2 \\ 0 & e^{\theta_4} & \theta_4 e^{\theta_4} & \theta_2+\theta_3 \\ 0 & 0 & e^{\theta_4} & \theta_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{Aut}:\begin{pmatrix} \alpha_{33}^2 & -\alpha_{33}\alpha_{34} & -m_{34}^{23}-\alpha_{33}\alpha_{34} & \alpha_{14} \\ 0 & \alpha_{33} & \alpha_{23} & \alpha_{24} \\ 0 & 0 & \alpha_{33} & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Der: $2E_{11}+E_{22}+E_{33},\ E_{14},\ E_{12}+E_{13}-E_{34},\ E_{23},\ E_{13}+E_{24}$

M$_0$: $\langle e_1\rangle$, $\langle e_1, e_2\rangle$, $\langle e_2, e_3; e_1\rangle$
Ch$_0$: —
I$_0$: —
S: 1-dim $\sim A_1$: $\langle e_1\rangle$, $\langle e_2\rangle$, $\langle e_3\rangle$, $\langle e_4\rangle$
2-dim $\sim 2A_1$: $\langle e_1, e_2\rangle$, $\langle e_1, e_3\rangle$
$\sim A_{2.1}$: $\langle e_4; e_1\rangle$, $\langle e_4; e_2\rangle$
3-dim $\sim A_{3.1}$: $\langle e_2, e_3; e_1\rangle$
$\sim A^{\frac{1}{2}}_{3.4}$: $\langle e_4; e_1, e_2\rangle$

$\boldsymbol{A^b_{4.8}}$: $[e_2,e_3]=e_1,\ [e_1,e_4]=(1+b)e_1,\ [e_2,e_4]=e_2,\ [e_3,e_4]=be_3,\ |b|\leq 1$ (indecomposable solvable)

$$\text{Int}:\begin{pmatrix} e^{(1+b)\theta_4} & -\theta_3 e^{\theta_4} & \theta_2 e^{b\theta_4} & (1+b)\theta_1+b\theta_2\theta_3 \\ 0 & e^{\theta_4} & 0 & \theta_2 \\ 0 & 0 & e^{b\theta_4} & b\theta_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$|b|<1,\ b\neq 0$:

$$\text{Aut}:\begin{pmatrix} \alpha_{22}\alpha_{33} & \alpha_{12} & \alpha_{33}\alpha_{24} & \alpha_{14} \\ 0 & \alpha_{22} & 0 & \alpha_{24} \\ 0 & 0 & \alpha_{33} & -b\alpha_{12}\alpha_{22}^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Der: $E_{12} - bE_{34},\ E_{14},\ E_{11} + E_{22},\ E_{13} + E_{24},\ E_{11} + E_{33}$

M_0: $\langle e_1\rangle,\ \langle e_1, e_2\rangle,\ \langle e_1, e_3\rangle,\ \langle e_2, e_3; e_1\rangle$
Ch_0: —
I_0: —
S: 1-dim $\sim A_1$: $\langle e_1\rangle,\ \langle e_2\rangle,\ \langle e_3\rangle,\ \langle e_4\rangle,\ \langle e_2 + \varepsilon e_3\rangle$
2-dim $\sim 2A_1$: $\langle e_1, e_2\rangle,\ \langle e_1, e_3\rangle,\ \langle e_1, e_2 + \varepsilon e_3\rangle$
$\sim A_{2.1}$: $\langle e_4; e_1\rangle,\ \langle e_4; e_2\rangle,\ \langle e_4; e_3\rangle$
3-dim $\sim A_{3.1}$: $\langle e_2, e_3; e_1\rangle$
$\sim A^a_{3.4}$: $\langle e_4; e_1, e_2\rangle^*,\ \langle e_4; e_1, e_3\rangle^{**}$

The parameter a in the algebra $A^a_{3.4}$ should satisfy the conditions:

$$a = \begin{cases} 1 + b, & |1 + b| < 1, \\ 1/(1 + b), & |1 + b| > 1 \end{cases} \quad \text{in the case *} \quad \text{and}$$

$$a = \begin{cases} b/(1 + b), & |b/(1 + b)| < 1 \text{ or b} = -\frac{1}{2}, \\ (1 + b)/b, & |b/(1 + b)| > 1 \end{cases} \quad \text{in the case **}.$$

$b = 0$:

$$\text{Aut}: \begin{pmatrix} \alpha_{22}\alpha_{33} & \alpha_{12} & \alpha_{33}\alpha_{24} & \alpha_{14} \\ 0 & \alpha_{22} & 0 & \alpha_{24} \\ 0 & 0 & \alpha_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Der: $E_{12},\ E_{14},\ E_{11} + E_{22},\ E_{13} + E_{24},\ E_{11} + E_{33}$

This automorphism group can be obtained from the general case $|b| < 1$, $b \neq 0$ by substitution $b = 0$, but these two cases can not be combined into one because of existence of additional (marked with *) megaideal in the case $b = 0$.

M_0: $\langle e_1\rangle,\ \langle e_1, e_2\rangle,\ \langle e_1, e_3\rangle,\ \langle e_2, e_3; e_1\rangle,\ \langle e_4; e_1, e_2\rangle^*$
Ch_0: —
I_0: $\langle e_4 + \varkappa e_3; e_1, e_2\rangle,\ \varkappa \neq 0$
S: 1-dim $\sim A_1$: $\langle e_1\rangle,\ \langle e_2\rangle,\ \langle e_3\rangle,\ \langle e_2 + \varepsilon e_3\rangle,\ \langle e_4 + \varkappa e_3\rangle$
2-dim $\sim 2A_1$: $\langle e_1, e_2\rangle,\ \langle e_1, e_3\rangle,\ \langle e_1, e_2 + \varepsilon e_3\rangle,\ \langle e_3, e_4\rangle$
$\sim A_{2.1}$: $\langle e_4 + \varkappa e_3; e_1\rangle,\ \langle e_4; e_2\rangle$
3-dim $\sim A_{2.1} \oplus A_1$: $\langle e_3, e_4; e_1\rangle$
$\sim A_{3.1}$: $\langle e_2, e_3; e_1\rangle$
$\sim A_{3.2}$: $\langle e_4 + \varkappa e_3; e_1, e_2\rangle,\ \varkappa \neq 0$
$\sim A_{3.3}$: $\langle e_4; e_1, e_2\rangle$

$b = 1$:

$$\text{Aut}: \begin{pmatrix} m^{23}_{23} & -m^{23}_{24} & -m^{23}_{34} & \alpha_{14} \\ 0 & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ 0 & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Der: $E_{12} - E_{34},\ E_{14},\ E_{11} + E_{22},\ E_{23},\ E_{13} + E_{24},\ E_{32},\ E_{11} + E_{33}$

M_0: $\langle e_1\rangle,\ \langle e_2, e_3; e_1\rangle$
Ch_0: —
I_0: $\langle e_1, pe_2 + qe_3\rangle$
S: 1-dim $\sim A_1$: $\langle e_1\rangle,\ \langle pe_2 + qe_3\rangle,\ \langle e_4\rangle$
2-dim $\sim 2A_1$: $\langle e_1, pe_2 + qe_3\rangle$
$\sim A_{2.1}$: $\langle e_4; e_1\rangle,\ \langle e_4; pe_2 + qe_3\rangle$
3-dim $\sim A_{3.1}$: $\langle e_2, e_3; e_1\rangle$
$\sim A^{1/2}_{3.4}$: $\langle e_4; e_1, pe_2 + qe_3\rangle$

$b=-1$:

Aut: $\begin{pmatrix} \alpha_{22}\alpha_{33} & \alpha_{22}\alpha_{34} & \alpha_{33}\alpha_{24} & \alpha_{14} \\ 0 & \alpha_{22} & 0 & \alpha_{24} \\ 0 & 0 & \alpha_{33} & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -\alpha_{32}\alpha_{23} & -\alpha_{32}\alpha_{24} & -\alpha_{23}\alpha_{34} & \alpha_{14} \\ 0 & 0 & \alpha_{23} & \alpha_{24} \\ 0 & \alpha_{32} & 0 & \alpha_{34} \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Der: $E_{12}+E_{34},\ E_{14},\ E_{11}+E_{22},\ E_{13}+E_{24},\ E_{11}+E_{33}$

M_0: $\langle e_1\rangle$, $\langle e_2,e_3;e_1\rangle$
Ch_0: $\langle e_1,e_3\rangle$, $\langle e_1,e_2\rangle$
I_0: —
S: 1-dim $\sim A_1$: $\langle e_1\rangle$, $\langle e_2\rangle$, $\langle e_2+\varepsilon e_3\rangle$, $\langle e_3\rangle$, $\langle e_4+\varkappa e_1\rangle$
2-dim $\sim 2A_1$: $\langle e_1,e_2\rangle$, $\langle e_1,e_3\rangle$, $\langle e_1,e_2+\varepsilon e_3\rangle$, $\langle e_4,e_1\rangle$
$\sim A_{2.1}$: $\langle e_4+\varkappa e_1;e_2\rangle$, $\langle e_4+\varkappa e_1;e_3\rangle$
3-dim $\sim A_{2.1}\oplus A_1$: $\langle e_4,e_1;e_2\rangle$, $\langle e_4,e_1;e_3\rangle$
$\sim A_{3.1}$: $\langle e_2,e_3;e_1\rangle$

$\boldsymbol{A^a_{4.9}}$: $[e_2,e_3]=e_1,\ [e_1,e_4]=2ae_1,\ [e_2,e_4]=ae_2-e_3,\ [e_3,e_4]=e_2+ae_3,\ a\geq 0$
(indecomposable solvable)

Int: $\begin{pmatrix} e^{2a\theta_4} & -(\theta_2\sin\theta_4+\theta_3\cos\theta_4)e^{a\theta_4} & (\theta_2\cos\theta_4-\theta_3\sin\theta_4)e^{a\theta_4} & 2a\theta_1+a\theta_2\theta_3-\frac{1}{2}\theta_2^2-\frac{1}{2}\theta_3^2 \\ 0 & \cos\theta_4 e^{a\theta_4} & \sin\theta_4 e^{a\theta_4} & a\theta_2+\theta_3 \\ 0 & -\sin\theta_4 e^{a\theta_4} & \cos\theta_4 e^{a\theta_4} & -\theta_2+a\theta_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$a>0$:

Aut: $\begin{pmatrix} s^{23}_{33} & (a^2+1)^{-1}(m^{23}_{34}-as^{23}_{34}) & -(a^2+1)^{-1}(am^{23}_{34}+s^{23}_{34}) & \alpha_{14} \\ 0 & \alpha_{33} & \alpha_{23} & \alpha_{24} \\ 0 & -\alpha_{23} & \alpha_{33} & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Der: $aE_{12}+E_{13}-(1+a^2)E_{34},\ E_{14},\ -E_{12}+E_{24}+aE_{34},\ -E_{23}+E_{32},\ 2E_{11}+E_{22}+E_{33}$

M_0: $\langle e_1\rangle$, $\langle e_2,e_3;e_1\rangle$
Ch_0: —
I_0: —
S: 1-dim $\sim A_1$: $\langle e_1\rangle$, $\langle e_2\rangle$, $\langle e_4\rangle$
2-dim $\sim 2A_1$: $\langle e_1,e_2\rangle$
$\sim A_{2.1}$: $\langle e_4;e_1\rangle$
3-dim $\sim A_{3.1}$: $\langle e_2,e_3;e_1\rangle$

$a=0$:

Aut: $\begin{pmatrix} \pm s^{23}_{33} & m^{23}_{34} & \mp s^{23}_{34} & \alpha_{14} \\ 0 & \pm\alpha_{33} & \alpha_{23} & \alpha_{24} \\ 0 & \mp\alpha_{23} & \alpha_{33} & \alpha_{34} \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}$

Der: $E_{13}-E_{34},\ E_{14},\ -E_{12}+E_{24},\ -E_{23}+E_{32},\ 2E_{11}+E_{22}+E_{33}$

M_0: $\langle e_1\rangle$, $\langle e_2,e_3;e_1\rangle$
Ch_0: —
I_0: —
S: 1-dim $\sim A_1$: $\langle e_1\rangle$, $\langle e_2\rangle$, $\langle e_4+\varkappa e_1\rangle$
2-dim $\sim 2A_1$: $\langle e_1,e_2\rangle$, $\langle e_1,e_4\rangle$
3-dim $\sim A_{3.1}$: $\langle e_2,e_3;e_1\rangle$

$\boldsymbol{A_{4.10}}$: $[e_1,e_3]=e_1,\ [e_2,e_3]=e_2,\ [e_1,e_4]=-e_2,\ [e_2,e_4]=e_1$ (indecomposable solvable)

$$\mathrm{Int}:\ \begin{pmatrix} \cos\theta_4 e^{\theta_3} & \sin\theta_4 e^{\theta_3} & \theta_1 & \theta_2 \\ -\sin\theta_4 e^{\theta_3} & \cos\theta_4 e^{\theta_3} & \theta_2 & -\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{Aut}:\ \begin{pmatrix} \pm\alpha_{22} & \alpha_{12} & \alpha_{13} & \pm\alpha_{23} \\ \mp\alpha_{12} & \alpha_{22} & \alpha_{23} & \mp\alpha_{13} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pm1 \end{pmatrix}$$

Der: $E_{11}+E_{22},\ E_{14}+E_{23},\ E_{13}-E_{24},\ E_{12}-E_{21}$

M_0: $\langle e_1,e_2\rangle,\ \langle e_2,e_3;e_1\rangle,\ \langle e_4;e_1,e_2\rangle$
Ch_0: $\langle e_4+\varkappa e_3;e_1,e_2\rangle,\ \ \varkappa\neq0$
I_0: —
S: 1-dim $\sim A_1$: $\langle e_1\rangle,\ \ \langle e_3\rangle,\ \ \langle e_4+\varkappa e_3\rangle$
2-dim $\sim 2A_1$: $\langle e_1,e_2\rangle,\ \ \langle e_3,e_4\rangle$
$\sim A_{2.1}$: $\langle e_3;e_1\rangle$
3-dim $\sim A_{3.3}$: $\langle e_3;e_1,e_2\rangle$
$\sim A_{3.5}^{|\varkappa|}$: $\langle e_4+\varkappa e_3;e_1,e_2\rangle$

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