THE LIE BRACKET AND THE CURVATURE TENSOR

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1. INTRODUCTION

The purpose of this paper is to present simple, coordinate-free proofs of well-known geometric interpretations (Theorems 1 and 2) of the Lie bracket and curvature tensor (in a C^∞-manifold with affine connection ∇). These pertain to the traversal of "parallelogram-like" circuits. The standard demonstrations of these interpretations usually make use of finite Taylor expansions in some special coordinate systems (cf. [1, pp. 135-138] for the Lie bracket; [5, pp. 106-108] for the curvature tensor), or repeated application of the multivariable chain rule (cf. [2, pp. 18-19] and [6, pp. 5-38 to 5-42] for the bracket). Spivak ([6, pp. 5-41]) refers to his proof as "an horrendous, but clever, calculation." An application to Lie group theory is given in Corollary 1.

All functions, curves, and vector fields are C^∞ on a C^∞ manifold M. If X is a vector field on M, then an *integral curve* of X is a curve γ (or γ_X) satisfying $\gamma'(t) = X(\gamma(t))$, for all t in domain (γ). If, in addition, $\gamma(0) = p$, we say that γ is an integral curve starting at p. We shall use X_t to denote the *flow* of X, so that $X_t(p) = \gamma(t)$, where γ is an integral curve of X starting at p.

2. THE LIE BRACKET

If f is a function on M, the following is immediate from applying Taylor's Theorem for functions of a real variable to the composition $f \cdot \gamma$, and observing that $(f \cdot \gamma)^{(k)} = X^k f \cdot \gamma$. Throughout this paper, $O(n)$ (n a positive integer) denotes a quantity for which $O(n) / t^n$ is bounded for small t.

LEMMA 1. (Taylor's Theorem for integral curves). If γ is an integral curve of a vector field X and if f is a real-valued function defined in a neighborhood of image (γ), then

$$f(\gamma(t)) - f(\gamma(0)) = \sum_{k=1}^{n} \frac{t^k}{k!} (X^k f)(\gamma(0)) + O(n+1)$$

THEOREM 1. Let X and Y be C^∞ vector fields on the C^∞ manifold M. Let $p \in M$ and let σ be the curve difined by

$$\sigma(u) = Y_u X_u Y_{-u} X_{-u} p$$

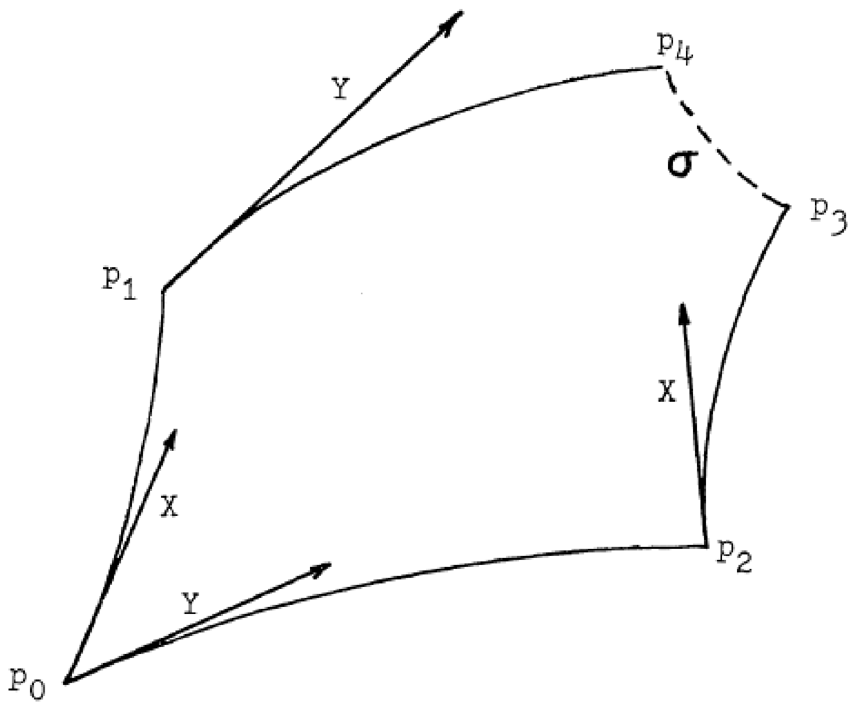
for u sufficiently small. Then for any C^∞ function f on M,

$$f(\sigma(t)) - f(\sigma(0)) = t^2 [X, Y]_p f + O(3).$$

Accordingly,

$$\lim_{t \to 0} \frac{f(\sigma(\sqrt{t})) - f(\sigma(0))}{t} = [X, Y]_p f$$

and the curve $\beta(u) = \sigma(\sqrt{u})$ satisfies $\beta'(0) = [X, Y]_p$.



Proof: In the figure, the four solid arcs are integral curves of X or Y, as depicted by the arrows, and all are parameterized on the interval $[0, t]$, for t sufficiently small. E.g., $p_2 = \gamma_X(0)$, $p_3 = \gamma_X(t) = X_t(p_2)$, etc. Subscripts denote the point of evaluation: f_i means $f(p_i)$; Xf_i or $X_i f$ means $(Xf)(p_i)$. The point p in the statement of Theorem 1 coincides with p_3 in the figure. We compute $f_4 - f_3$ by applying Lemma 1 to each arc.

(1) $$f_4 - f_1 = tYf_1 + \frac{t^2}{2} Y^2 f_1 + O(3)$$

(2) $$f_1 - f_0 = tXf_0 + \frac{t^2}{2} X^2 f_0 + O(3)$$

$$(3) \qquad f_3 - f_2 = tXf_2 + \frac{t^2}{2} X^2 f_2 + O(3)$$

$$(4) \qquad f_2 - f_0 = tYf_0 + \frac{t^2}{2} Y^2 f_0 + O(3)$$

Subtracting (3) and (4) from the sum of (1) and (2), and applying Lemma 1 again (up to $O(2)$ only), we obtain

$$f_4 - f_3 = t^2 (XYf - YXf)_0 + \frac{t^3}{2} (XY^2f - YX^2f)_0 + O(3),$$

or

$$(5) \qquad f_4 - f_3 = t^2 [X, Y]_0 f + O(3)$$

The meaning of this is that $[X, Y]$ measures the degree to which the circuit $p_3 \to p_2 \to p_0 \to p_1 \to p_4$ fails to be closed. Indeed, if $[X, Y] = 0$, then $p_3 = p_4$ (cf. [1, pp. 134-135]).

If we think of $p = p_3$ as the starting point, and (see figure) define $\sigma(u) = Y_u X_u Y_{-u} X_{-u} p$ (so that $p_4 = \sigma(t)$), we may re-express (5) as

$$f(\sigma(t)) - f(\sigma(0)) = t^2 [X, Y]_0 f + O(3) = t^2 [X, Y]_p f + O(3),$$

since switching to p changes $[X, Y] f$ by an amount which is only of order $O(1)$.

3. A Particular Case

As a special case, assume X and Y are left invariant vector fields on a Lie group G, i.e., elements of $L(G)$, the Lie algebra of G; and take p to be e, the identity element of the group. Since, in this context, $X_t(p) = p \exp(tX)$, for p in G, we have

$$\sigma(t) = \exp(-tX) \exp(-tY) \exp(tX) \exp(tY).$$

If we assume $f(e) = 0$, Theorem 1 yields

$$\begin{aligned} & f(\exp(-tX) \exp(-tY) \exp(tX) \exp(tY)) \\ &= t^2 [X, Y]_e f + O(3) \\ &= f(\exp\{t^2 [X, Y] + O(3)\}) \end{aligned}$$

and so

$$\exp(-tX) \exp(-tY) \exp(tX) \exp(tY) = \exp(t^2 [X, Y] + O(3)).$$

This formula is involved in proving that if H is (algebraically) a subgroup of a Lie group G and if H is a closed subset of G, then H is a topological Lie subgroup of G ([3, pp. 96, 105]). Specifically, it implies that $\{ V$ in $L(G) \mid \exp(tV)$ is in H, for all t real $\}$ is closed under the bracket. The formula also provides the following geometric interpretation of the bracket $[X, Y]$ on the Lie algebra $L(G)$ of a Lie group G.

COROLLARY 1. If X and Y belong to $L(G)$, then the curve

$$t \to \exp(-\sqrt{t}X)\exp(-\sqrt{t}Y)\exp(\sqrt{t}X)\exp(\sqrt{t}Y)$$

has velocity vector $[X, Y]$ at $t = 0$.

4. THE CURVATURE TENSOR

Now assume M is furnished with an affine connection (covariant differentiation operator) ∇.

The *curvature tensor* R on M is the $\binom{1}{3}$-tensor (equivalently, the linear-transformation-valued bilinear mapping) R defined by

$$\begin{aligned} R(X, Y)A &= \nabla_X \nabla_Y A - \nabla_Y \nabla_X A - \nabla_{[X,Y]} A \\ &= \left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right) A, \end{aligned}$$

for X, Y, and A vector fields on M. The relationship between this tensor and the Riemann curvature (in a Riemannian manifold) may be found in [4, pp. 72-73], [2, Chapter 9], and [5, pp. 125-127]. Here we shall show its relationship to parallel translation.

Consider the figure again, and let A be any vector field on M. We shall compare parallel translation along $p_0 \to p_1 \to p_4$ with that along $p_0 \to p_2 \to p_3$. Then, by adding the curve $\sigma(u) = Y_u X_u Y_{-u} X_{-u} p_3$ defined previously (the dotted curve in the figure), we obtain a closed circuit. We shall need the following.

LEMMA 2. (Taylor's Theorem for parallel translation). Let X be a vector field defined in a neighborhood of a curve γ, let $T = \gamma'(0)$, and for any t in domain (γ), let τ_t denote parallel translation along γ to $\gamma(t)$. Then

$$\tau_0 X(\gamma(t)) - X(\gamma(0)) = \sum_{k=1}^{n} \frac{t^k}{k!} \nabla_T^k X + O(n+1).$$

Proof. Apply the real-variable Taylor's Theorem to the function $f(t) = \tau_0 X(\gamma(t))$ which has values in a finite dimensional vector space.

$$f'(t) = \lim_{h \to 0} \frac{\tau_0 X(\gamma(t+h)) - \tau_0 X(\gamma(t))}{h}$$

$$= \tau_0 \lim_{h \to 0} \frac{\tau_t X(\gamma(t+h)) - X(\gamma(t))}{h} = \tau_0 \nabla_{\gamma'(t)} X.$$

Inductively, $f^{(n)}(t) = \tau_0 (\nabla_{\gamma'(t)}{}^n X)$ and $f^{(n)}(0) = \nabla_T{}^n X$.

THEOREM 2. Let X, Y, and A be C^∞ vector fields on the C^∞ manifold M with affine connection ∇. Let p belong to M and consider parallel translation of A_p around the closed circuit consisting of (in order) the integral curves of $-X$, $-Y$, X, and Y (each parameterized on $[0, t]$, t small), and (backwards along) the curve $\sigma(u) = Y_u X_u Y_{-u} X_{-u} p$, $0 \leqslant u \leqslant t$ (see figure). If ΔA is the change in A_p produced by parallel translation around this circuit, then

$$\Delta A = t^2 R(Y, X) A_p + O(3)$$

and hence

$$\lim_{t \to 0} \frac{\Delta A}{t^2} = R(Y, X) A_p .$$

Proof. The calculation is similar to that for the Lie bracket in Theorem 1, except that we must use parallel translation to compare vectors at different points. τ_i denotes parallel translation to p_i along the arc to p_i from the location of the tangent vector in question. Elsewhere, subscripts denote point of evaluation, as before. From Lemma 2, we have

$$\tau_1 A_4 - A_1 = t \nabla_Y A_1 + \frac{t^2}{2} \nabla_Y{}^2 A_1 + O(3) \tag{6}$$

$$\tau_0 A_1 - A_0 = t \nabla_X A_0 + \frac{t^2}{2} \nabla_X{}^2 A_0 + O(3) \tag{7}$$

$$\tau_2 A_3 - A_2 = t \nabla_X A_2 + \frac{t^2}{2} \nabla_X{}^2 A_2 + O(3) \tag{8}$$

$$\tau_0 A_2 - A_0 = t \nabla_Y A_0 + \frac{t^2}{2} \nabla_Y{}^2 A_0 + O(3) \tag{9}$$

Apply τ_0 to both sides of (6) and (8), obtaining (6′) and (8′), respectively. Subtracting (8′) and (9) from the sum of (6′) and (7), we obtain (via Lemma 2),

$$(10) \qquad \tau_0 \tau_1 A_4 - \tau_0 \tau_2 A_3 = t^2 [\nabla_X, \nabla_Y] A_0 + O(3)$$

As before, let $\beta(u) = \sigma(\sqrt{u})$, $0 \leqslant u \leqslant t^2$. Using $\beta'(0) = [X, Y]_3$ (from Theorem 1), we may, as in the proof of Lemma 2, show that

$$(11) \qquad \tau_3 A_4 - A_3 = t^2 \nabla_{[X,Y]} A_3 + O(4).$$

Now apply τ_4 to (11) and $\tau_4 \tau_1$ to (10). Taking the difference of the resulting equations and then applying τ_3 to both sides, we obtain

$$\begin{aligned} \Delta A &= \tau_3 \tau_4 \tau_1 \tau_0 \tau_2 A_3 - A_3 \\ &= t^2 (\tau_3 \tau_4 \nabla_{[X,Y]} A_3 - \tau_3 \tau_4 \tau_1 [\nabla_X, \nabla_Y] A_0) + O(3) \\ &= t^2 (\nabla_{[X,Y]} - [\nabla_X, \nabla_Y]) A_3 + O(3) = - t^2 R(X, Y) A_p + O(3), \end{aligned}$$

since the change produced by dropping the τ's and switching to p_3 may be absorbed in $O(3)$. Thus the theorem follows since $-R(X, Y) = R(Y, X)$.

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