1

Compact Lie Groups

1.1 Basic Notions

1.1.1 Manifolds

Lie theory is the study of symmetry springing from the intersection of algebra, analysis, and geometry. Less poetically, Lie groups are simultaneously groups and manifolds. In this section, we recall the definition of a manifold (see [8] or [88] for more detail). Let $n \in \mathbb{N}$.

Definition 1.1. An n-dimensional *topological manifold* is a second countable (i.e., possessing a countable basis for the topology) Hausdorff topological space M that is locally homeomorphic to an open subset of $\mathbb{R}^n$.

This means that for all $m \in M$ there exists a homeomorphism $\varphi : U \to V$ for some open neighborhood U of m and an open neighborhood V of $\mathbb{R}^n$. Such a homeomorphism φ is called a *chart*.

Definition 1.2. An n-dimensional smooth *manifold* is a topological manifold M along with a collection of charts, $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$, called an *atlas*, so that
(1) $M = \cup_\alpha U_\alpha$ and
(2) For all α, β with $U_\alpha \cap U_\beta \neq \emptyset$, the *transition map* $\varphi_{\alpha,\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a smooth map on $\mathbb{R}^n$.

It is an elementary fact that each atlas can be completed to a unique maximal atlas containing the original. By common convention, a manifold's atlas will always be extended to this completion.

Besides $\mathbb{R}^n$, common examples of manifolds include the *n-sphere*,

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\},$$

where $\|\cdot\|$ denotes the standard Euclidean norm, and the *n-torus*,

$$T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ copies}}.$$

Another important manifold is real *projective space*, $\mathbb{P}(\mathbb{R}^n)$, which is the n-dimensional compact manifold of all lines in $\mathbb{R}^{n+1}$. It may be alternately realized as $\mathbb{R}^{n+1}\backslash\{0\}$ modulo the equivalence relation $x \sim \lambda x$ for $x \in \mathbb{R}^{n+1}\backslash\{0\}$ and $\lambda \in \mathbb{R}\backslash\{0\}$, or as S^n modulo the equivalence relation $x \sim \pm x$ for $x \in S^n$. More generally, the *Grassmannian,* $\mathrm{Gr}_k(\mathbb{R}^n)$, consists of all k-planes in $\mathbb{R}^n$. It is a compact manifold of dimension $k(n-k)$ and reduces to $\mathbb{P}(\mathbb{R}^{n-1})$ when $k = 1$.

Write $M_{n,m}(\mathbb{F})$ for the set of $n \times m$ matrices over $\mathbb{F}$ where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. By looking at each coordinate, $M_{n,m}(\mathbb{R})$ may be identified with $\mathbb{R}^{nm}$ and $M_{n,m}(\mathbb{C})$ with $\mathbb{R}^{2nm}$. Since the determinant is continuous on $M_{n,n}(\mathbb{F})$, we see $\det^{-1}\{0\}$ is a closed subset. Thus the *general linear group*

$$GL(n,\mathbb{F}) = \{g \in M_{n,n}(\mathbb{F}) \mid g \text{ is invertible}\} \tag{1.3}$$

is an open subset of $M_{n,n}(\mathbb{F})$ and therefore a manifold. In a similar spirit, for any finite-dimensional vector space V over $\mathbb{F}$, we write $GL(V)$ for the set of invertible linear transformations on V.

1.1.2 Lie Groups

Definition 1.4. A *Lie group* G is a group and a manifold so that
(1) the *multiplication* map $\mu : G \times G \to G$ given by $\mu(g, g') = gg'$ is smooth and
(2) the *inverse* map $\iota : G \to G$ by $\iota(g) = g^{-1}$ is smooth.

A trivial example of a Lie group is furnished by $\mathbb{R}^n$ with its additive group structure. A slightly fancier example of a Lie group is given by S^1. In this case, the group structure is inherited from multiplication in $\mathbb{C}\backslash\{0\}$ via the identification

$$S^1 \cong \{z \in \mathbb{C} \mid |z| = 1\}.$$

However, the most interesting example of a Lie group so far is $GL(n,\mathbb{F})$. To verify $GL(n,\mathbb{F})$ is a Lie group, first observe that multiplication is smooth since it is a polynomial map in the coordinates. Checking that the inverse map is smooth requires the standard linear algebra formula $g^{-1} = \mathrm{adj}(g)/\det g$, where the $\mathrm{adj}(g)$ is the transpose of the matrix of cofactors. In particular, the coordinates of $\mathrm{adj}(g)$ are polynomial functions in the coordinates of g and $\det g$ is a nonvanishing polynomial on $GL(n,\mathbb{F})$ so the inverse is a smooth map.

Writing down further examples of Lie groups requires a bit more machinery. In fact, most of our future examples of Lie groups arise naturally as subgroups of $GL(n,\mathbb{F})$. To this end, we next develop the notion of a Lie subgroup.

1.1.3 Lie Subgroups and Homomorphisms

Recall that an (immersed) *submanifold* N of M is the image of a manifold N' under an injective immersion $\varphi : N' \to M$ (i.e., a one-to-one smooth map whose differential has full rank at each point of N') together with the manifold structure on N

making $\varphi : N' \to N$ a diffeomorphism. It is a familiar fact from differential geometry that the resulting topology on N may not coincide with the relative topology on N as a subset of M. A submanifold N whose topology agrees with the relative topology is called a *regular* (or *imbedded*) submanifold.

Defining the notion of a Lie subgroup is very similar. Essentially the word homomorphism needs to be thrown in.

Definition 1.5. A *Lie subgroup* H of a Lie group G is the image in G of a Lie group H' under an injective immersive homomorphism $\varphi : H' \to G$ together with the Lie group structure on H making $\varphi : H' \to H$ a diffeomorphism.

The map φ in the above definition is required to be smooth. However, we will see in Exercise 4.13 that it actually suffices to verify that φ is continuous.

As with manifolds, a Lie subgroup is *not* required to be a regular submanifold. A typical example of this phenomenon is constructed by wrapping a line around the torus at an irrational angle (Exercise 1.5). However, regular Lie subgroups play a special role and there happens to be a remarkably simple criterion for determining when Lie subgroups are regular.

Theorem 1.6. *Let G be a Lie group and $H \subseteq G$ a subgroup (with no manifold assumption). Then H is a regular Lie subgroup if and only if H is closed.*

The proof of this theorem requires a fair amount of effort. Although some of the necessary machinery is developed in §4.1.2, the proof lies almost entirely within the purview of a course on differential geometry. For the sake of clarity of exposition and since the result is only used to efficiently construct examples of Lie groups in §1.1.4 and §1.3.2, the proof of this theorem is relegated to Exercise 4.28. While we are busy putting off work, we record another useful theorem whose proof, for similar reasons, can also be left to a course on differential geometry (e.g., [8] or [88]). We note, however, that a proof of this result follows almost immediately once Theorem 4.6 is established.

Theorem 1.7. *Let H be a closed subgroup of a Lie group G. Then there is a unique manifold structure on the quotient space G/H so the projection map $\pi : G \to G/H$ is smooth, and so there exist local smooth sections of G/H into G.*

Pressing on, an immediate corollary of Theorem 1.6 provides an extremely useful method of constructing new Lie groups. The corollary requires the well-known fact that when $f : H \to M$ is a smooth map of manifolds with $f(H) \subseteq N$, N a *regular* submanifold of M, then $f : H \to N$ is also a smooth map (see [8] or [88]).

Corollary 1.8. *A closed subgroup of a Lie group is a Lie group in its own right with respect to the relative topology.*

Another common method of constructing Lie groups depends on the Rank Theorem from differential geometry.

Definition 1.9. A *homomorphism of Lie groups* is a smooth homomorphism between two Lie groups.

Theorem 1.10. *If G and G' are Lie groups and $\varphi : G \to G'$ is a homomorphism of Lie groups, then φ has constant rank and* $\ker \varphi$ *is a (closed) regular Lie subgroup of G of dimension* $\dim G - \operatorname{rk} \varphi$ *where* $\operatorname{rk} \varphi$ *is the rank of the differential of φ.*

Proof. It is well known (see [8]) that if a smooth map φ has constant rank, then $\varphi^{-1}\{e\}$ is a closed regular submanifold of G of dimension $\dim G - \operatorname{rk} \varphi$. Since $\ker \varphi$ is a subgroup, it suffices to show that φ has constant rank. Write l_g for left translation by g. Because φ is a homomorphism, $\varphi \circ l_g = l_{\varphi(g)} \circ \varphi$, and since l_g is a diffeomorphism, the rank result follows by taking differentials. □

1.1.4 Compact Classical Lie Groups

With the help of Corollary 1.8, it is easy to write down new Lie groups. The first is the *special linear group*

$$SL(n, \mathbb{F}) = \{g \in GL(n, \mathbb{F}) \mid \det g = 1\}.$$

As $SL(n, \mathbb{F})$ is a closed subgroup of $GL(n, \mathbb{F})$, it follows that it is a Lie group.

Using similar techniques, we next write down four infinite families of compact Lie groups collectively known as the *classical compact Lie groups:* $SO(2n + 1)$, $SO(2n)$, $SU(n)$, and $Sp(n)$.

1.1.4.1 $SO(n)$ The *orthogonal group* is defined as

$$O(n) = \{g \in GL(n, \mathbb{R}) \mid g^t g = I\},$$

where g^t denotes the transpose of g. The orthogonal group is a closed subgroup of $GL(n, \mathbb{R})$, so Corollary 1.8 implies that $O(n)$ is a Lie group. Since each column of an orthogonal matrix is a unit vector, we see that topologically $O(n)$ may be thought of as a closed subset of $S^{n-1} \times S^{n-1} \times \cdots \times S^{n-1} \subseteq \mathbb{R}^{n^2}$ (n copies). In particular, $O(n)$ is a compact Lie group.

The *special orthogonal group* (or *rotation group*) is defined as

$$SO(n) = \{g \in O(n) \mid \det g = 1\}.$$

This is a closed subgroup of $O(n)$, and so $SO(n)$ is also a compact Lie group.

Although not obvious at the moment, the behavior of $SO(n)$ depends heavily on the parity of n. This will become pronounced starting in §6.1.4. For this reason, the special orthogonal groups are considered to embody two separate infinite families: $SO(2n + 1)$ and $SO(2n)$.

1.1.4.2 $SU(n)$ The *unitary group* is defined as

$$U(n) = \{g \in GL(n, \mathbb{C}) \mid g^* g = I\},$$

where g^* denotes the complex conjugate transpose of g. The unitary group is a closed subgroup of $GL(n, \mathbb{C})$, and so $U(n)$ is a Lie group. As each column of a unitary matrix is a unit vector, we see that $U(n)$ may be thought of, topologically, as a closed subset of $S^{2n-1} \times S^{2n-1} \times \cdots \times^{2n-1} \subseteq \mathbb{R}^{2n^2}$ (n copies). In particular, $U(n)$ is a compact Lie group.

Likewise, the *special unitary group* is defined as

$$SU(n) = \{g \in U(n) \mid \det g = 1\}.$$

As usual, this is a closed subgroup of $U(n)$, and so $SU(n)$ is also a compact Lie group. The special case of $n = 2$ will play an especially important future role. It is straightforward to check (Exercise 1.8) that

$$SU(2) = \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \mid a, b \in \mathbb{C} \text{ and } |a|^2 + |b|^2 = 1 \right\} \tag{1.11}$$

so that topologically $SU(2) \cong S^3$.

1.1.4.3 $Sp(n)$ The final compact classical Lie group, the *symplectic group*, ought to be defined as

$$Sp(n) = \{g \in GL(n, \mathbb{H}) \mid g^* g = I\}, \tag{1.12}$$

where $\mathbb{H} = \{a + ib + jc + kd \mid a, b, c, d \in \mathbb{R}\}$ denotes the *quaternions* and g^* denotes the quaternionic conjugate transpose of g. However, $\mathbb{H}$ is a noncommutative division algebra, so understanding the meaning of $GL(n, \mathbb{H})$ takes a bit more work. Once this is done, Equation 1.12 will become the honest definition of $Sp(n)$.

To begin, view $\mathbb{H}^n$ as a *right* vector space with respect to scalar multiplication and let $M_{n,n}(\mathbb{H})$ denote the set of $n \times n$ matrices over $\mathbb{H}$. By using matrix multiplication on the *left*, $M_{n,n}(\mathbb{H})$ may therefore be identified with the set of $\mathbb{H}$-linear transformations of $\mathbb{H}^n$. Thus the old definition of $GL(n, \mathbb{F})$ in Equation 1.3 can be carried over to define $GL(n, \mathbb{H}) = \{g \in M_{n,n}(\mathbb{H}) \mid g \text{ is an invertible transformation of } \mathbb{H}^n\}$.

Verifying that $GL(n, \mathbb{H})$ is a Lie group, unfortunately, requires more work. In the case of $GL(n, \mathbb{F})$ in §1.1.2, that work was done by the determinant function which is no longer readily available for $GL(n, \mathbb{H})$. Instead, we embed $GL(n, \mathbb{H})$ into $GL(2n, \mathbb{C})$ as follows.

Observe that any $v \in \mathbb{H}$ can be uniquely written as $v = a + jb$ for $a, b \in \mathbb{C}$. Thus there is a well-defined $\mathbb{C}$-linear isomorphism $\vartheta : \mathbb{H}^n \to \mathbb{C}^{2n}$ given by $\vartheta(v_1, \ldots, v_n) = (a_1, \ldots, a_n, b_1, \ldots, b_n)$ where $v_p = a_p + jb_p$, $a_p, b_p \in \mathbb{C}$. Use this to define a $\mathbb{C}$-linear injection of algebras $\widetilde{\vartheta} : M_{n,n}(\mathbb{H}) \to M_{n,n}(\mathbb{C})$ by $\widetilde{\vartheta} X = \vartheta \circ X \circ \vartheta^{-1}$ for $X \in M_{n,n}(\mathbb{H})$ with respect to the usual identification of matrices as linear maps. It is straightforward to verify (Exercise 1.12) that when X is uniquely written as $X = A + jB$ for $A, B \in M_{n,n}(\mathbb{C})$, then

$$\widetilde{\vartheta}(A + jB) = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}, \tag{1.13}$$

where $\overline{A}$ denotes complex conjugation of A. Thus $\widetilde{\vartheta}$ is a $\mathbb{C}$-linear algebra isomorphism from $M_{n,n}(\mathbb{H})$ to

$$M_{2n,2n}(\mathbb{C})_{\mathbb{H}} \equiv \{\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \mid A, B \in M_{n,n}(\mathbb{C})\}.$$

An alternate way of checking this is to first let r_j denote scalar multiplication by j on $\mathbb{H}^n$, i.e., *right* multiplication by j. It is easy to verify (Exercise 1.12) that $\vartheta r_j \vartheta^{-1} z = J\overline{z}$ for $z \in \mathbb{C}^{2n}$ where

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Since ϑ is a $\mathbb{C}$-linear isomorphism, the image of $\widetilde{\vartheta}$ consists of all $Y \in M_{2n,2n}(\mathbb{C})$ commuting with $\vartheta r_j \vartheta^{-1}$ so that $M_{2n,2n}(\mathbb{C})_{\mathbb{H}} = \{Y \in M_{2n}(\mathbb{C}) \mid YJ = J\overline{Y}\}$.

Finally, observe that X is invertible if and only if $\widetilde{\vartheta} X$ is invertible. In particular, $M_{n,n}(\mathbb{H})$ may be thought of as $\mathbb{R}^{4n^2}$ and, since $\det \circ \widetilde{\vartheta}$ is continuous, $GL(n, \mathbb{H})$ is the open set in $M_{n,n}(\mathbb{H})$ defined by the complement of $(\det \circ \widetilde{\vartheta})^{-1}\{0\}$. Since $GL(n, \mathbb{H})$ is now clearly a Lie group, Equation 1.12 shows that $Sp(n)$ is a Lie group by Corollary 1.8. As with the previous examples, $Sp(n)$ is compact since each column vector is a unit vector in $\mathbb{H}^n \cong \mathbb{R}^{4n}$.

As an aside, Dieudonné developed the notion of determinant suitable for $M_{n,n}(\mathbb{H})$ (see [2], 151–158). This quaternionic determinant has most of the nice properties of the usual determinant and it turns out that elements of $Sp(n)$ always have determinant 1.

There is another useful realization for $Sp(n)$ besides the one given in Equation 1.12. The isomorphism is given by $\widetilde{\vartheta}$ and it remains only to describe the image of $Sp(n)$ under $\widetilde{\vartheta}$. First, it is easy to verify (Exercise 1.12) that $\widetilde{\vartheta}(X^*) = (\widetilde{\vartheta} X)^*$ for $X \in M_{n,n}(\mathbb{H})$, and thus $\widetilde{\vartheta} Sp(n) = U(2n) \cap M_{2n,2n}(\mathbb{C})_{\mathbb{H}}$. This answer can be reshaped further. Define

$$Sp(n, \mathbb{C}) = \{g \in GL(2n, \mathbb{C}) \mid g^t J g = J\}$$

so that $U(2n) \cap M_{2n,2n}(\mathbb{C})_{\mathbb{H}} = U(2n) \cap Sp(n, \mathbb{C})$. Hence $\widetilde{\vartheta}$ realizes the isomorphism:

$$\begin{aligned} Sp(n) &\cong U(2n) \cap M_{2n,2n}(\mathbb{C})_{\mathbb{H}} \\ &= U(2n) \cap Sp(n, \mathbb{C}). \end{aligned} \tag{1.14}$$

1.1.5 Exercises

Exercise 1.1 Show that S^n is a manifold that can be equipped with an atlas consisting of only two charts.

Exercise 1.2 **(a)** Show that $\mathrm{Gr}_k(\mathbb{R}^n)$ may be realized as the rank k elements of $M_{n,k}(\mathbb{R})$ modulo the equivalence relation $X \sim Xg$ for $X \in M_{n,k}(\mathbb{R})$ of rank k and $g \in GL(k,\mathbb{R})$. Find another realization showing that $\mathrm{Gr}_k(\mathbb{R}^n)$ is compact.
(b) For $S \subseteq \{1,2,\dots,n\}$ with $|S| = k$ and $X \in M_{n,k}(\mathbb{R})$, let $X|_S$ be the $k \times k$ matrix obtained from X by keeping only those rows indexed by an element of S, let $U_S = \{X \in M_{n,k}(\mathbb{R}) \mid X|_S \text{ is invertible}\}$, and let $\varphi_S : U_S \to M_{(n-k),k}(\mathbb{R})$ by $\varphi_S(X) = [X(X|_S)^{-1}]|_{S^c}$. Use these definitions to show that $\mathrm{Gr}_k(\mathbb{R}^n)$ is a $k(n-k)$ dimensional manifold.

Exercise 1.3 **(a)** Show that conditions (1) and (2) in Definition 1.4 may be replaced by the single condition that the map $(g_1, g_2) \to g_1 g_2^{-1}$ is smooth.
(b) In fact, show that condition (1) in Definition 1.4 implies condition (2).

Exercise 1.4 If U is an open set containing e in a Lie group G, show there exists an open set $V \subseteq U$ containing e, so $VV^{-1} \subseteq U$, where VV^{-1} is $\{vw^{-1} \mid v, w \in V\}$.

Exercise 1.5 Fix $a, b \in \mathbb{R}\backslash\{0\}$ and consider the subgroup of T^2 defined by $R_{a,b} = \{(e^{2\pi iat}, e^{2\pi ibt}) \mid t \in \mathbb{R}\}$.
(a) Suppose $\frac{a}{b} \in \mathbb{Q}$ and $\frac{a}{b} = \frac{p}{q}$ for relatively prime $p, q \in \mathbb{Z}$. As t varies, show that the first component of $R_{a,b}$ wraps around S^1 exactly p-times, while the second component wraps around q-times. Conclude that $R_{a,b}$ is closed and therefore a regular Lie subgroup diffeomorphic to S^1.
(b) Suppose $\frac{a}{b} \notin \mathbb{Q}$. Show that $R_{a,b}$ wraps around infinitely often without repeating. Conclude that $R_{a,b}$ is a Lie subgroup diffeomorphic to $\mathbb{R}$, but *not* a regular Lie subgroup (c.f. Exercise 5.*).
(c) What happens if a or b is 0?

Exercise 1.6 **(a)** Use Theorem 1.10 and the map $\det : GL(n,\mathbb{R}) \to \mathbb{R}$ to give an alternate proof that $SL(n,\mathbb{R})$ is a Lie group and has dimension $n^2 - 1$.
(b) Show the map $X \to XX^t$ from $GL(n,\mathbb{R})$ to $\{X \in M_{n,n}(\mathbb{R}) \mid X^t = X\}$ has constant rank $\frac{n(n+1)}{2}$. Use the proof of Theorem 1.10 to give an alternate proof that $O(n)$ is a Lie group and has dimension $\frac{n(n-1)}{2}$.
(c) Use the map $X \to XX^*$ on $GL(n,\mathbb{C})$ to give an alternate proof that $U(n)$ is a Lie group and has dimension n^2.
(d) Use the map $X \to XX^*$ on $GL(n,\mathbb{H})$ to give an alternate proof that $Sp(n)$ is a Lie group and has dimension $2n^2 + n$.

Exercise 1.7 For a Lie group G, write $Z(G) = \{z \in G \mid zg = gz, \text{ all } g \in G\}$ for the *center* of G. Show
(a) $Z(U(n)) \cong S^1$ and $Z(SU(n)) \cong \mathbb{Z}/n\mathbb{Z}$ for $n \geq 2$,
(b) $Z(O(2n)) \cong \mathbb{Z}/2\mathbb{Z}$, $Z(SO(2n)) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$, and $Z(SO(2)) = SO(2)$,
(c) $Z(O(2n+1)) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 1$, and $Z(SO(2n+1)) = \{I\}$ for $n \geq 1$,
(d) $Z(Sp(n)) \cong \mathbb{Z}/2\mathbb{Z}$.

Exercise 1.8 Verify directly Equation 1.11.

Exercise 1.9 **(a)** Let $A \subseteq GL(n,\mathbb{R})$ be the subgroup of diagonal matrices with positive elements on the diagonal and let $N \subseteq GL(n,\mathbb{R})$ be the subgroup of upper triangular matrices with 1's on the diagonal. Using Gram-Schmidt orthogonalization, show multiplication induces a diffeomorphism of $O(n)\times A\times N$ onto $GL(n,\mathbb{R})$. This is called the *Iwasawa* or KAN decomposition for $GL(n,\mathbb{R})$. As topological spaces, show that $GL(n,\mathbb{R}) \cong O(n) \times \mathbb{R}^{\frac{n(n+1)}{2}}$. Similarly, as topological spaces, show that $SL(n,\mathbb{R}) \cong SO(n) \times \mathbb{R}^{\frac{(n+2)(n-1)}{2}}$.
(b) Let $A \subseteq GL(n,\mathbb{C})$ be the subgroup of diagonal matrices with positive real elements on the diagonal and let $N \subseteq GL(n,\mathbb{C})$ be the subgroup of upper triangular matrices with 1's on the diagonal. Show that multiplication induces a diffeomorphism of $U(n) \times A \times N$ onto $GL(n,\mathbb{C})$. As topological spaces, show $GL(n,\mathbb{C}) \cong U(n)\times\mathbb{R}^{n^2}$. Similarly, as topological spaces, show that $SL(n,\mathbb{C}) \cong SU(n)\times\mathbb{R}^{n^2-1}$.

Exercise 1.10 Let $N \subseteq GL(n,\mathbb{C})$ be the subgroup of upper triangular matrices with 1's on the diagonal, let $\overline{N} \subseteq GL(n,\mathbb{C})$ be the subgroup of lower triangular matrices with 1's on the diagonal, and let W be the subgroup of permutation matrices (i.e., matrices with a single one in each row and each column and zeros elsewhere). Use Gaussian elimination to show $GL(n,\mathbb{C}) = \coprod_{w\in W} \overline{N}wN$. This is called the *Bruhat* decomposition for $GL(n,\mathbb{C})$.

Exercise 1.11 **(a)** Let $P \subseteq GL(n,\mathbb{R})$ be the set of positive definite symmetric matrices. Show that multiplication gives a bijection from $P \times O(n)$ to $GL(n,\mathbb{R})$.
(b) Let $H \subseteq GL(n,\mathbb{C})$ be the set of positive definite Hermitian matrices. Show that multiplication gives a bijection from $H \times U(n)$ to $GL(n,\mathbb{C})$.

Exercise 1.12 **(a)** Show that $\widetilde{\vartheta}$ is given by the formula in Equation 1.13.
(b) Show $\vartheta r_j \vartheta^{-1} z = J\overline{z}$ for $z \in \mathbb{C}^{2n}$.
(c) Show that $\widetilde{\vartheta}(X^*) = (\widetilde{\vartheta} X)^*$ for $X \in M_{n,n}(\mathbb{H})$.

Exercise 1.13 For $v, u \in \mathbb{H}^n$, let $(v,u) = \sum_{p=1}^n v_p\overline{u_p}$.
(a) Show that $(Xv,u) = (v, X^*u)$ for $X \in M_{n,n}(\mathbb{H})$.
(b) Show that $Sp(n) = \{g \in M_n(\mathbb{H}) \mid (gv, gu) = (v,u), \text{ all } v,u \in \mathbb{H}^n\}$.

1.2 Basic Topology

1.2.1 Connectedness

Recall that a topological space is *connected* if it is not the disjoint union of two nonempty open sets. A space is *path connected* if any two points can be joined by a continuous path. While in general these two notions are distinct, they are equivalent for manifolds. In fact, it is even possible to replace continuous paths with smooth paths.

The first theorem is a technical tool that will be used often.

Theorem 1.15. *Let G be a connected Lie group and U a neighborhood of e. Then U generates G, i.e., $G = \cup_{n=1}^{\infty} U^n$ where U^n consists of all n-fold products of elements of U.*

Proof. We may assume U is open without loss of generality. Let $V = U \cap U^{-1} \subseteq U$ where U^{-1} is the set of all inverses of elements in U. This is an open set since the inverse map is continuous. Let $H = \cup_{n=1}^{\infty} V^n$. By construction, H is an open subgroup containing e. For $g \in G$, write $gH = \{gh \mid h \in H\}$. The set gH contains g and is open since left multiplication by g^{-1} is continuous. Thus G is the union of all the open sets gH. If we pick a representative $g_\alpha H$ for each coset in G/H, then $G = \amalg_\alpha (g_\alpha H)$. Hence the connectedness of G implies that G/H contains exactly one coset, i.e., $eH = G$, which is sufficient to finish the proof. $\square$

We still lack general methods for determining when a Lie group G is connected. This shortcoming is remedied next.

Definition 1.16. If G is a Lie group, write G^0 for the connected component of G containing e.

Lemma 1.17. *Let G be a Lie group. The connected component G^0 is a regular Lie subgroup of G. If G^1 is any connected component of G with $g_1 \in G^1$, then $G^1 = g_1 G^0$.*

Proof. We prove the second statement of the lemma first. Since left multiplication by g_1 is a homeomorphism, it follows easily that $g_1 G^0$ is a connected component of G. But since $e \in G^0$, this means that $g_1 \in g_1 G^0$ so $g_1 G^0 \cap G^1 \neq \emptyset$. Since both are connected components, $G^1 = g_1 G^0$ and the second statement is finished.

Returning to the first statement of the lemma, it clearly suffices to show that G^0 is a subgroup. The inverse map is a homeomorphism, so $(G^0)^{-1}$ is a connected component of G. As above, $(G^0)^{-1} = G^0$ since both components contain e. Finally, if $g_1 \in G^0$, then the components $g_1 G^0$ and G^0 both contain g_1 since $e, g_1^{-1} \in G^0$. Thus $g_1 G^0 = G^0$, and so G^0 is a subgroup, as desired. $\square$

Theorem 1.18. *If G is a Lie group and H a connected Lie subgroup so that G/H is also connected, then G is connected.*

Proof. Since H is connected and contains e, $H \subseteq G^0$, so there is a continuous map $\pi : G/H \to G/G^0$ defined by $\pi(gH) = gG^0$. It is trivial that G/G^0 has the discrete topology with respect to the quotient topology. The assumption that G/H is connected forces $\pi(G/H)$ to be connected, and so $\pi(G/H) = eG^0$. However, π is a surjective map so $G/G^0 = eG^0$, which means $G = G^0$. $\square$

Definition 1.19. Let be G a Lie group and M a manifold.
(1) An *action* of G on M is a smooth map from $G \times M \to M$, denoted by $(g, m) \to g \cdot m$ for $g \in G$ and $m \in M$, so that:

  (i) $e \cdot m = m$, all $m \in M$ and
  (ii) $g_1 \cdot (g_2 \cdot m) = (g_1 g_2) \cdot m$ for all $g_1, g_2 \in G$ and $m \in M$.

(2) The action is called *transitive* if for each $m, n \in M$, there is a $g \in G$, so $g \cdot m = n$.
(3) The *stabilizer* of $m \in M$ is $G^m = \{g \in G \mid g \cdot m = m\}$.

If G has a transitive action on M and $m_0 \in M$, then it is clear (Theorem 1.7) that the action of G on m_0 induces a diffeomorphism from G/G^{m_0} onto M.

Theorem 1.20. *The compact classical groups, $SO(n)$, $SU(n)$, and $Sp(n)$, are connected.*

Proof. Start with $SO(n)$ and proceed by induction on n. As $SO(1) = \{1\}$, the case $n = 1$ is trivial. Next, observe that $SO(n)$ has a transitive action on S^{n-1} in $\mathbb{R}^n$ by matrix multiplication. For $n \geq 2$, the stabilizer of the north pole, $N = (1, 0, \dots, 0)$, is easily seen to be isomorphic to $SO(n-1)$ which is connected by the induction hypothesis. From the transitive action, it follows that $SO(n)/SO(n)^N \cong S^{n-1}$ which is also connected. Thus Theorem 1.18 finishes the proof.

For $SU(n)$, repeat the above argument with $\mathbb{R}^n$ replaced by $\mathbb{C}^n$ and start the induction with the fact that $SU(1) \cong S^1$. For $Sp(n)$, repeat the same argument with $\mathbb{R}^n$ replaced by $\mathbb{H}^n$ and start the induction with $Sp(1) \cong \{v \in \mathbb{H} \mid |v| = 1\} \cong S^3$. □

1.2.2 Simply Connected Cover

For a connected Lie group G, recall that the *fundamental group*, $\pi_1(G)$, is the homotopy class of all loops at a fixed base point. The Lie group G is called *simply connected* if $\pi_1(G)$ is trivial.

Standard covering theory from topology and differential geometry (see [69] and [8] or [88] for more detail) says that there exists a unique (up to isomorphism) *simply connected cover* $\widetilde{G}$ of G, i.e., a connected, simply connected manifold $\widetilde{G}$ with a *covering* (or *projection*) map $\pi : \widetilde{G} \to G$. Recall that being a covering map means π is a smooth surjective map with the property that each $g \in G$ has a connected neighborhood U of g in G so that the restriction of π to each connected component of $\pi^{-1}(U)$ is a diffeomorphism onto U.

Lemma 1.21. *If H is a discrete normal subgroup of a connected Lie group G, then H is contained in the center of G.*

Proof. For each $h \in H$, consider $C_h = \{ghg^{-1} \mid g \in G\}$. Since C_h is the continuous image of the connected set G, C_h is connected. Normality of H implies $C_h \subseteq H$. Discreteness of H and connectedness of C_h imply that C_h is a single point. As h is clearly in C_h, this shows that $C_h = \{h\}$, and so h is central. □

Theorem 1.22. *Let G be a connected Lie group.*
(1) The connected simply connected cover $\widetilde{G}$ is a Lie group.
(2) If π is the covering map and $\widetilde{Z} = \ker \pi$, then $\widetilde{Z}$ is a discrete central subgroup of $\widetilde{G}$.
(3) π induces a diffeomorphic isomorphism $G \cong \widetilde{G}/\widetilde{Z}$.
(4) $\pi_1(G) \cong \widetilde{Z}$.

Proof. Because coverings satisfy the *lifting property* (e.g., for any smooth map f of a connected simply connected manifold M to G with $m_0 \in M$ and $g_0 \in \pi^{-1}(f(m_0))$, there exists a unique smooth map $\widetilde{f} : M \to \widetilde{G}$ satisfying $\pi \circ \widetilde{f} = f$ and $\widetilde{f}(m_0) = g_0$), the Lie group structure on G lifts to a Lie group structure on $\widetilde{G}$, making π a homomorphism. To see this, consider the map $s : \widetilde{G} \times \widetilde{G} \to G$ by $f(\widetilde{g}, \widetilde{h}) =$

$\pi(\widetilde{g})\pi(\widetilde{h})^{-1}$ and fix some $\widetilde{e} \in \pi^{-1}(e)$. Then there is a unique lift $\widetilde{s} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ so that $\pi \circ \widetilde{s} = s$. To define the group structure $\widetilde{G}$, let $\widetilde{h}^{-1} = \widetilde{s}(\widetilde{e}, \widetilde{h})$ and $\widetilde{g}\widetilde{h} = \widetilde{s}(\widetilde{g}, \widetilde{h}^{-1})$. It is straightforward to verify that this structure makes $\widetilde{G}$ into a Lie group and π into a homomorphism (Exercise 1.21).

Hence we have constructed a connected simply connected Lie group $\widetilde{G}$ and a covering homomorphism $\pi : \widetilde{G} \to G$. Since π is a covering and a homomorphism, $\widetilde{Z} = \ker \pi$ is a discrete normal subgroup of $\widetilde{G}$ and so central by Lemma 1.21. Hence π induces a diffeomorphic isomorphism from $\widetilde{G}/\widetilde{Z}$ to G. The statement regarding $\pi_1(G)$ is a standard result from the covering theory of deck transformations (see [8]). □

Lemma 1.23. *$Sp(1)$ and $SU(2)$ are simply connected and isomorphic to each other. Either group is the simply connected cover of $SO(3)$, i.e., $SO(3)$ is isomorphic to $Sp(1)/\{\pm 1\}$ or $SU(2)/\{\pm I\}$.*

Proof. The isomorphism from $Sp(1)$ to $SU(2)$ is given by $\widetilde{\vartheta}$ in §1.1.4.3. Since either group is topologically S^3, the first statement follows.

For the second statement, write $(\cdot, \cdot)$ for the real inner product on $\mathbb{H}$ given by $(u, v) = \mathrm{Re}(u\overline{v})$ for $u, v \in \mathbb{H}$. By choosing an orthonormal basis $\{1, i, j, k\}$, we may identify $\mathbb{H}$ with $\mathbb{R}^4$ and $(\cdot, \cdot)$ with the standard Euclidean dot product on $\mathbb{R}^4$. Then $1^\perp = \{v \in \mathbb{H} \mid (1, v) = 0\}$ is the set of *imaginary* (or *pure*) *quaternions*, $\mathrm{Im}(\mathbb{H})$, spanned over $\mathbb{R}$ by $\{i, j, k\}$. In particular, we may identify $O(3)$ with $O(\mathrm{Im}(\mathbb{H})) \equiv \{\mathbb{R}\text{-linear maps } T : \mathrm{Im}(\mathbb{H}) \to \mathrm{Im}(\mathbb{H}) \mid (Tu, Tv) = (u, v) \text{ all } u, v \in \mathrm{Im}(\mathbb{H})\}$ and the connected component $O(\mathrm{Im}(\mathbb{H}))^0$ with $SO(3)$.

Define a smooth homomorphism $\mathrm{Ad} : Sp(1) \to O(\mathrm{Im}(\mathbb{H}))^0$ by $(\mathrm{Ad}(g))(u) = gu\overline{g}$ for $g \in Sp(1)$ and $u \in \mathrm{Im}(\mathbb{H})$. To see this is well defined, first view $\mathrm{Ad}(g)$ as an $\mathbb{R}$-linear transformation on $\mathbb{H}$. Using the fact that $g\overline{g} = 1$ for $g \in Sp(1)$, it follows immediately that $\mathrm{Ad}(g)$ leaves $(\cdot, \cdot)$ invariant. As $\mathrm{Ad}(g)$ fixes 1, $\mathrm{Ad}(g)$ preserves $\mathrm{Im}(\mathbb{H})$. Thus $\mathrm{Ad}(g) \in O(\mathrm{Im}(\mathbb{H}))^0$ since $Sp(1)$ is connected.

It is well known that $SO(3)$ consists of all rotations (Exercise 1.22). To show Ad is surjective, it therefore suffices to show that each rotation lies in the image of Ad. Let $v \in \mathrm{Im}(\mathbb{H})$ be a unit vector. Then v can be completed to a basis $\{v, u, w\}$ of $\mathrm{Im}(\mathbb{H})$ sharing the same properties as the $\{i, j, k\}$ basis. It is a simple calculation to show that $\mathrm{Ad}(\cos\theta + v \sin\theta)$ fixes v and is a rotation through an angel of 2θ in the uw plane (Exercise 1.23). Hence Ad is surjective. The same calculation also shows that $\ker \mathrm{Ad} = \{\pm 1\}$. Since the simply connected cover is unique, the proof is finished. □

In §6.3.3 we develop a direct method for calculating $\pi_1(G)$. For now we compute the fundamental group for the classical compact Lie groups by use of a higher homotopy exact sequence.

Theorem 1.24. *(1) $\pi_1(SO(2)) \cong \mathbb{Z}$ and $\pi_1(SO(n)) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$.*
(2) $SU(n)$ is simply connected for $n \geq 2$.
(3) $Sp(n)$ is simply connected for $n \geq 1$.

Proof. Start with $SO(n)$. As $SO(2) \cong S^1$, $\pi_1(SO(2)) \cong \mathbb{Z}$. Recall from the proof of Theorem 1.20 that $SO(n)$ has a transitive action on S^{n-1} with stabilizer isomorphic to $SO(n-1)$. From the resulting exact sequence, $\{1\} \to SO(n-1) \to SO(n) \to S^{n-1} \to \{1\}$, there is a long exact sequence of higher homotopy groups (e.g., see [51] p. 296)

$$\cdots \to \pi_2(S^{n-1}) \to \pi_1(SO(n-1)) \to \pi_1(SO(n)) \to \pi_1(S^{n-1}) \to \cdots.$$

For $n \geq 3$, $\pi_1(S^{n-1})$ is trivial, so there is an exact sequence

$$\pi_2(S^{n-1}) \to \pi_1(SO(n-1)) \to \pi_1(SO(n)) \to \{1\}.$$

Since $\pi_2(S^{n-1})$ is trivial for $n \geq 4$, induction on the exact sequence implies $\pi_1(SO(n)) \cong \pi_1(SO(3))$ for $n \geq 4$. It only remains to show that $\pi_1(SO(3)) \cong \mathbb{Z}/2\mathbb{Z}$, but this follows from Lemma 1.23 and Theorem 1.22.

For $SU(n)$, as in the proof of Theorem 1.20, there is an exact sequence $\{1\} \to SU(n-1) \to SU(n) \to S^{2n-1} \to \{1\}$. Since $\pi_1(S^{2n-1})$ and $\pi_2(S^{2n-1})$ are trivial for $n \geq 3$ (actually for $n = 2$ as well, though not useful here), the long exact sequence of higher homotopy groups implies that $\pi_1(SU(n)) \cong \pi_1(SU(2))$ for $n \geq 2$. By Lemma 1.23, $\pi_1(SU(2))$ is trivial.

For $Sp(n)$, the corresponding exact sequence is $\{1\} \to Sp(n-1) \to Sp(n) \to S^{4n-1} \to \{1\}$. Since $\pi_1(S^{4n-1})$ and $\pi_2(S^{4n-1})$ are trivial for $n \geq 2$ (actually for $n = 1$ as well), the resulting long exact sequence implies $\pi_1(Sp(n)) \cong \pi_1(Sp(1))$ for $n \geq 1$. By Lemma 1.23, $\pi_1(Sp(1))$ is trivial. $\square$

As an immediate corollary of Theorems 1.22 and 1.24, there is a connected simply connected double cover of $SO(n)$, $n \geq 3$. That simply connected Lie group is called $\mathrm{Spin}_n(\mathbb{R})$ and it fits in the following exact sequence:

$$\{1\} \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}_n(\mathbb{R}) \to SO(n) \to \{I\}. \tag{1.25}$$

Lemma 1.23 shows $\mathrm{Spin}_3(\mathbb{R}) \cong SU(2) \cong Sp(1)$. For larger n, an explicit construction of $\mathrm{Spin}_n(\mathbb{R})$ is given in §1.3.2.

1.2.3 Exercises

Exercise 1.14 For a connected Lie group G, show that even if the second countable hypothesis is omitted from the definition of manifold, G is still second countable.

Exercise 1.15 Show that an open subgroup of a Lie group is closed.

Exercise 1.16 Show that $GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$ are connected.

Exercise 1.17 Show that $GL(n, \mathbb{R})$ has two connected components: $GL(n, \mathbb{R})^0 = \{g \in GL(n, \mathbb{R}) \mid \det g > 0\}$ and $\{g \in GL(n, \mathbb{R}) \mid \det g < 0\}$. Prove $SL(n, \mathbb{R})$ is connected.

Exercise 1.18 Show $O(2n+1) \cong SO(2n+1) \times (\mathbb{Z}/2\mathbb{Z})$ as both a manifold and a group. In particular, $O(2n+1)$ has two connected components with $O(2n+1)^0 = SO(2n+1)$.

Exercise 1.19 **(a)** Show $O(2n) \cong SO(2n) \times (\mathbb{Z}/2\mathbb{Z})$ as a manifold. In particular, $O(2n)$ has two connected components with $O(2n)^0 = SO(2n)$.
(b) Show that $O(2n)$ is *not* isomorphic to $SO(2n) \times (\mathbb{Z}/2\mathbb{Z})$ as a group. Instead show that $O(2n)$ is isomorphic to a semidirect product $SO(2n) \rtimes (\mathbb{Z}/2\mathbb{Z})$. Describe explicitly the multiplication structure on $SO(2n) \rtimes (\mathbb{Z}/2\mathbb{Z})$ under its isomorphism with $O(2n)$.

Exercise 1.20 Show $U(n) \cong (SU(n) \times S^1)/(\mathbb{Z}/n\mathbb{Z})$ as both a manifold and a group. In particular, $U(n)$ is connected.

Exercise 1.21 Check the details in the proof of Theorem 1.22 to carefully show that the Lie group structure on G lifts to a Lie group structure on $\widetilde{G}$, making the covering map $\pi : \widetilde{G} \to G$ a homomorphism.

Exercise 1.22 Let $\mathcal{R}_3 \subseteq GL(3, \mathbb{R})$ be the set of rotations in $\mathbb{R}^3$ about the origin. Show that $\mathcal{R}_3 = SO(3)$.

Exercise 1.23 **(a)** Let $v \in \mathrm{Im}(\mathbb{H})$ be a unit vector. Show that v can be completed to a basis $\{v, u, w\}$ of $\mathrm{Im}(\mathbb{H})$, sharing the same properties as the $\{i, j, k\}$ basis.
(b) Show $\mathrm{Ad}(\cos\theta + v\sin\theta)$ from the proof of Lemma 1.23 fixes v and acts by a rotation through an angle 2θ on the $\mathbb{R}$-span of $\{u, w\}$.

Exercise 1.24 Let $\mathfrak{su}(2) = \left\{ \begin{pmatrix} ix & -\overline{b} \\ b & -ix \end{pmatrix} \mid b \in \mathbb{C},\ x \in \mathbb{R} \right\}$ and $(X, Y) = \frac{1}{2}\operatorname{tr}(XY^*)$ for $X, Y \in \mathfrak{su}(2)$. Define $(\mathrm{Ad}\, g)X = gXg^{-1}$ for $g \in SU(2)$ and $X \in \mathfrak{su}(2)$. Modify the proof of Lemma 1.23 to directly show that the map $\mathrm{Ad} : SU(2) \to SO(3)$ is well defined and realizes the simply connected cover of $SO(3)$ as $SU(2)$.

1.3 The Double Cover of $SO(n)$

At the end of §1.2.2 we saw that $SO(n)$, $n \geq 3$, has a simply connected double cover called $\mathrm{Spin}_n(\mathbb{R})$. The proof of Lemma 1.23 gave an explicit construction of $\mathrm{Spin}_3(\mathbb{R})$ as $Sp(1)$ or $SU(2)$. The key idea was to first view $SO(3)$ as the set of rotations in $\mathbb{R}^3$ and then use the structure of the quaternion algebra, $\mathbb{H}$, along with a conjugation action to realize each rotation uniquely up to a $\pm$-sign.

This section gives a general construction of $\mathrm{Spin}_n(\mathbb{R})$. The algebra that takes the place of $\mathbb{H}$ is called the Clifford algebra, $\mathcal{C}_n(\mathbb{R})$, and instead of simply constructing rotations, it is more advantageous to use a conjugation action that constructs all reflections.

1.3.1 Clifford Algebras

Alhough the entire theory of Clifford algebras easily generalizes (Exercise 1.30), it is sufficient for our purposes here to work over $\mathbb{R}^n$ equipped with the standard Euclidean dot product $(\cdot,\cdot)$. Recall that the *tensor algebra* over $\mathbb{R}^n$ is $\mathcal{T}_n(\mathbb{R}) = \bigoplus_{k=0}^{\infty} \mathbb{R}^n \otimes \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$ (k copies) with a basis $\{1\} \cup \{x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k} \mid 1 \leq i_k \leq n\}$, where $\{x_1, x_2, \ldots, x_n\}$ is a basis of $\mathbb{R}^n$.

Definition 1.26. The *Clifford algebra* is

$$\mathcal{C}_n(\mathbb{R}) = \mathcal{T}_n(\mathbb{R})/\mathcal{I}$$

where $\mathcal{I}$ is the ideal of $\mathcal{T}_n(\mathbb{R})$ generated by

$$\{(x \otimes x + |x|^2) \mid x \in \mathbb{R}^n\}.$$

By way of notation for Clifford multiplication, write

$$x_1 x_2 \cdots x_k$$

for the element $x_1 \otimes x_2 \otimes \cdots \otimes x_k + \mathcal{I} \in \mathcal{C}_n(\mathbb{R})$, where $x_1, x_2, \ldots, x_n \in \mathbb{R}^n$.

In particular,

$$x^2 = -|x|^2 \tag{1.27}$$

in $\mathcal{C}_n(\mathbb{R})$ for $x \in \mathbb{R}^n$. Starting with the equality $xy + yx = (x+y)^2 - x^2 - y^2$ for $x, y \in \mathbb{R}^n$, it follows that Equation 1.27 is equivalent to

$$xy + yx = -2(x, y) \tag{1.28}$$

in $\mathcal{C}_n(\mathbb{R})$ for $x, y \in \mathbb{R}^n$.

It is a straightforward exercise (Exercise 1.25) to show that

$$\mathcal{C}_0(\mathbb{R}) \cong \mathbb{R},\ \mathcal{C}_1(\mathbb{R}) \cong \mathbb{C},\ \text{and}\ \mathcal{C}_2(\mathbb{R}) \cong \mathbb{H}.$$

More generally, define the *standard basis* for $\mathbb{R}^n$ to be $\{e_1, e_2, \ldots, e_n\}$, where $e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 appearing in the k^{th} entry. Clearly $\{1\} \cup \{e_{i_1} e_{i_2} \cdots e_{i_k} \mid k > 0, 1 \leq i_k \leq n\}$ spans $\mathcal{C}_n(\mathbb{R})$, but this is overkill. First, observe that $\mathcal{C}_n(\mathbb{R})$ inherits a filtration from $\mathcal{T}_n(\mathbb{R})$ by degree. Up to lower degree terms, Equation 1.28 can be used to commute adjacent e_{i_j} and Equation 1.27 can be used to remove multiple copies of e_{i_j} within a product $e_{i_1} e_{i_2} \cdots e_{i_k}$. An inductive argument on filtration degree therefore shows that

$$\{1\} \cup \{e_{i_1} e_{i_2} \cdots e_{i_k} \mid 1 \leq i_1 < i_2 < \ldots i_k \leq n\} \tag{1.29}$$

spans $\mathcal{C}_n(\mathbb{R})$, so $\dim \mathcal{C}_n(\mathbb{R}) \leq 2^n$. In fact, we will shortly see Equation 1.29 provides a *basis* for $\mathcal{C}_n(\mathbb{R})$ and so $\dim \mathcal{C}_n(\mathbb{R}) = 2^n$. This will be done by constructing a *linear* isomorphism $\Psi : \mathcal{C}_n(\mathbb{R}) \to \bigwedge \mathbb{R}^n$, where $\bigwedge \mathbb{R}^n = \bigoplus_{k=0}^{n} \bigwedge^k \mathbb{R}^n$ is the *exterior algebra* of $\mathbb{R}^n$.

To begin, we recall some multilinear algebra.

Definition 1.30. **(1)** For $x \in \mathbb{R}^n$, let *exterior multiplication* be the map $\epsilon(x) : \bigwedge^k \mathbb{R}^n \to \bigwedge^{k+1} \mathbb{R}^n$ given by

$$(\epsilon(x))(y) = x \wedge y$$

for $y \in \bigwedge^k \mathbb{R}^n$.
(2) For $x \in \mathbb{R}^n$, let *interior multiplication* be the map $\iota(x) : \bigwedge^k \mathbb{R}^n \to \bigwedge^{k-1} \mathbb{R}^n$ given by

$$(\iota(x))(y_1 \wedge y_2 \wedge \cdots \wedge y_k) = \sum\nolimits_{i=1}^{k} (-1)^{i+1} (x, y_i)\, y_1 \wedge y_2 \wedge \cdots \wedge \widehat{y_i} \wedge \cdots \wedge y_k$$

for $y_i \in \mathbb{R}^n$, where $\widehat{y_i}$ means to omit the term.

It is straightforward (Exercise 1.26) from multilinear algebra that $\iota(x)$ is the adjoint of $\epsilon(x)$ with respect to the natural form on $\bigwedge \mathbb{R}^n$. In particular, $\epsilon(x)^2 = \iota(x)^2 = 0$ for $x \in \mathbb{R}^n$. It is also straightforward (Exercise 1.26) that

$$\epsilon(x)\iota(x) + \iota(x)\epsilon(x) = m_{|x|^2}, \tag{1.31}$$

where $m_{|x|^2}$ is the operator that multiplies by $|x|^2$.

Definition 1.32. **(1)** For $x \in \mathbb{R}^n$, let $L_x : \bigwedge \mathbb{R}^n \to \bigwedge \mathbb{R}^n$ be given by $L_x = \epsilon(x) - \iota(x)$.
(2) Let $\Phi : \mathcal{T}_n(\mathbb{R}) \to \operatorname{End}(\bigwedge \mathbb{R}^n)$ be the natural map of algebras determined by setting $\Phi(x) = L_x$ for $x \in \mathbb{R}^n$.

Observe that Equation 1.31 implies that $L_x^2 + m_{|x|^2} = 0$ so that $\Phi(\mathcal{I}) = 0$. In particular, Φ descends to $\mathcal{C}_n(\mathbb{R})$.

Definition 1.33. **(1)** Abusing notation, let $\Phi : \mathcal{C}_n(\mathbb{R}) \to \operatorname{End}(\bigwedge \mathbb{R}^n)$ be the map induced on $\mathcal{C}_n(\mathbb{R})$ by the original map $\Phi : \mathcal{T}_n(\mathbb{R}) \to \operatorname{End}(\bigwedge \mathbb{R}^n)$.
(2) Let $\Psi : \mathcal{C}_n(\mathbb{R}) \to \bigwedge \mathbb{R}^n$ by $\Psi(v) = (\Phi(v))(1)$.

Explicitly, for $x_i \in \mathbb{R}^n$, $\Psi(x_1) = (\epsilon(x_1) - \iota(x_1))\, 1 = x_1$, and

$$\begin{aligned}\Psi(x_1 x_2) &= (\epsilon(x_1) - \iota(x_1))\,(\epsilon(x_2) - \iota(x_2))\, 1 \\ &= (\epsilon(x_1) - \iota(x_1))\, x_2 = x_1 \wedge x_2 - (x_1, x_2).\end{aligned}$$

In general,

$$\Psi(x_1 x_2 \cdots x_k) = x_1 \wedge x_2 \wedge \cdots \wedge x_k + \text{ terms in } \bigoplus\nolimits_{i \geq 1} \bigwedge^{k-2i} \mathbb{R}^n. \tag{1.34}$$

Equation 1.34 is easily established (Exercise 1.27) by induction on k. Also by induction on degree, it is an immediate corollary of Equation 1.34 that Ψ is surjective. A dimension count therefore shows that Ψ is a linear isomorphism. In summary:

Theorem 1.35. *The map $\Psi : \mathcal{C}_n(\mathbb{R}) \to \bigwedge \mathbb{R}^n$ is a linear isomorphism of vector spaces, and so* $\dim \mathcal{C}_n(\mathbb{R}) = 2^n$ *and Equation 1.29 provides a basis for* $\mathcal{C}_n(\mathbb{R})$.

Thus, with respect to the standard basis (or any orthonormal basis for that matter), $\mathcal{C}_n(\mathbb{R})$ has a particularly simple algebra structure. Namely, $\mathcal{C}_n(\mathbb{R})$ is the $\mathbb{R}$-span of the basis $\{1\} \cup \{e_{i_1} e_{i_2} \cdots e_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$ with the algebraic relations generated by $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ when $i \neq j$.

1.3.2 $\mathrm{Spin}_n(\mathbb{R})$ and $\mathrm{Pin}_n(\mathbb{R})$

For the next definition, observe that $\mathcal{T}_n(\mathbb{R})$ breaks into a direct sum of the subalgebra generated by the tensor product of any even number of elements of $\mathbb{R}^n$ and the subspace generated the tensor product of any odd number of elements of $\mathbb{R}^n$. Since $\mathcal{I}$ is generated by elements of even degree, it follows that this decomposition descends to $\mathcal{C}_n(\mathbb{R})$.

Definition 1.36. (1) Let $\mathcal{C}_n^+(\mathbb{R})$ be the subalgebra of $\mathcal{C}_n(\mathbb{R})$ spanned by all products of an *even* number of elements of $\mathbb{R}^n$.
(2) Let $\mathcal{C}_n^-(\mathbb{R})$ be the subspace of $\mathcal{C}_n(\mathbb{R})$ spanned by all products of an *odd* number of elements of $\mathbb{R}^n$ so $\mathcal{C}_n(\mathbb{R}) = \mathcal{C}_n^+(\mathbb{R}) \oplus \mathcal{C}_n^-(\mathbb{R})$ as a vector space.
(3) Let the automorphism α, called the *main involution*, of $\mathcal{C}_n(\mathbb{R})$ act as multiplication by ± 1 on $\mathcal{C}_n^\pm(\mathbb{R})$.
(4) *Conjugation*, an anti-involution on $\mathcal{C}_n(\mathbb{R})$, is defined by

$$(x_1 x_2 \cdots x_k)^* = (-1)^k \, x_k \cdots x_2 x_1$$

for $x_i \in \mathbb{R}^n$.

The next definition makes sense for $n \geq 1$. However, because of Equation 1.25, we are really only interested in the case of $n \geq 3$ (see Exercise 1.34 for details when $n = 1, 2$).

Definition 1.37. (1) Let $\mathrm{Spin}_n(\mathbb{R}) = \{g \in \mathcal{C}_n^+(\mathbb{R}) \mid gg^* = 1 \text{ and } gxg^* \in \mathbb{R}^n \text{ for all } x \in \mathbb{R}^n\}$.
(2) Let $\mathrm{Pin}_n(\mathbb{R}) = \{g \in \mathcal{C}_n(\mathbb{R}) \mid gg^* = 1 \text{ and } \alpha(g)xg^* \in \mathbb{R}^n \text{ for all } x \in \mathbb{R}^n\}$. Note $\mathrm{Spin}_n(\mathbb{R}) \subseteq \mathrm{Pin}_n(\mathbb{R})$.
(3) For $g \in \mathrm{Pin}_n(\mathbb{R})$ and $x \in \mathbb{R}^n$, define the homomorphism $\mathcal{A} : \mathrm{Pin}_n(\mathbb{R}) \to GL(n, \mathbb{R})$ by $(\mathcal{A}g)x = \alpha(g)xg^*$. Note $(\mathcal{A}g)\, x = gxg^*$ when $g \in \mathrm{Spin}_n(\mathbb{R})$.

Viewing left multiplication by $v \in \mathcal{C}_n(\mathbb{R})$ as an element of $\mathrm{End}(\mathcal{C}_n(\mathbb{R}))$, use of the determinant shows that the set of invertible elements of $\mathcal{C}_n(\mathbb{R})$ is an open subgroup of $\mathcal{C}_n(\mathbb{R})$. It follows fairly easily that the set of invertible elements is a Lie group. As both $\mathrm{Spin}_n(\mathbb{R})$ and $\mathrm{Pin}_n(\mathbb{R})$ are closed subgroups of this Lie group, Corollary 1.8 implies that $\mathrm{Spin}_n(\mathbb{R})$ and $\mathrm{Pin}_n(\mathbb{R})$ are Lie groups as well.

Lemma 1.38. *$\mathcal{A}$ is a covering map of $\mathrm{Pin}_n(\mathbb{R})$ onto $O(n)$ with $\ker \mathcal{A} = \{\pm 1\}$, so there is an exact sequence*

$$\{1\} \to \{\pm 1\} \to \mathrm{Pin}_n(\mathbb{R}) \xrightarrow{\mathcal{A}} O(n) \to \{I\}.$$

Proof. $\mathcal{A}$ maps $\mathrm{Pin}_n(\mathbb{R})$ into $O(n)$: Let $g \in \mathrm{Pin}_n(\mathbb{R})$ and $x \in \mathbb{R}^n$. Using Equation 1.27 and the fact that conjugation on $\mathbb{R}^n$ is multiplication by -1, we calculate

$$\begin{aligned}
|(\mathcal{A}g)x|^2 &= -\left(\alpha(g)xg^*\right)^2 = -\left(\alpha(g)xg^*\right)\left(\alpha(g)xg^*\right) = \alpha(g)xg^*\left(\alpha(g)xg^*\right)^* \\
&= \alpha(g)xg^*gx^*\alpha(g)^* = \alpha(g)xx^*\alpha(g)^* = -\alpha(g)x^2\alpha(g)^* = |x|^2\, \alpha(gg^*) \\
&= |x|^2 .
\end{aligned}$$

Thus $\mathcal{A}g \in O(n)$.

$\mathcal{A}$ maps $\mathrm{Pin}_n(\mathbb{R})$ *onto* $O(n)$: It is well known (Exercise 1.32) that each orthogonal matrix is a product of reflections. Thus it suffices to show that each reflection lies in the image of $\mathcal{A}$. Let $x \in S^{n-1}$ be any unit vector in $\mathbb{R}^n$ and write r_x for the reflection across the plane perpendicular to x. Observe $xx^* = -x^2 = |x|^2 = 1$. Thus $\alpha(x)xx^* = -xxx^* = -x$. If $y \in \mathbb{R}^n$ and $(x, y) = 0$, then Equation 1.28 says $xy = -yx$ so that $\alpha(x)yx^* = xyx = -x^2y = y$. Hence $x \in \mathrm{Pin}_n(\mathbb{R})$ and $\mathcal{A}x = r_x$.

$\ker \mathcal{A} = \{\pm 1\}$: Since $\mathbb{R} \cap \mathrm{Pin}_n(\mathbb{R}) = \{\pm 1\}$ and both elements are clearly in $\ker \mathcal{A}$, it suffices to show that $\ker \mathcal{A} \subseteq \mathbb{R}$. So suppose $g \in \mathrm{Pin}_n(\mathbb{R})$ with $\mathcal{A}g = I$. As $g^* = g^{-1}$, $\alpha(g)x = xg$ for all $x \in \mathbb{R}^n$. Expanding g with respect to the standard basis from Equation 1.29, we may uniquely write $g = e_1a + b$, where a, b are linear combinations of 1 and monomials in $e_2, e_3, \ldots, e_n$. Looking at the special case of $x = e_1$, we have $\alpha(e_1a+b)e_1 = e_1\,(e_1a + b)$ so that $-e_1\alpha(a)e_1+\alpha(b)e_1 = -a+e_1b$. Since a and b contain no e_1's, $\alpha(a)e_1 = e_1a$ and $\alpha(b)e_1 = e_1b$. Thus $a + e_1b_1 = -a + e_1b$ which implies that $a = 0$ so that g contains no e_1. Induction similarly shows that g contains no e_k, $1 \leq k \leq n$, and so $g \in \mathbb{R}$.

$\mathcal{A}$ is a covering map: From Theorem 1.10, π has constant rank with $N = \mathrm{rk}\,\pi = \dim \mathrm{Pin}_n(\mathbb{R})$ since $\ker \pi = \{\pm 1\}$. For any $g \in \mathrm{Pin}_n(\mathbb{R})$, the Rank Theorem from differential geometry ([8]) says there exists cubical charts (U, φ) of g and (V, ψ) of $\pi(g)$ so that $\psi \circ \pi \circ \varphi^{-1}(x_1, \ldots, x_N) = (x_1, \ldots, x_N, 0, \ldots, 0)$ with $\dim O(n) - N$ zeros. Using the second countability of $\mathrm{Pin}_n(\mathbb{R})$ and the Baire category theorem, surjectivity of π implies $\dim O(n) = N$. In particular, π restricted to U is a diffeomorphism onto V. Since $\ker \pi = \{\pm 1\}$, π is also a diffeomorphism of $-U$ onto V. Finally, injectivity of π on U implies that $(-U) \cap U = \emptyset$ so that the connected components of $\pi^{-1}(V)$ are U and $-U$. □

Lemma 1.39. $\mathrm{Pin}_n(\mathbb{R})$ *and* $\mathrm{Spin}_n(\mathbb{R})$ *are compact Lie groups with*

$$\mathrm{Pin}_n(\mathbb{R}) = \{x_1 \cdots x_k \mid x_i \in S^{n-1} \textit{ for } 1 \leq k \leq 2n\}$$
$$\mathrm{Spin}_n(\mathbb{R}) = \{x_1x_2 \cdots x_{2k} \mid x_i \in S^{n-1} \textit{ for } 2 \leq 2k \leq 2n\}$$

and $\mathrm{Spin}_n(\mathbb{R}) = \mathcal{A}^{-1}(SO(n))$.

Proof. We know from the proof of Lemma 1.38 that $\mathcal{A}x = r_x$ for each $x \in S^{n-1} \subseteq \mathrm{Pin}_n(\mathbb{R})$. Since elements of $O(n)$ are products of at most $2n$ reflections and $\mathcal{A}$ is surjective with kernel $\{\pm 1\}$, this implies that $\mathrm{Pin}_n(\mathbb{R}) = \{x_1 \cdots x_k \mid x_i \in S^{n-1}$ for $1 \leq k \leq 2n\}$. The equality $\mathrm{Spin}_n(\mathbb{R}) = \mathrm{Pin}_n(\mathbb{R}) \cap \mathcal{C}_n^+(\mathbb{R})$ then implies $\mathrm{Spin}_n(\mathbb{R}) = \{x_1x_2 \cdots x_{2k} \mid x_i \in S^{n-1}$ for $2 \leq 2k \leq 2n\}$. In particular, $\mathrm{Pin}_n(\mathbb{R})$ and $\mathrm{Spin}_n(\mathbb{R})$ are compact. Moreover because $\det r_x = -1$, the last equality is equivalent to the equality $\mathrm{Spin}_n(\mathbb{R}) = \mathcal{A}^{-1}(SO(n))$. □

Theorem 1.40. *(1)* $\mathrm{Pin}_n(\mathbb{R})$ *has two connected $(n \geq 2)$ components with* $\mathrm{Spin}_n(\mathbb{R}) = \mathrm{Pin}_n(\mathbb{R})^0$.
(2) $\mathrm{Spin}_n(\mathbb{R})$ *is the connected $(n \geq 2)$ simply connected $(n \geq 3)$ two-fold cover of $SO(n)$. The covering homomorphism is given by $\mathcal{A}$ with* $\ker \mathcal{A} = \{\pm 1\}$*, i.e., there is an exact sequence*

$$\{1\} \to \{\pm 1\} \to \mathrm{Spin}_n(\mathbb{R}) \xrightarrow{\mathcal{A}} SO(n) \to \{I\}.$$

Proof. For $n \geq 2$, consider the path $t \to \gamma(t) = \cos t + e_1 e_2 \sin t$. Since $\gamma(t) = e_1(-e_1 \cos t + e_2 \sin t)$, it follows that $\gamma(t) \in \mathrm{Spin}_n(\mathbb{R})$ and so $\{\pm 1\}$ are path connected in $\mathrm{Spin}_n(\mathbb{R})$. From Lemmas 1.38 and 1.39, we know that $\mathrm{Spin}_n(\mathbb{R})$ is a double cover of $SO(n)$ and so $\mathrm{Spin}_n(\mathbb{R})$ is connected. Thus, for $n \geq 3$, Theorem 1.24 and the uniqueness of connected simply connected coverings implies $\mathrm{Spin}_n(\mathbb{R})$ is the connected simply connected cover of $SO(n)$.

Finally, let $x_0 \in S^{n-1}$. Clearly $\mathrm{Pin}_n(\mathbb{R}) = x_0 \, \mathrm{Spin}_n(\mathbb{R}) \amalg \mathrm{Spin}_n(\mathbb{R})$. We know that $\mathcal{A}$ is a continuous map of $\mathrm{Pin}_n(\mathbb{R})$ onto $O(n)$. Since $O(n)$ is not connected but $x_0 \, \mathrm{Spin}_n(\mathbb{R})$ and $\mathrm{Spin}_n(\mathbb{R})$ are connected, $x_0 \, \mathrm{Spin}_n(\mathbb{R}) \amalg \mathrm{Spin}_n(\mathbb{R})$ cannot be connected. Thus $x_0 \, \mathrm{Spin}_n(\mathbb{R})$ and $\mathrm{Spin}_n(\mathbb{R})$ are the connected components of $\mathrm{Pin}_n(\mathbb{R})$. □

1.3.3 Exercises

Exercise 1.25 Show $\mathcal{C}_0(\mathbb{R}) \cong \mathbb{R}$, $\mathcal{C}_1(\mathbb{R}) \cong \mathbb{C}$, and $\mathcal{C}_2(\mathbb{R}) \cong \mathbb{H}$.

Exercise 1.26 **(a)** Show $\iota(x) = \epsilon(x)^*$ with respect to the inner product on $\bigwedge \mathbb{R}^n$ induced by defining $(x_1 \wedge x_2 \wedge \cdots \wedge x_k,\ y_1 \wedge y_2 \wedge \cdots \wedge y_l)$ to be 0 when $k \neq l$ and to be $\det(x_i, y_j)$ when $k = l$.
(b) Show $\epsilon(x)\iota(x) + \iota(x)\epsilon(x) = m_{|x|^2}$ for any $x \in \mathbb{R}^n$.

Exercise 1.27 **(a)** Prove Equation 1.34.
(b) Prove Theorem 1.35.

Exercise 1.28 For $u, v \in \mathcal{C}_n(\mathbb{R})$, show that $uv = 1$ if and only if $vu = 1$.

Exercise 1.29 For $n \geq 3$, show that the polynomial $x_1^2 + \cdots + x_n^2$ is irreducible over $\mathbb{C}$. However, show that $x_1^2 + \cdots + x_n^2$ is a product of linear factors over $\mathcal{C}_n(\mathbb{R})$.

Exercise 1.30 Let $(\cdot, \cdot)$ be *any* symmetric bilinear form on $\mathbb{R}^n$ or $\mathbb{C}^n$. Generalize the notion of Clifford algebra in Definition 1.26 by replacing $x \otimes x + |x|^2$ by $x \otimes x - (x, x)$ in the definition of $\mathcal{I}$. Prove the analogue of Theorem 1.35 still holds. If $(\cdot, \cdot)$ has signature p, q on $\mathbb{R}^n$, the resulting Clifford algebra is denoted $\mathcal{C}_{p,q}(\mathbb{R})$ (so $\mathcal{C}_n(\mathbb{R}) = \mathcal{C}_{0,n}(\mathbb{R})$) and if $(\cdot, \cdot)$ is the negative dot product on $\mathbb{C}^n$, the resulting Clifford algebra is denoted by $\mathcal{C}_n(\mathbb{C})$.

Exercise 1.31 Show that there is an algebra isomorphism $\mathcal{C}_{n-1}(\mathbb{R}) \cong \mathcal{C}_n^+(\mathbb{R})$ induced by mapping $a + b$, $a \in \mathcal{C}_{n-1}^+(\mathbb{R})$ and $b \in \mathcal{C}_{n-1}^-(\mathbb{R})$ to $a + be_n$. Conclude that $\dim \mathcal{C}_n^+(\mathbb{R}) = 2^{n-1}$.

Exercise 1.32 Use induction on n to show that any $g \in O(n)$ may be written as a product of at most $2n$ reflections. *Hint:* If $g \in O(n)$ and $ge_1 \neq e_1$, show that there is a reflection r_1 so that $r_1 g e_1 = e_1$. Now use orthogonality and induction.

Exercise 1.33 Show that $\mathcal{A}(\cos t + e_1 e_2 \sin t) = \begin{pmatrix} \cos 2t & -\sin 2t & 0 \\ \sin 2t & \cos 2t & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$.

Exercise 1.34 **(a)** Under the isomorphism $\mathcal{C}_1(\mathbb{R}) \cong \mathbb{C}$ induced by $e_1 \to i$, show that $\mathrm{Pin}(1) = \{\pm 1, \pm i\}$ and $\mathrm{Spin}(1) = \{\pm 1\}$ with $\mathcal{A}(\pm 1) = I$ and $\mathcal{A}(\pm i) = -I$ on $\mathbb{R}i$.
(b) Under the isomorphism $\mathcal{C}_2(\mathbb{R}) \cong \mathbb{H}$ induced by $e_1 \to i$, $e_2 \to j$, and $e_1 e_2 \to k$, show that $\mathrm{Pin}(2) = \{\cos\theta + k\sin\theta, i\sin\theta + j\cos\theta\}$ and $\mathrm{Spin}(2) = \{\cos\theta + k\sin\theta\}$ with $\mathcal{A}(\cos\theta + k\cos\theta)$ acting as rotation by 2θ in the ij-plane.

Exercise 1.35 **(a)** For n odd, show that the center of $\mathrm{Spin}_n(\mathbb{R})$ is $\{\pm 1\}$.
(b) For n even, show that the center of $\mathrm{Spin}_n(\mathbb{R})$ is $\{\pm 1, \pm e_1 e_2 \cdots e_n\}$.

Exercise 1.36 **(a)** Replace $\mathbb{R}$ by $\mathbb{C}$ in Definitions 1.36 and 1.37 to define $\mathrm{Spin}_n(\mathbb{C})$ (c.f. Exercise 1.30). Modify the proof of Theorem 1.40 to show $\mathcal{A}$ realizes $\mathrm{Spin}_n(\mathbb{C})$ as a connected double cover of $SO(n,\mathbb{C}) = \{g \in SL(n,\mathbb{C}) \mid (gx, gy) = (x, y)$ for all $x, y \in \mathbb{C}^n\}$, where $(\cdot,\cdot)$ is the negative dot product on $\mathbb{C}^n$.
(b) Replace $\mathcal{C}_n(\mathbb{R})$ by $\mathcal{C}_{p,q}(\mathbb{R})$ (Exercise 1.30) in Definitions 1.36 and 1.37 to define $\mathrm{Spin}_{p,q}(\mathbb{R})$. Modify the proof of Theorem 1.40 to show that $\mathcal{A}$ realizes $\mathrm{Spin}_{p,q}(\mathbb{R})$ as a double cover of $SO(p,q)^0$, where $SO(p,q) = \{g \in SL(n,\mathbb{R}) \mid (gx, gy) = (x, y)$ for all $x, y \in \mathbb{C}^n\}$ and $(\cdot,\cdot)$ has signature p, q on $\mathbb{R}^n$.
(c) For $p, q > 0$ but not both 1, show that $\mathrm{Spin}_{p,q}(\mathbb{R})$ is connected. For $p = q = 1$, show that $\mathrm{Spin}_{1,1}(\mathbb{R})$ has two connected components.

Exercise 1.37 **(a)** Let $\mathfrak{so}(n) = \{X \in M_{n,n}(\mathbb{R}) \mid X^t = -X\}$ and $\mathfrak{q} = \sum_{i \neq j} \mathbb{R} e_i e_j \subseteq \mathcal{C}_n(\mathbb{R})$. Show that $\mathfrak{so}(n)$ and $\mathfrak{q}$ are closed under the bracket (Lie) algebra structure given by $[x, y] = xy - yx$.
(b) Show that there is a (Lie) bracket algebra isomorphism from $\mathfrak{so}(n)$ to $\mathfrak{q}$ induced by the map $E_{i,j} - E_{j,i} \to \frac{1}{2} e_i e_j$ where $\{E_{i,j}\}$ is the set of standard basis elements for $M_{n,n}(\mathbb{R})$.

1.4 Integration

1.4.1 Volume Forms

If $\Phi : M \to N$ is a smooth map of manifolds, write $d\Phi : T_p(M) \to T_{\Phi(p)}(N)$ for the *differential* of Φ where $T_p(M)$ is the *tangent space* of M at p. Write $\Phi^* : T^*_{\Phi(p)}(N) \to T^*_p(M)$ for the *pullback* of Φ where $T^*_p(M)$ is the *cotangent space* of M at p. As usual, extend the definition of the pullback to the exterior algebra, $\Phi^* : \bigwedge T^*_{\Phi(p)}(N) \to \bigwedge T^*_p(M)$, as a map of algebras.

If M is an n-dimensional manifold, M is said to be *orientable* if there exists a nonvanishing element $\omega_M \in \bigwedge^*_n(M)$ where $\bigwedge^*_n(M)$ is the *exterior n-bundle* of the cotangent bundle of M. When this happens, ω_M determines an *orientation* on M that permits integration of n-forms on M.

Suppose ω_M (ω_N) is a nonvanishing n-form providing an orientation on M (N). If Φ is a diffeomorphism, $\Phi : M \to N$, then $\Phi^*\omega_N = c\omega_M$ where c is a nonvanishing function on M. When $c > 0$, Φ is called *orientation preserving* and when $c < 0$, Φ is called *orientation reversing*. Similarly, a chart (U, φ) of M is said to be an *orientation preserving chart* if U is open and if φ is orientation preserving with respect to the orientations provided by $\omega|_U$, i.e., ω restricted to U, and by the standard volume form on $\varphi(U) \subseteq \mathbb{R}^n$, i.e., $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n|_{\varphi(U)}$.

If ω is a continuous n-form compactly supported in U where (U, φ) is an orientation preserving chart, recall the *integral of* ω with respect to the orientation on M induced by ω_M is defined as

$$\int_M \omega = \int_{\varphi(U)} (\varphi^{-1})^* \omega.$$

As usual (see [8] or [88] for more detail), the requirement that ω be supported in U is removed by covering M with orientation preserving charts, multiplying ω by a partition of unity subordinate to that cover, and summing over the partition using the above definition on each chart.

The *change of variables* formula from differential geometry is well known. If $\Phi : M \to N$ is a diffeomorphism of oriented manifolds and ω' is any continuous compactly supported n-form on N, then

$$\int_N \omega' = \pm \int_M \Phi^* \omega' \tag{1.41}$$

with the sign being a $+$ when Φ is orientation preserving and a $-$ when Φ is orientation reversing. A simple generalization of Equation 1.41 applicable to covering maps is also useful. Namely, if $\Psi : M \to N$ is an m-fold covering map of oriented manifolds and ω' is any continuous compactly supported n-form on N, then

$$m \int_N \omega' = \pm \int_M \Psi^* \omega' \tag{1.42}$$

with the sign determined by whether Ψ is orientation preserving or orientation reversing. The proof follows immediately from Equation 1.41 by using a partition of unity argument and the definition of a covering (Exercise 1.39).

Finally, functions on M can be integrated by fixing a *volume form* on M. A volume form is simply a fixed choice of a nonvanishing n-form, ω_M, defining the orientation on M. If f is a continuous compactly supported function on M, integration is defined with respect to this volume form by

$$\int_M f = \int_M f \omega_M.$$

It is easy to see (Exercise 1.40) that switching the volume form ω_M to $c\omega_M$, for some $c \in \mathbb{R}\backslash\{0\}$, multiplies the value of $\int_M f$ by $|c|$ (for negative c, the orientation is switched as well as the form against which f is integrated). In particular, the value of $\int_M f$ depends only on the choice of volume form modulo $\pm\omega_M$.

1.4.2 Invariant Integration

Let G be a Lie group of dimension n.

Definition 1.43. **(1)** Write l_g and r_g for *left* and *right translation* by $g \in G$, i.e., $l_g(h) = gh$ and $r_g(h) = hg$ for $h \in G$.
(2) A volume form, ω_G, on G is called *left invariant* if $l_g^*\omega_G = \omega_G$ and *right invariant* if $r_g^*\omega_G = \omega_G$ for all $g \in G$.

Lemma 1.44. *(1) Up to multiplication by a nonzero scalar, there is a unique left invariant volume form on G.*
(2) If G is compact, up to multiplication by ± 1, there is a unique left invariant volume form, ω_G, on G, so $\int_G 1 = 1$ with respect to ω_G.

Proof. Since $\dim \bigwedge_n^*(G)_e = 1$, up to multiplication by a nonzero scalar, there is a unique choice of $\omega_e \in \bigwedge_n^*(G)_e$. This choice uniquely extends to a left invariant n-form, ω, by defining $\omega_g = l_{g^{-1}}^*\omega_e$. For part (2), recall that replacing the volume form ω by $c\omega$ multiplies the value of the resulting integral by $|c|$. Because G is compact, $\int_G 1$ is finite with respect to the volume form ω. Thus there is a unique c, up to multiplication by ± 1, so that $\int_G 1 = 1$ with respect to the volume form $c\omega$. $\square$

Definition 1.45. For compact G, let ω_G be a left invariant volume form on G normalized so $\int_G 1 = 1$ with respect to ω_G. For any $f \in C(G)$, define

$$\int_G f(g)\,dg = \int_G f = \int_G f\omega_G$$

with respect to the orientation given by ω_G. By using the Riesz Representation Theorem, dg is also used to denote its completion to a Borel measure on G called *Haar measure* (see [37] or [73] for details).

If G has a suitably nice parametrization, it is possible to use the relation $\omega_g = l_{g^{-1}}^*\omega_e$ to pull the volume form back to an explicit integral over Euclidean space (see Exercise 1.44).

Theorem 1.46. *Let G be compact. The measure dg is left invariant, right invariant, and invariant under inversion, i.e.,*

$$\int_G f(hg)\,dg = \int_G f(gh)\,dg = \int_G f(g^{-1})\,dg = \int_G f(g)\,dg$$

for $h \in G$ and f a Borel integrable function on G.

Proof. It suffices to work with continuous f. Left invariance follows from the left invariance of ω_G and the change of variables formula in Equation 1.41 (l_h is clearly orientation preserving):

$$\begin{aligned}\int_G f(hg)\,dg &= \int_G (f \circ l_h)\omega_G = \int_G (f \circ l_h)(l_h^*\omega_G) \\ &= \int_G l_h^*(f\omega_G) = \int_G f\omega_G = \int_G f(g)\,dg.\end{aligned}$$

To address right invariance, first observe that l_g and r_g commute. Thus the n-form $r_g^*\omega_G$ is still left invariant. By Lemma 1.44, this means $r_g^*\omega_G = c(g)^{-1}\omega_G$ for some $c(g) \in \mathbb{R}\backslash\{0\}$. Because $r_g \circ r_h = r_{hg}$, it follows that the *modular function* $c : G \to \mathbb{R}\backslash\{0\}$ is a homomorphism. The compactness of G clearly forces $|c(g)| = 1$ (Exercise 1.41).

Since r_g is orientation preserving if and only if $c(g) > 0$, the definitions and Equation 1.41 imply that

$$\begin{aligned}\int_G f(gh)\,dg &= \int_G (f \circ r_h)\omega_G = c(h) \int_G (f \circ r_h)(r_h^*\omega_G) \\ &= c(h) \int_G r_h^*(f\omega_G) = c(h)\,\mathrm{sgn}\,(c(h)) \int_G f\omega_G = \int_G f(g)\,dg.\end{aligned}$$

Invariance of the measure under the transformation $g \to g^{-1}$ is handled similarly (Exercise 1.42). □

We already know that ω_G is the unique (up to ± 1) left invariant normalized volume form on G. More generally, the corresponding measure dg is the unique left invariant normalized Borel measure on G.

Theorem 1.47. *For compact G, the measure dg is the unique left invariant Borel measure on G normalized so G has measure* 1.

Proof. Suppose dh is a left invariant Borel measure on G normalized so G has measure 1. Then for nonnegative measurable f, definitions and the Fubini–Tonelli Theorem show that

$$\begin{aligned}\int_G f(g)\,dg &= \int_G \int_G f(g)\,dg\,dh = \int_G \int_G f(gh)\,dg\,dh \\ &= \int_G \int_G f(gh)\,dh\,dg = \int_G \int_G f(h)\,dh\,dg = \int_G f(h)\,dh,\end{aligned}$$

which is sufficient to establish $dg = dh$. □

1.4.3 Fubini's Theorem

Part of the point of Fubini's Theorem is to reduce integration in multiple variables to more simple iterated integrals. Here we examine a variant that is appropriate for compact Lie groups. In the special case where the H_i are compact Lie groups and $G = H_1 \times H_2$, Fubini's Theorem will simply say that

$$\int_{H_1 \times H_2} f(g)\,dg = \int_{H_1} \left(\int_{H_2} f(h_1 h_2)\,dh_2 \right) dh_1$$

for integrable f on G.

More generally, let G be a Lie group and H a closed subgroup of G, so (Theorem 1.7) G/H is a manifold. In general, G/H may not be orientable (Exercise 1.38). The next theorem tells us when G/H *is* orientable and how its corresponding measure relates to dg and dh, the invariant measures on G and H. Abusing notation, continue to write l_g for left translation by $g \in G$ on G/H.

Theorem 1.48. *Let G be a compact Lie group and H a closed subgroup of G. If l_h^* is the identity map on $\bigwedge_{\text{top}}^*(G/H)_{eH}$ for all $h \in H$ (which is always true when H is connected), then, up to scalar, G/H possesses a unique left G-invariant volume form, $\omega_{G/H}$, and a corresponding left invariant Borel measure, $d(gH)$. Up to ± 1, $\omega_{G/H}$ can be uniquely normalized, so*

$$\int_{G/H} F = \int_G F \circ \pi,$$

where $\pi : G \to G/H$ is the canonical projection and F is an integrable function on G/H. In this case,

$$\int_G f(g)\, dg = \int_{G/H} \left(\int_H f(gh)\, dh \right) d(gH),$$

where f is an integrable function on G.

Proof. Consider first the question of the existence of a left invariant volume form on G/H. As in the proof of Lemma 1.44, let $\omega_{eH} \in \bigwedge_{\text{top}}^*(G/H)_{eH}$. If it makes sense to define the form ω by setting $\omega_{gH} = l_{g^{-1}}^* \omega_{eH}$, then ω is clearly left invariant and unique up to scalar multiplication. However, this process is well defined if and only if $l_{g^{-1}}^* = l_{(gh)^{-1}}^*$ on $\bigwedge_{\text{top}}^*(G/H)_{eH}$ for all $h \in H$ and $g \in G$. Since $l_{gh} = l_g \circ l_h$, it follows that $\omega_{G/H}$ exists if and only if l_h^* is the identity map on $\bigwedge_{\text{top}}^*(G/H)_{eH}$ for all $h \in H$.

Since $\bigwedge_{\text{top}}^*(G/H)_{eH}$ is one-dimensional, $l_h^* \omega_{eH} = c(h) \omega_{eH}$ for $h \in H$ and some $c(h) \in \mathbb{R}\backslash\{0\}$. The equality $l_{hh'} = l_h \circ l_{h'}$ shows that $c : H \to \mathbb{R}\backslash\{0\}$ is a homomorphism. The compactness of G shows that $c(h) \in \{\pm 1\}$. If H is connected, the image of H under c must be connected and so $c(h) = 1$, which shows that $\omega_{G/H}$ exists.

Suppose that $\omega_{G/H}$ exists. Since dh is invariant, the function $g \to \int_H f(gh)\, dh$ may be viewed as a function on G/H. Working with characteristic functions, the assignment $f \to \int_{G/H} \left(\int_H f(gh)\, dh \right) d(gH)$ defines a normalized left invariant Borel measure on G. By Theorem 1.47, this measure must be dg and so the second displayed formula of this theorem is established. To see that the first displayed equation holds, let $f = F \circ \pi$. □

1.4.4 Exercises

Exercise 1.38 **(a)** Show that the antipode map, $x \to -x$, on S^{2n} is orientation reversing.
(b) Show $\mathbb{P}(\mathbb{R}^{2n})$ is not orientable.
(c) Find a compact Lie group G with a closed subgroup H, so $G/H \cong \mathbb{P}(\mathbb{R}^{2n})$.

Exercise 1.39 If $\Psi : M \to N$ is an m-fold covering map of oriented manifolds and ω' is any continuous compactly supported n-form on N, show that

$$m \int_N \omega' = \pm \int_M \Psi^* \omega'$$

with the sign determined by whether Ψ is orientation preserving or orientation reversing.

Exercise 1.40 If f is a continuous compactly supported function on an orientable manifold M, show that switching the volume form from ω_M to $c\omega_M$, for some $c \in \mathbb{R}\backslash\{0\}$, multiplies the value of $\int_M f$ by $|c|$.

Exercise 1.41 If G is a compact Lie group and $c : G \to \mathbb{R}\backslash\{0\}$ is a homomorphism, show that $c(g) \in \{\pm 1\}$ for all $g \in G$ and that $c(g) = 1$ if G is connected.

Exercise 1.42 **(a)** For f a continuous function on a compact Lie group G, show that $\int_G f(g^{-1})\,dg = \int_G f(g)\,dg$.
(b) If φ is a smooth automorphism of G, show that $\int_G f \circ \varphi = \int_G f$.

Exercise 1.43 Let G be the Lie group $\{\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{R} \text{ and } x > 0\}$. Show that the left invariant measure is $x^{-2}dxdy$ but the right invariant measure is $x^{-1}dxdy$.

Exercise 1.44 Let G be a Lie group and $\varphi : U \to V \subseteq \mathbb{R}^n$ a chart of G with $e \in U$, $0 \in V$, and $\varphi(e) = 0$. Suppose f is any integrable function on G supported in U.
(a) For $x \in V$, write $g = g(x) = \varphi^{-1}(x) \in U$. Show the function $l_x = \varphi \circ l_{g^{-1}} \circ \varphi^{-1}$ is well defined on a neighborhood of x.
(b) Write $\left|\frac{\partial l_x}{\partial x}|_x\right|$ for the absolute value of the determinant of the Jacobian matrix of l_x evaluated at x, i.e., $\left|\frac{\partial l_x}{\partial x}|_x\right| = |\det J|$, where the Jacobian matrix J is given by $J_{i,j} = \frac{\partial (l_x)_j}{\partial x_i}|_x$. Pull back the relation $\omega_g = l^*_{g^{-1}}\omega_e$ to show that the left invariant measure dg can be *scaled* so that

$$\int_G f\,dg = \int_V (f \circ \varphi^{-1})(x) \left|\frac{\partial l_x}{\partial x}|_x\right| dx_1 \dots dx_n.$$

(c) Show that changing l_x to $r_x = \varphi \circ r_{g^{-1}} \circ \varphi^{-1}$ in part (b) gives an expression for the right invariant measure.
(d) Write $\{(\frac{\partial}{\partial x_i}|_y)\}_{i=1}^n$ for the standard basis of of $T_y(\mathbb{R}^n)$. Show that the Jacobian matrix J is the change of basis matrix for the bases $\{d(l_{g^{-1}} \circ \varphi^{-1})(\frac{\partial}{\partial x_i}|_x)\}_{i=1}^n$ and $\{d\varphi^{-1}(\frac{\partial}{\partial x_i}|_0)\}_{i=1}^n$ of $T_e(G)$, i.e., $d(l_{g^{-1}} \circ \varphi^{-1})(\frac{\partial}{\partial x_i}|_x) = \sum_j J_{i,j} d\varphi^{-1}(\frac{\partial}{\partial x_j}|_0)$.
(e) Fix a basis $\{v_i\}_{i=1}^n$ of $T_e(G)$. Let C be the change of basis matrix for the bases $\{d(l_{g^{-1}} \circ \varphi^{-1})(\frac{\partial}{\partial x_i}|_x)\}_{i=1}^n$ and $\{v\}_{i=1}^n$, i.e., $d(l_{g^{-1}} \circ \varphi^{-1})(\frac{\partial}{\partial x_i}|_x) = \sum_j C_{i,j} v_j$. After *rescaling* dg, conclude that

$$\int_G f\,dg = \int_V (f \circ \varphi^{-1})(x)\, |\det C|\; dx_1 \cdots dx_n.$$

(f) Let H be a closed subgroup of a compact Lie group G and now suppose $\varphi : U \to V \subseteq \mathbb{R}^n$ a chart of G/H with $eH \in U$, $0 \in V$, and $\varphi(e) = 0$. Suppose l_h^* is the identity map on $\bigwedge^*_{\text{top}}(G/H)_{eH}$ for all $h \in H$ (which is always true when H is connected) and F is any integrable function on G/H supported in U. Fix a basis

$\{v_i\}_{i=1}^n$ of $T_{eH}(G/H)$ and define C as in part (e). Show that that $d(gH)$ can be *scaled* so that

$$\int_{G/H} F\, d(gH) = \int_V (F \circ \varphi^{-1})(x)\, |\det C|\; dx_1 \cdots dx_n.$$

Exercise 1.45 **(a)** View $GL(n, \mathbb{R})$ as an open dense set in $M_{n,n}(\mathbb{R})$ and identify functions on $GL(n, \mathbb{R})$ with functions on $M_{n,n}(\mathbb{R})$ that vanish on the complement of $GL(n, \mathbb{R})$. Show that the left and right invariant measure on $GL(n, \mathbb{R})$ is given by

$$\int_{GL(n,\mathbb{R})} f(g)\, dg = \int_{M_{n,n}(\mathbb{R})} f(X)\, |\det X|^{-n}\; dX,$$

where dX is the standard Euclidean measure on $M_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$. In particular, the invariant measure for the multiplicative group $\mathbb{R}^\times = \mathbb{R}\backslash\{0\}$ is $\frac{dx}{|x|}$.
(b) Show that the invariant measure for the multiplicative group $\mathbb{C}^\times$ is $\frac{dxdy}{x^2+y^2}$ with respect to the usual embedding of $\mathbb{C}^\times$ into $\mathbb{C} \cong \mathbb{R}^2$.
(c) Show that the invariant measure for the multiplicative group $\mathbb{H}^\times$ is $\frac{dxdydudv}{\left(x^2+y^2+u^2+v^2\right)^2}$ with respect to the usual embedding of $\mathbb{H}^\times$ into $\mathbb{H} \cong \mathbb{R}^4$.

Exercise 1.46 **(a)** On S^2, show that the $SO(3)$ normalized invariant measure is given by the integral $\frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} F(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi) \sin\phi\, d\theta d\phi$.
(b) Let f be the function on $SO(3)$ that maps a matrix to the determinant of the lower right 2×2 submatrix. Evaluate $\int_{SO(3)} f$.

Exercise 1.47 Let

$$\alpha(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \beta(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

$$\text{and } \gamma(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}.$$

(a) Verify that $(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi) = \alpha(\theta)\beta(\phi)e_3$ where $e_3 = (0, 0, 1)$. Use the isomorphism $S^2 \cong SO(3)/SO(2)$ to show that each element $g \in SO(3)$ can be written as $g = \alpha(\theta)\beta(\phi)\alpha(\psi)$ for $0 \leq \theta, \psi < 2\pi$ and $0 \leq \phi \leq \pi$ and that (θ, ϕ, ψ) is unique when $\phi \neq 0, \pi$. The coordinates (θ, ϕ, ψ) for $SO(3)$ are called the *Euler angles*.
(b) Viewing the map $(\theta, \phi, \psi) \to g = \alpha(\theta)\beta(\phi)\alpha(\psi)$ as a map into $M_{3,3}(\mathbb{R}) \cong \mathbb{R}^9$, show that

$$\begin{aligned} g^{-1}\frac{\partial g}{\partial\theta} &= \beta'(0)\sin\phi\cos\psi + \gamma'(0)\sin\phi\sin\psi + \alpha'(0)\cos\phi \\ g^{-1}\frac{\partial g}{\partial\phi} &= \beta'(0)\sin\psi - \gamma'(0)\cos\psi \\ g^{-1}\frac{\partial g}{\partial\psi} &= \alpha'(0). \end{aligned}$$

For $0 < \theta, \psi < 2\pi$ and $0 < \phi < \pi$, conclude that the inverse of the map $(\theta, \phi, \psi) \to \alpha(\theta)\beta(\phi)\alpha(\psi)$ is a chart for an open dense subset of $SO(3)$.
(c) Use Exercise 1.44 to show that the invariant integral on $SO(3)$ is given by

$$\int_{SO(3)} f(g)\,dg = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{\pi} \int_0^{2\pi} f(\alpha(\theta)\beta(\phi)\alpha(\psi)) \sin\phi \, d\theta d\phi d\psi$$

for integrable f on $SO(3)$.

Exercise 1.48 Let

$$\alpha(\theta) = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} \text{ and } \beta(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

As in Exercise 1.47, show that the invariant integral on $SU(2)$ is given by

$$\int_{SU(2)} f(g)\,dg = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{\pi} \int_0^{2\pi} f(\alpha(\theta)\beta(\phi)\alpha(\psi)) \sin\phi \, d\theta d\phi d\psi$$

for integrable f on $SU(2)$.