2

Representations

Lie groups are often the abstract embodiment of symmetry. However, most frequently they manifest themselves through an action on a vector space which will be called a representation. In this chapter we confine ourselves to the study of finite-dimensional representations.

2.1 Basic Notions

2.1.1 Definitions

Definition 2.1. A *representation* of a Lie group G on a finite-dimensional complex vector space V is a homomorphism of Lie groups $\pi : G \to GL(V)$. The *dimension* of a representation is $\dim V$.

Technically, a representation should be denoted by the pair (π, V). When no ambiguity exists, it is customary to relax this requirement by referring to a representation (π, V) as simply π or as V. Some synonyms for expressing the fact that (π, V) is a representation of G include the phrases V is a *G-module* or *G acts on V*. As evidence of further laziness, when a representation π is clearly understood it is common to write

$$gv \text{ or } g \cdot v \quad \text{in place of} \quad (\pi(g))(v)$$

for $g \in G$ and $v \in V$.

Although smoothness is part of the definition of a homomorphism (Definition 1.9), in fact we will see that continuity of π is sufficient to imply smoothness (Exercise 4.13). We will also eventually need to deal with infinite-dimensional vector spaces. The additional complexity of infinite-dimensional spaces will require a slight tweaking of our definition (Definition 3.11), although the changes will not affect the finite-dimensional case.

Two representations will be called equivalent if they are the same up to, basically, a change of basis. Recall that $\mathrm{Hom}(V, V')$ is the set of all linear maps from V to V'.

Definition 2.2. Let (π, V) and (π', V') be finite-dimensional representations of a Lie group G.
(1) $T \in \text{Hom}(V, V')$ is called an *intertwining operator* or *G-map* if $T \circ \pi = \pi' \circ T$.
(2) The set of all G-maps is denoted by $\text{Hom}_G(V, V')$.
(3) The representations V and V' are *equivalent*, $V \cong V'$, if there exists a bijective G-map from V to V'.

2.1.2 Examples

Let G be a Lie group. A representation of G on a finite-dimensional vector space V smoothly assigns to each $g \in G$ an invertible linear transformation of V satisfying

$$\pi(g)\pi(g') = \pi(gg')$$

for all $g, g' \in G$. Although surprisingly important at times, the most boring example of a representation is furnished by the map $\pi : G \to GL(1, \mathbb{C}) = \mathbb{C}\backslash\{0\}$ given by $\pi(g) = 1$. This one-dimensional representation is called the *trivial representation*. More generally, the action of G on a vector space is called *trivial* if each $g \in G$ acts as the identity operator.

2.1.2.1 Standard Representations Let G be $GL(n, \mathbb{F})$, $SL(n, \mathbb{F})$, $U(n)$, $SU(n)$, $O(n)$, or $SO(n)$. The *standard representation* of G is the representation on $\mathbb{C}^n$ where $\pi(g)$ is given by matrix multiplication on the left by the matrix $g \in G$. It is clear that this defines a representation.

2.1.2.2 *SU*(2) This example illustrates a general strategy for constructing new representations. Namely, if a group G acts on a space M, then G can be made to act on the space of functions on M (or various generalizations of functions).

Begin with the standard two-dimensional representation of $SU(2)$ on $\mathbb{C}^2$ where $g\eta$ is simply left multiplication of matrices for $g \in SU(2)$ and $\eta \in \mathbb{C}^2$. Let

$$V_n(\mathbb{C}^2)$$

be the vector space of holomorphic polynomials on $\mathbb{C}^2$ that are homogeneous of degree n. A basis for $V_n(\mathbb{C}^2)$ is given by $\{z_1^k z_2^{n-k} \mid 0 \le k \le n\}$, so $\dim V_n(\mathbb{C}^2) = n + 1$.

Define an action of $SU(2)$ on $V_n(\mathbb{C}^2)$ by setting

$$(g \cdot P)(\eta) = P(g^{-1}\eta)$$

for $g \in SU(2)$, $P \in V_n(\mathbb{C}^2)$, and $\eta \in \mathbb{C}^2$. To verify that this is indeed a representation, calculate that

$$\begin{aligned}[g_1 \cdot (g_2 \cdot P)](\eta) &= (g_2 \cdot P)(g_1^{-1}\eta) = P(g_2^{-1}g_1^{-1}\eta) = P((g_1g_2)^{-1}\eta) \\ &= [(g_1g_2) \cdot P](\eta)\end{aligned}$$

so that $g_1 \cdot (g_2 \cdot P) = (g_1g_2) \cdot P$. Since smoothness and invertibility are clear, this action yields an $n + 1$-dimensional representation of $SU(2)$ on $V_n(\mathbb{C}^2)$.

Although these representations are fairly simple, they turn out to play an extremely important role as a building blocks in representation theory. With this in mind, we write them out in all their glory. If $g = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \in SU(2)$, then $g^{-1} = \begin{pmatrix} \overline{a} & \overline{b} \\ -b & a \end{pmatrix}$, so that $g^{-1}\eta = (\overline{a}\eta_1 + \overline{b}\eta_2, -b\eta_1 + a\eta_2)$ where $\eta = (\eta_1, \eta_2)$. In particular, if $P = z_1^k z_2^{n-k}$, then $(g \cdot P)(\eta) = (\overline{a}\eta_1 + \overline{b}\eta_2)^k(-b\eta_1 + a\eta_2)^{n-k}$, so that

$$\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \cdot (z_1^k z_2^{n-k}) = (\overline{a}z_1 + \overline{b}z_2)^k(-bz_1 + az_2)^{n-k}. \tag{2.3}$$

Let us now consider another family of representations of $SU(2)$. Define

$$V_n'$$

to be the vector space of holomorphic functions in one variable of degree less than or equal to n. As such, V_n' has a basis consisting of $\{z^k \mid 0 \leq k \leq n\}$, so V_n' is also $n+1$-dimensional. In this case, define an action of $SU(2)$ on V_n' by

$$(g \cdot Q)(u) = (-bu + a)^n \, Q\left(\frac{\overline{a}u + \overline{b}}{-bu + a}\right) \tag{2.4}$$

for $g = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \in SU(2)$, $Q \in V_n'$, and $u \in \mathbb{C}$. It is easy to see that (Exercise 2.1) this yields a representation of $SU(2)$.

In fact, this apparently new representation is old news since it turns out that $V_n' \cong V_n(\mathbb{C}^2)$. To see this, we need to construct a bijective intertwining operator from $V_n(\mathbb{C}^2)$ to V_n'. Let $T : V_n(\mathbb{C}^2) \to V_n'$ be given by $(TP)(u) = P(u, 1)$ for $P \in V_n(\mathbb{C}^2)$ and $u \in \mathbb{C}$. This map is clearly bijective. To see that T is a G-map, use the definitions to calculate that

$$\begin{aligned}
[T\,(g \cdot P)]\,(u) = (g \cdot P)\,(u, 1) &= P(\overline{a}u + \overline{b}, -bu + a) \\
&= (-bu + a)^n P\left(\frac{\overline{a}u + \overline{b}}{-bu + a}, 1\right) \\
&= (-bu + a)^n \,(TP)\,(u) = [g \cdot (TP)]\,(u),
\end{aligned}$$

so $T\,(g \cdot P) = g \cdot (TP)$ as desired.

2.1.2.3 *O(n)* and Harmonic Polynomials Let

$$V_m(\mathbb{R}^n)$$

be the vector space of complex-valued polynomials on $\mathbb{R}^n$ that are homogeneous of degree m. Since $V_m(\mathbb{R}^n)$ has a basis consisting of $\{x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \mid k_i \in \mathbb{N}$ and $k_1 + k_2 + \cdots + k_n = m\}$, $\dim V_m(\mathbb{R}^n) = \binom{m+n-1}{m}$ (Exercise 2.4). Define an action of $O(n)$ on $V_m(\mathbb{R}^n)$ by

$$(g \cdot P)(x) = P(g^{-1}x)$$

for $g \in O(n)$, $P \in V_m(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$. As in §2.1.2.2, this defines a representation. As fine and natural as this representation is, it actually contains a smaller, even nicer, representation.

Write $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ for the *Laplacian* on $\mathbb{R}^n$. It is a well-known corollary of the chain rule and the definition of $O(n)$ that Δ commutes with this action, i.e., $\Delta(g \cdot P) = g \cdot (\Delta P)$ (Exercise 2.5).

Definition 2.5. Let $\mathcal{H}_m(\mathbb{R}^n)$ be the subspace of all *harmonic polynomials* of degree m, i.e., $\mathcal{H}_m(\mathbb{R}^n) = \{P \in V_m(\mathbb{R}^n) \mid \Delta P = 0\}$.

If $P \in \mathcal{H}_m(\mathbb{R}^n)$ and $g \in O(n)$, then $\Delta(g \cdot P) = g \cdot (\Delta P) = 0$ so that $g \cdot P \in \mathcal{H}_m(\mathbb{R}^n)$. In particular, the action of $O(n)$ on $V_m(\mathbb{R}^n)$ descends to a representation of $O(n)$ (or $SO(n)$, of course) on $\mathcal{H}_m(\mathbb{R}^n)$. It will turn out that these representations do not break into any smaller pieces.

2.1.2.4 Spin and Half-Spin Representations Any representation (π, V) of $SO(n)$ automatically yields a representation of $\mathrm{Spin}_n(\mathbb{R})$ by looking at $(\pi \circ \mathcal{A}, V)$ where $\mathcal{A}$ is the covering map from $\mathrm{Spin}_n(\mathbb{R})$ to $SO(n)$. The set of representations of $\mathrm{Spin}_n(\mathbb{R})$ constructed this way is exactly the set of representations in which $-1 \in \mathrm{Spin}_n(\mathbb{R})$ acts as the identity operator. In this section we construct an important representation, called the spin representation, of $\mathrm{Spin}_n(\mathbb{R})$ that is *genuine*, i.e., one that does not originate from a representation of $SO(n)$ in this manner.

Let $(\cdot, \cdot)$ be the symmetric bilinear form on $\mathbb{C}^n$ given by the dot product. Write $n = 2m$ when n is even and write $n = 2m + 1$ when n is odd. Recall a subspace $W \subseteq \mathbb{C}^n$ is called *isotropic* if $(\cdot, \cdot)$ vanishes on W. It is well known that $\mathbb{C}^n$ can be written as a direct sum

$$\mathbb{C}^n = \begin{cases} W \oplus W' & n \text{ even} \\ W \oplus W' \oplus \mathbb{C}e_0 & n \text{ odd} \end{cases} \tag{2.6}$$

for W, W' maximal isotropic subspaces (of dimension m) and e_0 a vector that is perpendicular to $W \oplus W'$ and satisfies $(e_0, e_0) = 1$. Thus, when n is even, take $W = \{(z_1, \ldots, z_m, iz_1, \ldots, iz_m) \mid z_k \in \mathbb{C}\}$ and $W' = \{(z_1, \ldots, z_m, -iz_1, \ldots, -iz_m) \mid z_k \in \mathbb{C}\}$. For n odd, take $W = \{(z_1, \ldots, z_m, iz_1, \ldots, iz_m, 0) \mid z_k \in \mathbb{C}\}$, $W' = \{(z_1, \ldots, z_m, -iz_1, \ldots, -iz_m, 0) \mid z_k \in \mathbb{C}\}$, and $e_0 = (0, \ldots, 0, 1)$.

Compared to our previous representations, the action of the spin representation is fairly complicated. We state the necessary definition below, although it will take some work to provide appropriate motivation and to show that everything is well defined. Recall (Lemma 1.39) that one realization of $\mathrm{Spin}_n(\mathbb{R})$ is $\{x_1 x_2 \cdots x_{2k} \mid x_i \in S^{n-1} \text{ for } 2 \le 2k \le 2n\}$.

Definition 2.7. **(1)** The elements of $S = \bigwedge W$ are called *spinors* and $\mathrm{Spin}_n(\mathbb{R})$ has a representation on S called the *spin representation*.
(2) For *n even*, the action for the spin representation of $\mathrm{Spin}_n(\mathbb{R})$ on S is induced by the map

$$x \to \epsilon(w) - 2\iota(w'),$$

where $x \in S^{n-1}$ is uniquely written as $x = w + w'$ according to the decomposition $\mathbb{R}^n \subseteq \mathbb{C}^n = W \oplus W'$.
(3) Let $S^+ = \bigwedge^+ W = \bigoplus_k \bigwedge^{2k} W$ and $S^- = \bigwedge^- W = \bigoplus_k \bigwedge^{2k+1} W$. As vector spaces $S = S^+ \oplus S^-$.
(4) For *n even*, the spin representation action of $\mathrm{Spin}_n(\mathbb{R})$ on S preserves the subspaces S^+ and S^-. These two spaces are therefore representations of $\mathrm{Spin}_n(\mathbb{R})$ in their own right and called the *half-spin representations*.
(5) For *n odd*, the action for the spin representation of $\mathrm{Spin}_n(\mathbb{R})$ on S is induced by the map

$$x \to \epsilon(w) - 2\iota(w') + (-1)^{\deg} m_{i\zeta},$$

where $x \in S^{n-1}$ is uniquely written as $x = w + w' + \zeta e_0$ according to the decomposition $\mathbb{R}^n \subseteq \mathbb{C}^n = W \oplus W' \oplus \mathbb{C}e_0$, $(-1)^{\deg}$ is the linear operator acting by ± 1 on $\bigwedge^{\pm} W$, and $m_{i\zeta}$ is multiplication by $i\zeta$.

To start making proper sense of this definition, let $\mathcal{C}_n(\mathbb{C}) = \mathcal{C}_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. From the definition of $\mathcal{C}_n(\mathbb{R})$, it is easy to see that $\mathcal{C}_n(\mathbb{C})$ is simply $\mathcal{T}(\mathbb{C}^n)$ modulo the ideal generated by either $\{(z \otimes z + (z, z)) \mid z \in \mathbb{C}^n\}$ or equivalently by $\{(z_1 \otimes z_2 + z_2 \otimes z_1 + 2(z_1, z_2)) \mid z_i \in \mathbb{C}^n\}$ (c.f. Exercise 1.30).

Since $\mathrm{Spin}_n(\mathbb{R}) \subseteq \mathcal{C}_n(\mathbb{C})$, $\mathcal{C}_n(\mathbb{C})$ itself becomes a representation for $\mathrm{Spin}_n(\mathbb{R})$ under left multiplication. Under this action, $-1 \in \mathrm{Spin}_n(\mathbb{R})$ acts as m_{-1}, and so this representation is genuine. However, the spin representations turn out to be much smaller than $\mathcal{C}_n(\mathbb{C})$. One way to find these smaller representations is to restrict left multiplication of $\mathrm{Spin}_n(\mathbb{R})$ to certain left ideals in $\mathcal{C}_n(\mathbb{C})$. While this method works (e.g., Exercise 2.12), we take an equivalent path that realizes $\mathcal{C}_n(\mathbb{C})$ as a certain endomorphism ring.

Theorem 2.8. *As algebras,*

$$\mathcal{C}_n(\mathbb{C}) \cong \begin{cases} \mathrm{End} \bigwedge W & n \text{ even} \\ \left(\mathrm{End} \bigwedge W\right) \bigoplus \left(\mathrm{End} \bigwedge W\right) & n \text{ odd.} \end{cases}$$

Proof. *n even*: For $z = w + w' \in \mathbb{C}^n$, define $\widetilde{\Phi} : \mathbb{C}^n \to \mathrm{End} \bigwedge W$ by

$$\widetilde{\Phi}(z) = \epsilon(w) - 2\iota(w').$$

As an algebra map, extend $\widetilde{\Phi}$ to $\widetilde{\Phi} : \mathcal{T}_n(\mathbb{C}) \to \mathrm{End} \bigwedge W$. A simple calculation (Exercise 2.6) shows $\widetilde{\Phi}(z)^2 = m_{-2(w,w')} = m_{-(z,z)}$ so that $\widetilde{\Phi}$ descends to a map $\widetilde{\Phi} : \mathcal{C}_n(\mathbb{C}) \to \mathrm{End} \bigwedge W$.

To see that $\widetilde{\Phi}$ is an isomorphism, it suffices to check that $\widetilde{\Phi}$ is surjective since $\mathcal{C}_n(\mathbb{C})$ and $\mathrm{End} \bigwedge W$ both have dimension 2^n. Pick a basis $\{w_1, \ldots, w_m\}$ of W and let $\{w'_1, \ldots, w'_m\}$ be the dual basis for W, i.e., (w_i, w'_j) is 0 when $i \neq j$ and 1 when $i = j$. With respect to this basis, $\widetilde{\Phi}$ acts in a particularly simple fashion. If $1 \leq i_1 < \cdots < i_k \leq m$, then $\widetilde{\Phi}(w_{i_1} \cdots w_{i_k} w'_{i_1} \cdots w'_{i_k})$ kills $\bigwedge^p W$ for $p < k$, maps

$\bigwedge^k W$ onto $\mathbb{C}w_{i_1} \wedge \cdots \wedge w_{i_k}$, and preserves $\bigwedge^p W$ for $p > k$. An inductive argument on $n - k$ therefore shows that the image of $\widetilde{\Phi}$ contains each projection of $\bigwedge W$ onto $\mathbb{C}w_{i_1} \wedge \cdots \wedge w_{i_k}$. Successive use of the operators $\widetilde{\Phi}(w_i)$ and $\widetilde{\Phi}(w'_j)$ can then be used to map $w_{i_1} \wedge \cdots \wedge w_{i_k}$ to any other $w_{j_1} \wedge \cdots \wedge w_{j_l}$. This implies that $\widetilde{\Phi}$ is surjective (in familiar matrix notation, this shows that the image of $\widetilde{\Phi}$ contains all endomorphisms corresponding to each matrix basis element $E_{i,j}$).

n odd: For $z = w + w' + \zeta e_0 \in \mathbb{C}^n$, let $\widetilde{\Phi}^{\pm} : \mathbb{C}^n \to \operatorname{End} \bigwedge W$ by

$$\widetilde{\Phi}^{\pm}(z) = \epsilon(w) - 2\iota(w') \pm (-1)^{\deg} m_{i\zeta}.$$

As an algebra map, extend $\widetilde{\Phi}^{\pm}$ to $\widetilde{\Phi}^{\pm} : \mathcal{T}_n(\mathbb{C}) \to \operatorname{End} \bigwedge W$. A simple calculation (Exercise 2.6) shows that $\widetilde{\Phi}^{\pm}(z)^2 = m_{-(z,z)}$ so that $\widetilde{\Phi}^{\pm}$ descends to a map $\widetilde{\Phi}^{\pm} : \mathcal{C}_n(\mathbb{C}) \to \operatorname{End} \bigwedge W$. Thus the map $\widetilde{\Phi} : \mathcal{C}_n(\mathbb{C}) \to (\operatorname{End} \bigwedge W) \bigoplus (\operatorname{End} \bigwedge W)$ given by $\widetilde{\Phi}(v) = (\widetilde{\Phi}^+(v), \widetilde{\Phi}^-(v))$ is well defined.

To see that $\widetilde{\Phi}$ is an isomorphism, it suffices to verify that $\widetilde{\Phi}$ is surjective since $\mathcal{C}_n(\mathbb{C})$ and $(\operatorname{End} \bigwedge W) \bigoplus (\operatorname{End} \bigwedge W)$ both have dimension 2^n. The argument is similar to the one given for the even case and left as an exercise (Exercise 2.7). □

Theorem 2.9. *As algebras,*

$$\mathcal{C}_n^+(\mathbb{C}) \cong \begin{cases} (\operatorname{End} \bigwedge^+ W) \bigoplus (\operatorname{End} \bigwedge^- W) & n \text{ even} \\ (\operatorname{End} \bigwedge W) & n \text{ odd.} \end{cases}$$

Proof. n even: From the definition of $\widetilde{\Phi}$ in the proof of Theorem 2.8, it is clear that the operators in $\widetilde{\Phi}(\mathcal{C}_n^+(\mathbb{C}))$ preserve $\bigwedge^{\pm} W$. Thus restricted to $\mathcal{C}_n^+(\mathbb{C})$, $\widetilde{\Phi}$ may be viewed as a map to $(\operatorname{End} \bigwedge^+ W) \bigoplus (\operatorname{End} \bigwedge^- W)$. Since this map is already known to be injective, it suffices to show that $\dim [(\operatorname{End} \bigwedge^+ W) \bigoplus (\operatorname{End} \bigwedge^- W)] = \dim \mathcal{C}_n^+(\mathbb{C})$. In fact, it is a simple task (Exercise 2.9) to see that both dimensions are 2^{n-1}. For instance, Equation 1.34 and Theorem 1.35 show $\dim \mathcal{C}_n^+(\mathbb{C}) = 2^{n-1}$. Alternatively, use Exercise 1.3.1 to show that $\mathcal{C}_{n-1}(\mathbb{R}) \cong \mathcal{C}_n^+(\mathbb{R})$.

n odd: In this case, operators in $\widetilde{\Phi}(\mathcal{C}_n^+(\mathbb{C}))$ no longer have to preserve $\bigwedge^{\pm} W$. However, restriction of the map $\widetilde{\Phi}^+$ to $\mathcal{C}_n^+(\mathbb{C})$ yields a map of $\mathcal{C}_n^+(\mathbb{C})$ to $\operatorname{End} \bigwedge W$ (alternatively, $\widetilde{\Phi}^-$ could have been used). By construction, this map is known to be surjective. Thus to see that the map is an isomorphism, it again suffices to show that $\dim (\operatorname{End} \bigwedge W) = \dim \mathcal{C}_n^+(\mathbb{C})$. As before, it is simple (Exercise 2.9) to see that both dimensions are 2^{n-1}. □

At long last, the origin of the spin representations can be untangled. Since $\operatorname{Spin}_n(\mathbb{R}) \subseteq \mathcal{C}_n^+(\mathbb{C})$, Definition 2.7 uses the homomorphism $\widetilde{\Phi}$ from Theorem 2.9 for n even and the homomorphism $\widetilde{\Phi}^+$ for n odd and restricts the action to $\operatorname{Spin}_n(\mathbb{R})$. In the case of n even, $\widetilde{\Phi}$ can be further restricted to either $\operatorname{End} \bigwedge^{\pm} W$ to construct the two half-spin representations. Finally, $-1 \in \operatorname{Spin}_n(\mathbb{R})$ acts by m_{-1}, so the spin representations are genuine as claimed.

2.1.3 Exercises

Exercise 2.1 Show that Equation 2.4 defines a representation of $SU(2)$ on V'_n.

Exercise 2.2 **(a)** Find the joint eigenspaces for the action of $\{\text{diag}(e^{i\theta}, e^{-i\theta}) \mid \theta \in \mathbb{R}\} \subseteq SU(2)$ on $V_n(\mathbb{C}^2)$. That is, find all nonzero $P \in V_n(\mathbb{C}^2)$, so that $\left(\text{diag}(e^{i\theta}, e^{-i\theta})\right) \cdot P = \lambda_\theta P$ for all $\theta \in \mathbb{R}$ and some $\lambda_\theta \in \mathbb{C}$.
(b) Find the joint eigenspaces for the action of $SO(2)$ on $V_m(\mathbb{R}^2)$ and on $\mathcal{H}_m(\mathbb{R}^2)$.
(c) Find the joint eigenspaces for the action of

$$\{(\cos\theta_1 + e_1e_2 \sin\theta_1)(\cos\theta_2 + e_3e_4 \sin\theta_2) \mid \theta_i \in \mathbb{R}\} \subseteq \text{Spin}(4)$$

on the half-spin representations $S^{\pm}$.

Exercise 2.3 Define a Hermitian inner product on $V_n(\mathbb{C}^2)$ by

$$\left(\sum_k a_k z_1^k z_2^{n-k}, \sum_k b_k z_1^k z_2^{n-k}\right) = \sum k!(n-k)!\, a_k \overline{b_k}.$$

For $g \in SU(2)$, show that $\left(gP, gP'\right) = (P, P')$ for $P, P' \in V_n(\mathbb{C}^2)$.

Exercise 2.4 Show that $|\ \{x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \mid k_i \in \mathbb{N} \text{ and } k_1 + k_2 + \cdots + k_n = m\}\ | = \binom{m+n-1}{m}$.

Exercise 2.5 For $g \in O(n)$ and f a smooth function on $\mathbb{R}^n$, show that $\Delta(f \circ l_g) = (\Delta f) \circ l_g$ where $l_g(x) = gx$.

Exercise 2.6 **(a)** For n even and $\widetilde{\Phi}(z) = \epsilon(w) - 2\iota(w')$ for $z = w + w' \in \mathbb{C}^n$ with $w \in W$ and $w' \in W'$, show $\widetilde{\Phi}(z)^2 = m_{-2(w,w')} = m_{-(z,z)}$.
(b) For n odd and $\widetilde{\Phi}(z) = \epsilon(w) - 2\iota(w') \pm (-1)^{\deg} m_{i\zeta}$ for $z = w + w' + \zeta e_0 \in \mathbb{C}^n$ with $w \in W$, $w' \in W'$, and $\zeta \in \mathbb{C}$, show that $\widetilde{\Phi}(z)^2 = m_{-2(w,w')-\zeta^2} = m_{-(z,z)}$.

Exercise 2.7 In the proof of Theorem 2.8, show that the map $\widetilde{\Phi}$ is surjective when n is odd.

Exercise 2.8 Use Theorem 2.8 to compute the center of $\mathcal{C}_n(\mathbb{C})$.

Exercise 2.9 From Theorem 2.9, show directly that $\dim \mathcal{C}_n^+(\mathbb{C})$ and

$$\dim\left[\left(\text{End} \bigwedge^+ W\right) \bigoplus \left(\text{End} \bigwedge^- W\right)\right]$$

are both 2^{n-1} for n even, and $\dim\left(\text{End} \bigwedge W\right) = 2^{n-1}$ for n odd.

Exercise 2.10 Up to equivalence, show that the spin and half-spin representations are independent of the choice of the maximal isotropic decomposition for $\mathbb{C}^n$ (as in Equation 2.6).

Exercise 2.11 For n odd, $\widetilde{\Phi}^+$ was used to define the spin representation of $\text{Spin}_n(\mathbb{R})$ on $\bigwedge W$. Show that an equivalent representation is constructed by using $\widetilde{\Phi}^-$ in place of Φ^+.

Exercise 2.12 **(a)** Use the same notation as in the proof of Theorem 2.8. For n even, let $w_0' = w_1' \cdots w_m' \in \mathcal{C}_n(\mathbb{C})$, let $\mathcal{J}$ be the left ideal of $\mathcal{C}_n(\mathbb{C})$ generated by w_0', and let $T : \bigwedge W \to \mathcal{J}$ be the linear map satisfying $T(w_{i_1} \wedge \cdots \wedge w_{i_k}) = w_{i_1} \cdots w_{i_k} w_0'$. Show that T is a well-defined and $\mathrm{Spin}_n(\mathbb{R})$-intertwining isomorphism with respect to the spin action on $\bigwedge W$ and left Clifford multiplication on $\mathcal{J}$.
(b) For n odd, let $w_0' = (1-ie_0)w_1' \cdots w_m'$. Show that there is an analogous $\mathrm{Spin}_n(\mathbb{R})$-intertwining isomorphism with respect to the spin action on $\bigwedge W$ and left Clifford multiplication on the appropriate left ideal of $\mathcal{C}_n(\mathbb{C})$.

Exercise 2.13 **(a)** Define a nondegenerate bilinear form $(\cdot, \cdot)$ on $\bigwedge W$ by setting $\left(\bigwedge^k W, \bigwedge^l W\right) = 0$ when $k+l \neq m$ and requiring $\alpha(u^*) \wedge v = (u, v)\, w_1 \wedge \cdots \wedge w_m$ for $u \in \bigwedge^k W$ and $v \in \bigwedge^{m-k} W$ (see §1.3.2 for notation). Show that the form is symmetric when $m \equiv 0, 3 \bmod(4)$ and that it is skew-symmetric when $m \equiv 1, 2, \bmod(4)$.
(b) With respect to the spin representation action, show that $(g \cdot u, g \cdot v) = (u, v)$ for $u, v \in S = \bigwedge W$ and $g \in \mathrm{Spin}_n(\mathbb{R})$.
(c) For n even, show that $(\cdot, \cdot)$ restricts to a nondegenerate form on $S^\pm = \bigwedge^\pm W$ when m is even, but restricts to zero when m is odd.

2.2 Operations on Representations

2.2.1 Constructing New Representations

Given one or two representations, it is possible to form many new representations using standard constructions from linear algebra. For instance, if V and W are vector spaces, one can form new vector spaces via the direct sum, $V \oplus W$, the tensor product, $V \otimes W$, or the set of linear maps from V to W, $\mathrm{Hom}(V, W)$. The tensor product leads to the construction of the tensor algebra, $\mathcal{T}(V) = \bigoplus_{k=0}^\infty \left(\bigotimes^k V\right)$, and its quotients, the *exterior algebra*, $\bigwedge(V) = \bigoplus_{k=0}^{\dim V} \bigwedge^k V$, and the *symmetric algebra*, $S(V) = \bigoplus_{k=0}^\infty S^k(V)$. Further constructions include the *dual* (or *contragradient*) space, $V^* = \mathrm{Hom}(V, \mathbb{C})$, and the *conjugate* space, $\overline{V}$, which has the same underlying additive structure as V, but is equipped with a new scalar multiplication structure, $\cdot'$, given by $z \cdot' v = \bar{z} v$ for $z \in \mathbb{C}$ and $v \in V$. Each of these new vector spaces also carries a representation as defined below.

Definition 2.10. Let V and W be finite-dimensional representations of a Lie group G.
(1) G acts on $V \oplus W$ by $g(v, w) = (gv, gw)$.
(2) G acts on $V \otimes W$ by $g \sum v_i \otimes w_j = \sum gv_i \otimes gw_j$.
(3) G acts on $\mathrm{Hom}(V, W)$ by $(gT)(v) = g\left[T\left(g^{-1}v\right)\right]$.
(4) G acts on $\bigotimes^k V$ by $g \sum v_{i_1} \otimes \cdots \otimes v_{i_k} = \sum \left(gv_{i_1}\right) \otimes \cdots \otimes \left(gv_{i_k}\right)$.
(5) G acts on $\bigwedge^k V$ by $g \sum v_{i_1} \wedge \cdots \wedge v_{i_k} = \sum \left(gv_{i_1}\right) \wedge \cdots \wedge \left(gv_{i_k}\right)$.
(6) G acts on $S^k(V)$ by $g \sum v_{i_1} \cdots v_{i_k} = \sum \left(gv_{i_1}\right) \cdots \left(gv_{i_k}\right)$.

(7) G acts on V^* by $(gT)(v) = T\left(g^{-1}v\right)$.
(8) G acts on $\overline{V}$ by the same action as it does on V.

It needs to be verified that each of these actions define a representation. All are simple. We check numbers (3) and (5) and leave the rest for Exercise 2.14. For number (3), smoothness and invertibility are clear. It remains to verify the homomorphism property so we calculate

$$[g_1(g_2T)](v) = g_1\left[(g_2T)\left(g_1^{-1}v\right)\right] = g_1g_2\left[T(g_2^{-1}g_1^{-1}v)\right] = [(g_1g_2)T](v)$$

for $g_i \in G$, $T \in \text{Hom}(V, W)$, and $v \in V$. For number (5), recall that $\bigwedge^k V$ is simply $\bigotimes^k V$ modulo $\mathcal{I}_k$, where $\mathcal{I}_k$ is $\bigotimes^k V$ intersect the ideal generated by $\{v \otimes v \mid v \in V\}$. Since number (4) is a representation, it therefore suffices to show that the action of G on $\bigotimes^k V$ preserves $\mathcal{I}_k$—but this is clear.

Some special notes are in order. For number (1) dealing with $V \oplus W$, choose in the obvious way a basis for $V \oplus W$ that is constructed from a basis for V and a basis for W. With respect to this basis, the action of G can be realized on $V \oplus W$ by multiplication by a matrix of the form $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ where the upper left block is given by the action of G on V and the lower right block is given by the action of G on W.

For number (7) dealing with V^*, fix a basis $\mathcal{B} = \{v_i\}_{i=1}^n$ for V and let $\mathcal{B}^* = \{v_i^*\}_{i=1}^n$ be the *dual basis* for V^*, i.e., $v_i^*(v_j)$ is 1 when $i = j$ and is 0 when $i \neq j$. Using these bases, identify V and V^* with $\mathbb{C}^n$ by the coordinate maps $\left[\sum_i c_i v_i\right]_{\mathcal{B}} = (c_1, \ldots, c_n)$ and $\left[\sum_i c_i v_i^*\right]_{\mathcal{B}^*} = (c_1, \ldots, c_n)$. With respect to these bases, realize the action of g on V and V^* by a matrices M_g and M_g' so that $[g \cdot v]_{\mathcal{B}} = M_g [v]_{\mathcal{B}}$ and $[g \cdot T]_{\mathcal{B}^*} = M_g' [T]_{\mathcal{B}^*}$ for $v \in V$ and $T \in V^*$. In particular, $\left[M_g\right]_{i,j} = v_i^*\left(g \cdot v_j\right)$ and $\left[M_g'\right]_{i,j} = \left(g \cdot v_j^*\right)(v_i)$. Thus $\left[M_g'\right]_{i,j} = v_j^*(g^{-1} \cdot v_i) = \left[M_{g^{-1}}\right]_{j,i}$ so that $M_g' = M_g^{-1,t}$. In other words, once appropriate bases are chosen and the G action is realized by matrix multiplication, the action of G on V^* is obtained from the action of G on V simply by taking the *inverse transpose* of the matrix.

For number (8) dealing with $\overline{V}$, fix a basis for V and realize the action of g by a matrix M_g as above. To examine the action of g on $v \in \overline{V}$, recall that scalar multiplication is the conjugate of the original scalar multiplication in V. In particular, in $\overline{V}$, $g \cdot v$ is therefore realized by the matrix $\overline{M_g}$. In other words, once a basis is chosen and the G action is realized by matrix multiplication, the action of G on $\overline{V}$ is obtained from the action of G on V simply by taking the *conjugate* of the matrix.

It should also be noted that few of these constructions are independent of each other. For instance, the action in number (7) on V^* is just the special case of the action in number (3) on $\text{Hom}(V, W)$ in which $W = \mathbb{C}$ is the trivial representation. Also the actions in (4), (5), and (6) really only make repeated use of number (2). Moreover, as representations, it is the case that $V^* \otimes W \cong \text{Hom}(V, W)$ (Exercise 2.15) and, for compact G, $V^* \cong \overline{V}$ (Corollary 2.20).

2.2.2 Irreducibility and Schur's Lemma

Now that we have many ways to glue representations together, it makes sense to seek some sort of classification. For this to be successful, it is necessary to examine the smallest possible building blocks.

Definition 2.11. Let G be a Lie group and V a finite-dimensional representation of G.
(1) A subspace $U \subseteq V$ is *G-invariant* (also called a *submodule* or a *subrepresentation*) if $gU \subseteq U$ for $g \in G$. Thus U is a representation of G in its own right.
(2) A nonzero representation V is *irreducible* if the only G-invariant subspaces are $\{0\}$ and V. A nonzero representation is called *reducible* if there is a proper (i.e., neither zero nor all of V) G-invariant subspace of V.

It follows that a nonzero finite-dimensional representation V is irreducible if and only if

$$V = \text{span}_{\mathbb{C}}\{gv \mid g \in G\}$$

for each nonzero $v \in V$, since this property is equivalent to excluding proper G-invariant subspaces. For example, it is well known from linear algebra that this condition is satisfied for each of the standard representations in §2.1.2.1 and so each is irreducible.

For more general representations, this approach is often impossible to carry out. In those cases, other tools are needed. One important tool is based on the next result.

Theorem 2.12 (Schur's Lemma). *Let V and W be finite-dimensional representations of a Lie group G. If V and W are irreducible, then*

$$\dim \text{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \ncong W. \end{cases}$$

Proof. If nonzero $T \in \text{Hom}_G(V, W)$, then $\ker T$ is not all of V and G-invariant so irreducibility implies T is injective. Similarly, the image of T is nonzero and G-invariant, so irreducibility implies T is surjective and therefore a bijection. Thus there exists a nonzero $T \in \text{Hom}_G(V, W)$ if and only if $V \cong W$.

In the case $V \cong W$, fix a bijective $T_0 \in \text{Hom}_G(V, W)$. If also $T \in \text{Hom}_G(V, W)$, then $T \circ T_0^{-1} \in \text{Hom}_G(V, V)$. Since V is a finite-dimensional vector space over $\mathbb{C}$, there exists an eigenvalue λ for $T \circ T_0^{-1}$. As $\ker(T \circ T_0^{-1} - \lambda I)$ is nonzero and G-invariant, irreducibility implies $T \circ T_0^{-1} - \lambda I = 0$, and so $\text{Hom}_G(V, W) = \mathbb{C}T_0$. □

Note Schur's Lemma implies that

$$\text{Hom}_G(V, V) = \mathbb{C}I \tag{2.13}$$

for irreducible V.

2.2.3 Unitarity

Definition 2.14. **(1)** Let V be a representation of a Lie group G. A form $(\cdot, \cdot) : V \times V \to \mathbb{C}$ is called *G-invariant* if $(gv, gv') = (v, v')$ for $g \in G$ and $v, v' \in V$.
(2) A representation V of a Lie group G is called *unitary* if there exists a G-invariant (Hermitian) inner product on V.

Noncompact groups abound with nonunitary representations (Exercise 2.18). However, compact groups are much more nicely behaved.

Theorem 2.15. *Every representation of a compact Lie group is unitary.*

Proof. Begin with any inner product $\langle \cdot, \cdot \rangle$ on V and define

$$(v, v') = \int_G \left\langle gv, gv' \right\rangle \, dg.$$

This is well defined since G is compact and $g \to \left\langle gv, gv' \right\rangle$ is continuous. The new form is clearly Hermitian and it is G-invariant since dg is right invariant. It remains only to see it is definite, but by definition, $(v, v) = \int_G \langle gv, gv \rangle \, dg$ which is positive for $v \neq 0$ since $\langle gv, gv \rangle > 0$. □

Theorem 2.15 provides the underpinning for much of the representation theory of compact Lie groups. It also says a representation (π, V) of a compact Lie group is better than a homomorphism $\pi : G \to GL(V)$; it is a homomorphism to the *unitary group* on V with respect to the G-invariant inner product (Exercise 2.20).

Definition 2.16. A finite-dimensional representation of a Lie group is called *completely reducible* if it is a direct sum of irreducible submodules.

Reducible but not completely reducible representations show up frequently for noncompact groups (Exercise 2.18), but again, compact groups are much simpler. We note that an analogous result will hold even in the infinite-dimensional setting of unitary representations of compact groups (Corollary 3.15).

Corollary 2.17. *Finite-dimensional representations of compact Lie groups are completely reducible.*

Proof. Suppose V is a representation of a compact Lie group G that is reducible. Let $(\cdot, \cdot)$ be a G-invariant inner product. If $W \subseteq V$ is a proper G-invariant subspace, then $V = W \oplus W^\perp$. Moreover, $W^\perp$ is also a proper G-invariant subspace since $(gw', w) = (w', g^{-1}w) = 0$ for $w' \in W^\perp$ and $w \in W$. By the finite dimensionality of V and induction, the proof is finished. □

As a result, any representation V of a compact Lie group G may be written as

$$V \cong \bigoplus_{i=1}^{N} n_i V_i, \tag{2.18}$$

where $\{V_i \mid 1 \leq i \leq N\}$ is a collection of inequivalent irreducible representations of G and $n_i V_i = V_i \oplus \cdots \oplus V_i$ (n_i copies). To study any representation of G, it therefore suffices to understand each irreducible representation and to know how to compute the n_i. In §2.2.4 we will find a formula for n_i.

Understanding the set of irreducible representations will take much more work. The bulk of the remaining text is, in one way or another, devoted to answering this question. In §3.3 we will derive a large amount of information on the set of all irreducible representations by studying functions on G. However, we will not be able to classify and construct all irreducible representations individually until §7.3.5 and §**??** where we study highest weights and associated structures.

Corollary 2.19. *If V is a finite-dimensional representation of a compact Lie group G, V is irreducible if and only if* $\dim \operatorname{Hom}_G(V, V) = 1$.

Proof. If V is irreducible, then Schur's Lemma (Theorem 2.12) implies that $\dim \operatorname{Hom}_G(V, V) = 1$. On the other hand, if V is reducible, then $V = W \oplus W'$ for proper submodules W, W' of V. In particular, this shows that $\dim \operatorname{Hom}_G(V, V) \geq 2$ since it contains the projection onto either summand. Hence $\dim \operatorname{Hom}_G(V, V) = 1$ implies that V is irreducible. □

The above result also has a corresponding version that holds even in the infinite-dimensional setting of unitary representations of compact groups (Theorem 3.12).

Corollary 2.20. *(1) If V is a finite-dimensional representation of a compact Lie group G, then $\overline{V} \cong V^*$.*
(2) If V is irreducible, then the G-invariant inner product is unique up to scalar multiplication by a positive real number.

Proof. For part (1), let $(\cdot, \cdot)$ be a G-invariant inner product on V. Define the bijective linear map $T : \overline{V} \to V^*$ by $Tv = (\cdot, v)$ for $v \in V$. To see that it is a G-map, calculate that $g(Tv) = (g^{-1}\cdot, v) = (\cdot, gv) = T(gv)$.

For part (2), assume V is irreducible. If $(\cdot, \cdot)'$ is another G-invariant inner product on V, define a second bijective linear map $T' : \overline{V} \to V^*$ by $T'v = (\cdot, v)'$. Schur's Lemma (Theorem 2.12) shows that $\dim \operatorname{Hom}_G(\overline{V}, V^*) = 1$. Since $T, T' \in \operatorname{Hom}_G(\overline{V}, V^*)$, there exists $c \in \mathbb{C}$, so $T' = cT$. Thus $(\cdot, v)' = c(\cdot, v)$ for all $v \in V$. It is clear that c must be in $\mathbb{R}$ and positive. □

Corollary 2.21. *Let V be a finite-dimensional representation of a compact Lie group G with a G-invariant inner product $(\cdot, \cdot)$. If V_1, V_2 are inequivalent irreducible submodules of V, then $V_1 \perp V_2$,* i.e., $(V_1, V_2) = 0$.

Proof. Consider $W = \{v_1 \in V_1 \mid (v_1, V_2) = 0\}$. Since $(\cdot, \cdot)$ is G-invariant, W is a submodule of V_1. If $(V_1, V_2) \neq 0$, i.e., $W \neq V_1$, then irreducibility implies that $W = \{0\}$ and $(\cdot, \cdot)$ yields a nondegenerate pairing of V_1 and V_2. Thus the map $v_1 \to (\cdot, v_1)$ exhibits an equivalence $\overline{V_1} \cong V_2^*$. This implies that $V_1 \cong \overline{V_2}^* \cong V_2$. □

2.2.4 Canonical Decomposition

Definition 2.22. (1) Let G be a compact Lie group. Denote the set of equivalence classes of irreducible (unitary) representations of G by $\widehat{G}$. When needed, choose a representative representation (π, E_π) for each $[\pi] \in \widehat{G}$.
(2) Let V be a finite-dimensional representation of G. For $[\pi] \in \widehat{G}$, let $V_{[\pi]}$ be the largest subspace of V that is a direct sum of irreducible submodules equivalent to E_π. The submodule $V_{[\pi]}$ is called the *π-isotypic component* of V.
(3) The *multiplicity* of π in V, m_π, is $\frac{\dim V_{[\pi]}}{\dim E_\pi}$, i.e., $V_{[\pi]} \cong m_\pi E_\pi$.

First, we verify that $V_{[\pi]}$ is well defined. The following lemma does that as well as showing that $V_{[\pi]}$ is the sum of *all* submodules of V equivalent to E_π.

Lemma 2.23. *If V_1, V_2 are direct sums of irreducible submodules isomorphic to E_π, then so is $V_1 + V_2$.*

Proof. By finite dimensionality, it suffices to check the following: if $\{W_i\}$ are G-submodules of a representation and W_1 is irreducible satisfying $W_1 \not\subseteq W_2 \oplus \cdots \oplus W_n$, then $W_1 \cap (W_2 \oplus \cdots \oplus W_n) = \{0\}$. However, $W_1 \cap (W_2 \oplus \cdots \oplus W_n)$ is a G-invariant submodule of W_1, so the initial hypothesis and irreducibility finish the argument. □

If V, W are representations of a Lie group G and $V \cong W \oplus W$, note this decomposition is not canonical. For example, if $c \in \mathbb{C}\backslash\{0\}$, then $W' = \{(w, cw) \mid w \in W\}$ and $W'' = \{(w, -cw) \mid w \in W\}$ are two other submodules both equivalent to W and satisfying $V \cong W' \oplus W''$. The following result gives a uniform method of handling this ambiguity as well as giving a formula for the n_i in Equation 2.18.

Theorem 2.24 (Canonical Decomposition). *Let V be a finite-dimensional representation of a compact Lie group G.*
(1) There is a G-intertwining isomorphism ι_π

$$\mathrm{Hom}_G(E_\pi, V) \otimes E_\pi \xrightarrow{\cong} V_{[\pi]}$$

induced by mapping $T \otimes v \to T(v)$ for $T \in \mathrm{Hom}_G(E_\pi, V)$ and $v \in V$. In particular, the multiplicity of π is

$$m_\pi = \dim \mathrm{Hom}_G(E_\pi, V).$$

(2) There is a G-intertwining isomorphism

$$\bigoplus_{[\pi]\in\widehat{G}} \mathrm{Hom}_G(E_\pi, V) \otimes E_\pi \xrightarrow{\cong} V = \bigoplus_{[\pi]\in\widehat{G}} V_{[\pi]}.$$

Proof. For part (1), let $T \in \mathrm{Hom}_G(E_\pi, V)$ be nonzero. Then $\ker T = \{0\}$ by the irreducibility of E_π. Thus T is an equivalence of E_π with $T(E_\pi)$, and so $T(E_\pi) \subseteq V_{[\pi]}$. Thus ι_π is well defined. Next, by the definition of the G-action on $\mathrm{Hom}(E_\pi, V)$

and the definition of $\mathrm{Hom}_G(E_\pi, V)$, it follows that G acts trivially on $\mathrm{Hom}_G(E_\pi, V)$. Thus $g(T \otimes v) = T \otimes (gv)$, so $\iota_\pi(g(T \otimes v)) = T(gv) = gT(v) = g\iota_\pi(T \otimes v)$, and so ι_π is a G-map. To see that ι_π is surjective, let $V_1 \cong E_\pi$ be a direct summand in $V_{[\pi]}$ with equivalence given by $T : E_\pi \to V_1$. Then $T \in \mathrm{Hom}_G(E_\pi, V)$ and V_1 clearly lies in the image of ι_π. Finally, make use of a dimension count to show that ι_π is injective. Write $V_{[\pi]} = V_1 \oplus \cdots \oplus V_{m_\pi}$ with $V_i \cong E_\pi$. Then

$$\begin{aligned}\dim \mathrm{Hom}_G(E_\pi, V) &= \dim \mathrm{Hom}_G(E_\pi, V_{[\pi]}) = \dim \mathrm{Hom}_G(E_\pi, V_1 \oplus \cdots \oplus V_{m_\pi}) \\ &= \sum_{i=1}^{m_\pi} \dim \mathrm{Hom}_G(E_\pi, V_i) = m_\pi\end{aligned}$$

by Schur's Lemma (Theorem 2.12). Thus $\dim \mathrm{Hom}_G(E_\pi, V) \otimes E_\pi = m_\pi \dim E_\pi = \dim V_{[\pi]}$.

For part (2), it only remains to show that $V = \bigoplus_{[\pi]\in\widehat{G}} V_{[\pi]}$. By Equation 2.18, $V = \sum_{[\pi]\in\widehat{G}} V_{[\pi]}$ and by Corollary 2.21 the sum is direct. □

See Theorem 3.19 for the generalization to the infinite-dimensional setting of unitary representations of compact groups.

2.2.5 Exercises

Exercise 2.14 Verify that the actions given in Definition 2.10 are representations.

Exercise 2.15 **(a)** Let V and W be finite-dimensional representations of a Lie group G. Show that $V^* \otimes W$ is equivalent to $\mathrm{Hom}(V, W)$ by mapping $T \otimes w$ to the linear map $wT(\cdot)$.
(b) Show, as representations, that $V \otimes V \cong S^2(V) \oplus \bigwedge^2(V)$.

Exercise 2.16 If V is an irreducible finite-dimensional representation of a Lie group G, show that V^* is also irreducible.

Exercise 2.17 This exercise considers a natural generalization of $V_n(\mathbb{C}^2)$. Let W be a representation of a Lie group G. Define $V_n(W)$ to be the space of holomorphic polynomials on W that are homogeneous of degree n and let $(gP)(\eta) = P(g^{-1}\eta)$. Show that there is an equivalence of representations $S^n(W^*) \cong V_n(W)$ induced by viewing $T_1 \cdots T_n$, $T_i \in W^*$, as a function on W.

Exercise 2.18 **(a)** Show that the map $\pi : t \to \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ produces a representation of $\mathbb{R}$ on $\mathbb{C}^2$.
(b) Show that this representation is not unitary.
(c) Find all invariant submodules.
(b) Show that the representation is reducible and yet not completely reducible.

Exercise 2.19 Use Schur's Lemma (Theorem 2.12) to quickly calculate the centers of the groups having standard representations listed in §2.1.2.1.

Exercise 2.20 Let $(\cdot,\cdot)$ be an inner product on $\mathbb{C}^n$. Show that $U(n) \cong \{g \in GL(n,\mathbb{C}) \mid (gv, gv') = (v, v')$ for $v, v' \in \mathbb{C}^n\}$.

Exercise 2.21 **(a)** Use Equation 2.13 to show that all irreducible finite-dimensional representations of an Abelian Lie group are 1-dimensional (c.f. Exercise 3.18).
(b) Classify all irreducible representations of S^1 and show that $\widehat{S^1} \cong \mathbb{Z}$.
(c) Find the irreducible summands of the representation of S^1 on $\mathbb{C}^2$ generated by the isomorphism $S^1 \cong SO(2)$.
(d) Show that a smooth homomorphism $\varphi : \mathbb{R} \to \mathbb{C}$ satisfies the differential equation $\varphi' = \left[\varphi'(0)\right] \varphi$. Use this to show that the set of irreducible representations of $\mathbb{R}$ is indexed by $\mathbb{C}$ and that the unitary ones are indexed by $i\mathbb{R}$.
(e) Use part (d) to show that the set of irreducible representations of $\mathbb{R}^+$ under its multiplicative structure is indexed by $\mathbb{C}$ and that the unitary ones are indexed by $i\mathbb{R}$.
(f) Classify all irreducible representations of $\mathbb{C} \cong \mathbb{R}^2$ under its additive structure and of $\mathbb{C}\backslash\{0\}$ under its multiplicative structure.

Exercise 2.22 Let V be a finite-dimensional representation of a compact Lie group G. Show the set of G-invariant inner products on G is isomorphic to $\mathrm{Hom}_G(V^*, V^*)$.

Exercise 2.23 **(a)** Let $\pi_i : V_i \to U(n)$ be two (unitary) equivalent irreducible representations of a compact Lie group G. Use Corollary 2.20 to show that there exists a *unitary* transformation intertwining π_1 and π_2.
(b) Repeat part (a) without the hypothesis of irreducibility.

Exercise 2.24 Let V be a finite-dimensional representation of a compact Lie group G and let $W \subseteq V$ be a subrepresentation. Show that $W_{[\pi]} \subseteq V_{[\pi]}$ for $[\pi] \in \widehat{G}$.

Exercise 2.25 Suppose V is a finite-dimensional representation of a compact Lie group G. Show that the set of G-intertwining automorphisms of V is isomorphic to $\prod_{[\pi]\in\widehat{G}} GL(m_\pi, \mathbb{C})$ where m_π is the multiplicity of the isotypic component $V_{[\pi]}$.

2.3 Examples of Irreducibility

2.3.1 $SU(2)$ and $V_n(\mathbb{C}^2)$

In this section we show that the representation $V_n(\mathbb{C}^2)$ from §2.1.2.2 of $SU(2)$ is irreducible. In fact, we will later see (Theorem 3.32) these are, up to equivalence, the only irreducible representations of $SU(2)$. The trick employed here points towards the powerful techniques that will be developed in §4 where derivatives, i.e., the tangent space of G, are studied systematically (c.f. Lemma 6.6).

Let $H \subseteq V_n(\mathbb{C}^2)$ be a nonzero invariant subspace. From Equation 2.3,

$$\mathrm{diag}(e^{i\theta}, e^{-i\theta}) \cdot (z_1^k z_2^{n-k}) = e^{i(n-2k)\theta}\, z_1^k z_2^{n-k}. \tag{2.25}$$

As the joint eigenvalues $e^{i(n-2k)\theta}$ are distinct and since H is preserved by $\{\mathrm{diag}(e^{i\theta}, e^{-i\theta})\}$, H is spanned by some of the joint eigenvectors $z_1^k z_2^{n-k}$. In particular, there is a k_0, so $z_1^{k_0} z_2^{n-k_0} \in H$.

Let $K_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in SU(2)$ and let $\eta_t = \begin{pmatrix} \cos t & i \sin t \\ i \sin t & \cos t \end{pmatrix} \in SU(2)$. Since H is $SU(2)$ invariant, $\frac{1}{2}(K_t \pm i\eta_t)\, z_1^{k_0} z_2^{n-k_0} \in H$. Thus, when the limits exist, $\frac{d}{dt}\left[\frac{1}{2}(K_t \pm i\eta_t)\, z_1^{k_0} z_2^{n-k_0}\right]|_{t=0} \in H$. Using Equation 2.3, a simple calculation (Exercise 2.26) shows that

$$\text{(2.26)} \qquad \frac{1}{2}\frac{d}{dt}\left[(K_t \pm i\eta_t)\, z_1^{k_0} z_2^{n-k_0}\right]|_{t=0} = \begin{cases} k_0\, z_1^{k_0-1} z_2^{n-k_0+1} & \text{for } + \\ (k_0 - n)\, z_1^{k_0+1} z_2^{n-k_0-1} & \text{for } - . \end{cases}$$

Induction therefore implies that $V_n(\mathbb{C}) \subseteq H$, and so $V_n(\mathbb{C})$ is irreducible.

2.3.2 *SO(n)* and Harmonic Polynomials

In this section we show that the representation of $SO(n)$ on the harmonic polynomials $\mathcal{H}_m(\mathbb{R}^n) \subseteq V_m(\mathbb{R}^n)$ is irreducible (see §2.1.2.3 for notation). Let $D_m(\mathbb{R}^n)$ be the space of complex constant coefficient differential operators on $\mathbb{R}^n$ of degree m. Recall that the algebra isomorphism from $\bigoplus_m V_m(\mathbb{R}^n)$ to $\bigoplus_m D_m(\mathbb{R}^n)$ is generated by mapping $x_i \to \partial_{x_i}$. In general, if $q \in \bigoplus_m V_m(\mathbb{R}^n)$, write ∂_q for the corresponding element of $\bigoplus_m D_m(\mathbb{R}^n)$.

Define $\langle \cdot, \cdot \rangle$ a Hermitian form on $V_m(\mathbb{R}^n)$ by $\langle p, q \rangle = \partial_{\overline{q}}(p) \in \mathbb{C}$ for $p, q \in V_m(\mathbb{R}^n)$. Since $\{x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} \mid k_i \in \mathbb{N} \text{ and } k_1 + k_2 + \cdots + k_n = m\}$ turns out to be an orthogonal basis for $V_m(\mathbb{R}^n)$, it is easy to see that $\langle \cdot, \cdot \rangle$ is an inner product. In fact, $\langle \cdot, \cdot \rangle$ is actually $O(n)$-invariant (Exercise 2.27), although we will not need this fact.

Lemma 2.27. *With respect to the inner product $\langle \cdot, \cdot \rangle$ on $V_m(\mathbb{R}^n)$, $\mathcal{H}_m(\mathbb{R}^n)^\perp = |x|^2\, V_{m-2}(\mathbb{R}^n)$ where $|x|^2 = \sum_{i=1}^n x_i^2 \in V_2(\mathbb{R}^n)$. As $O(n)$-modules,*

$$V_m(\mathbb{R}^n) \cong \mathcal{H}_m(\mathbb{R}^n) \oplus \mathcal{H}_{m-2}(\mathbb{R}^n) \oplus \mathcal{H}_{m-4}(\mathbb{R}^n) \oplus \cdots .$$

Proof. Let $p \in V_m(\mathbb{R}^n)$ and $q \in V_{m-2}(\mathbb{R}^n)$. Then $\langle p, |x|^2 q \rangle = \partial_{|x|^2 \overline{q}}\, p = \partial_{\overline{q}} \Delta p = \langle \Delta p, q \rangle$. Thus $\left[|x|^2\, V_{m-2}(\mathbb{R}^n)\right]^\perp = \mathcal{H}_m(\mathbb{R}^n)$ so that

$$\text{(2.28)} \qquad V_m(\mathbb{R}^n) = \mathcal{H}_m(\mathbb{R}^n) \oplus |x|^2\, V_{m-2}(\mathbb{R}^n).$$

Induction therefore shows that

$$V_m(\mathbb{R}^n) = \mathcal{H}_m(\mathbb{R}^n) \oplus |x|^2\, \mathcal{H}_{m-2}(\mathbb{R}^n) \oplus |x|^4\, \mathcal{H}_{m-4}(\mathbb{R}^n) \oplus \cdots .$$

The last statement of the lemma follows by observing that $O(n)$ fixes $|x|^{2k}$. □

By direct calculation, $\dim \mathcal{H}_m(\mathbb{R}^1) = 0$ for $m \geq 2$. For $n \geq 2$, however, it is clear that $\dim V_m(\mathbb{R}^n) > \dim V_{m-1}(\mathbb{R}^n)$ so that $\dim \mathcal{H}_m(\mathbb{R}^n) \geq 1$.

Lemma 2.29. *If G is a compact Lie group with finite-dimensional representations U, V, W satisfying $U \oplus V \cong U \oplus W$, then $V \cong W$.*

Proof. Using Equation 2.18, decompose $U \cong \bigoplus_{[\pi]\in\widehat{G}} m_\pi E_\pi$, $V \cong \bigoplus_{[\pi]\in\widehat{G}} m'_\pi E_\pi$, and $W \cong \bigoplus_{[\pi]\in\widehat{G}} m''_\pi E_\pi$. The condition $U \oplus V \cong U \oplus W$ therefore implies that $m_\pi + m'_\pi = m_\pi + m''_\pi$ so that $m'_\pi = m''_\pi$ and $V \cong W$. □

Definition 2.30. If H is a Lie subgroup of a Lie group G and V is a representation of G, write $V|_H$ for the representation of H on V given by restricting the action of G to H.

For the remainder of this section, view $O(n-1)$ as a Lie subgroup of $O(n)$ via the embedding $g \to \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$.

Lemma 2.31.

$$\mathcal{H}_m(\mathbb{R}^n)|_{O(n-1)} \cong \mathcal{H}_m(\mathbb{R}^{n-1}) \oplus \mathcal{H}_{m-1}(\mathbb{R}^{n-1}) \oplus \cdots \oplus \mathcal{H}_0(\mathbb{R}^{n-1}).$$

Proof. Any $p \in V_m(\mathbb{R}^n)$ may be uniquely written as $p = \sum_{k=0}^m x_1^k p_k$ with $p_k \in V_{m-k}(\mathbb{R}^{n-1})$ where $\mathbb{R}^n$ is viewed as $\mathbb{R} \times \mathbb{R}^{n-1}$. Since $O(n-1)$ acts trivially on x_1^k,

$$V_m(\mathbb{R}^n)|_{O(n-1)} \cong \bigoplus_{k=0}^m V_{m-k}(\mathbb{R}^{n-1}). \tag{2.32}$$

Applying Equation 2.28 first (restricted to $O(n-1)$) and then Equation 2.32, we get

$$V_m(\mathbb{R}^n)|_{O(n-1)} \cong \mathcal{H}_m(\mathbb{R}^n)|_{O(n-1)} \oplus \bigoplus_{k=0}^{m-2} V_{m-2-k}(\mathbb{R}^{n-1}).$$

Applying Equation 2.32 first and then Equation 2.28 yields

$$\begin{aligned} V_m(\mathbb{R}^n)|_{O(n-1)} &\cong \bigoplus_{k=0}^m \left[\mathcal{H}_{m-k}(\mathbb{R}^{n-1}) \oplus V_{m-2-k}(\mathbb{R}^{n-1})\right] \\ &= \left[\bigoplus_{k=0}^m \mathcal{H}_{m-k}(\mathbb{R}^{n-1})\right] \oplus \left[\bigoplus_{k=0}^{m-2} V_{m-2-k}(\mathbb{R}^{n-1})\right]. \end{aligned}$$

The proof is now finished by Lemma 2.29. □

Theorem 2.33. *$\mathcal{H}_m(\mathbb{R}^n)$ is an irreducible $O(n)$-module and, in fact, is irreducible under $SO(n)$ for $n \geq 3$.*

Proof. See Exercise 2.31 for the case of $n = 2$. In this proof assume $n \geq 3$.

$\mathcal{H}_m(\mathbb{R}^n)|_{SO(n-1)}$ contains, up to scalar multiplication, a unique $SO(n-1)$-invariant function: If $f \in \mathcal{H}_m(\mathbb{R}^n)$ is nonzero and $SO(n)$-invariant, then it is constant on each sphere in $\mathbb{R}^n$ and thus a function of the radius. Homogeneity implies that $f(x) = C\,|x|^m$ for some nonzero constant. It is trivial to check the condition that $\Delta f = 0$ now forces $m = 0$. Thus only $\mathcal{H}_0(\mathbb{R}^n)$ contains a nonzero $SO(n)$-invariant function. The desired result now follows from the previous observation and Lemma 2.31.

If V is a finite-dimensional $SO(n)$-invariant subspace of continuous functions on S^{n-1}, then V contains a nonzero $SO(n-1)$-invariant function: Here the action of $SO(n)$ on V is, as usual, given by $(gf)(s) = f(g^{-1}s)$. Since $SO(n)$ acts transitively on S^{n-1} and V is nonzero invariant, there exists $f \in V$, so $f(1, 0, \ldots, 0) \neq 0$.

Define $\widetilde{f}(s) = \int_{SO(n-1)} f(gs)\,dg$. If $\{f_i\}$ is a basis of V, then $f(gs) = \left(g^{-1}f\right)(s)$ and so may be written as $f(gs) = \sum_i c_i(g) f_i(s)$ for some smooth functions c_i. By integrating, it follows that $\widetilde{f} \in V$. From the definition, it is clear that $\widetilde{f}$ is $SO(n-1)$-invariant. It is nonzero since $\widetilde{f}(1, 0, \dots, 0) = f(1, 0, \dots, 0)$.

$\mathcal{H}_m(\mathbb{R}^n)$ *is an irreducible* $SO(n)$*-module:* Suppose $\mathcal{H}_m(\mathbb{R}^n) = V_1 \oplus V_2$ for proper $SO(n)$-invariant subspaces. By homogeneity, restricting functions in V_i from $\mathbb{R}^n$ to S^{n-1} is injective. Hence, both V_1 and V_2 contain independent $SO(n-1)$-invariant functions. But this contradicts the fact that $\mathcal{H}_m(\mathbb{R}^n)$ has only one independent $SO(n-1)$-invariant function. □

A relatively small dose of functional analysis (Exercise 3.14) can be used to further show that $L^2(S^{n-1}) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(\mathbb{R}^n)|_{S^{n-1}}$ (Hilbert space direct sum) and that $\mathcal{H}_m(\mathbb{R}^n)|_{S^{n-1}}$ is the eigenspace of the Laplacian on S^{n-1} with eigenvalue $-m(n+m-2)$.

2.3.3 Spin and Half-Spin Representations

The spin representation $S = \bigwedge W$ of $\mathrm{Spin}_n(\mathbb{R})$ for n odd and the half-spin representations $S^{\pm} = \bigwedge^{\pm} W$ for n even were constructed in §2.1.2.4, where W is a maximal isotropic subspace of $\mathbb{C}^n$. This section shows that these representations are irreducible.

For n even with $n = 2m$, let $W = \{(z_1, \dots, z_m, iz_1, \dots, iz_m) \mid z_k \in \mathbb{C}\}$ and $W' = \{(z_1, \dots, z_m, -iz_1, \dots, -iz_m) \mid z_k \in \mathbb{C}\}$. Identify W with $\mathbb{C}^m$ by projecting onto the first m coordinates. For $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ in $\mathbb{R}^m$, let $(x, y) = (x_1, \dots, x_m, y_1, \dots, y_m) \in \mathbb{R}^n$. In particular, $(x, y) = \frac{1}{2}(x - iy, i(x - iy)) + \frac{1}{2}(x + iy, -i(x + iy))$. Using Definition 2.7, the identification of $\mathbb{C}^m$ with W, and noting $((a, -ia), (b, ib)) = 2(a, b)$, the spin action of $\mathrm{Spin}_{2m}(\mathbb{R})$ on $\bigwedge^{\pm} \mathbb{C}^m \cong S^{\pm}$ is induced by having (x, y) act as

$$\frac{1}{2}\epsilon(x - iy) - 2\iota(x + iy). \tag{2.34}$$

For n odd with $n = 2m + 1$, take $W = \{(z_1, \dots, z_m, iz_1, \dots, iz_m, 0) \mid z_k \in \mathbb{C}\}$, $W' = \{(z_1, \dots, z_m, -iz_1, \dots, -iz_m, 0) \mid z_k \in \mathbb{C}\}$, and $e_0 = (0, \dots, 0, 1)$. As above, identify W with $\mathbb{C}^m$ by projecting onto the first m coordinates. For $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ in $\mathbb{R}^m$ and $u \in \mathbb{R}$, let $(x, y, u) = (x_1, \dots, x_m, y_1, \dots, y_m, u) \in \mathbb{R}^n$. In particular, $(x, y, u) = \frac{1}{2}(x - iy, i(x - iy), 0) + \frac{1}{2}(x + iy, -i(x + iy), 0) + (0, 0, u)$. Using Definition 2.7 and the identification of $\mathbb{C}^m$ with W, the spin action of $\mathrm{Spin}_{2m+1}(\mathbb{R})$ on $\bigwedge \mathbb{C}^m \cong S$ is induced by having (x, y, u) act as

$$\frac{1}{2}\epsilon(x - iy) - 2\iota(x + iy) + (-1)^{\deg} m_{iu}. \tag{2.35}$$

Theorem 2.36. *For n even, the half-spin representations $S^{\pm}$ of* $\mathrm{Spin}_n(\mathbb{R})$ *are irreducible. For n odd, the spin representation S of* $\mathrm{Spin}_n(\mathbb{R})$ *is irreducible.*

Proof. Using the standard basis $\{e_j\}_{j=1}^n$, calculate

$$(e_j \pm ie_{j+m})(e_k \pm ie_{k+m}) = e_j e_k \pm i(e_j e_{k+m} + e_{j+m} e_k) - e_{j+m} e_{k+m}$$

for $1 \leq j, k \leq m$. Since $e_j e_k$, $e_j e_{k+m}$, $e_{j+m} e_k$, and $e_{j+m} e_{k+m}$ lie in $\mathrm{Spin}_n(\mathbb{R})$, Equations 2.34 and 2.35 imply that the operators $\epsilon(e_j)\epsilon(e_k)$ and $\iota(e_j)\iota(e_k)$ on $\bigwedge \mathbb{C}^m$ are achieved by linear combinations of the action of elements of $\mathrm{Spin}_n(\mathbb{R})$ on $\bigwedge \mathbb{C}^m$.

For *n even*, let W be a nonzero $\mathrm{Spin}_n(\mathbb{R})$-invariant subspace contained in either $S^+ \cong \bigwedge^+ \mathbb{C}^m$ or $S^- \cong \bigwedge^- \mathbb{C}^m$. The operators $\epsilon(e_j)\epsilon(e_k)$ can be used to show that W contains a nonzero element in either $\bigwedge^m \mathbb{C}^m$ or $\bigwedge^{m-1} \mathbb{C}^m$, depending on the parity of m. In the first case, since $\dim \bigwedge^m \mathbb{C}^m = 1$, the operators $\iota(e_j)\iota(e_k)$ can be used to generate all of $\bigwedge^\pm \mathbb{C}^m$. In the second case, the operators $\iota(e_j)\iota(e_k)$ and $\epsilon(e_{j'})\epsilon(e_k)$ can be used to generate all of $\bigwedge^{m-1} \mathbb{C}^m$ after which the operators $\iota(e_j)\iota(e_k)$ can be used to generate all of $S^\pm$. Thus both half-spin representation are irreducible.

Similarly, for *n odd*, examination of the element $(e_j \pm ie_{j+m})e_n$ shows that the operators $\epsilon(e_j)(-1)^{\deg}$ and $\iota(e_j)(-1)^{\deg}$ are obtainable as linear combinations of the action of elements of $\mathrm{Spin}_n(\mathbb{R})$ on $\bigwedge \mathbb{C}^m$. Hence any nonzero $\mathrm{Spin}_n(\mathbb{R})$-invariant subspace W of $\bigwedge \mathbb{C}^m$ contains $\bigwedge^m \mathbb{C}^m$ by use of the operators $\epsilon(e_j)(-1)^{\deg}$. Finally,the operators $\iota(e_j)(-1)^{\deg}$ can then be used to show that $W = \bigwedge \mathbb{C}^m$ so that S is irreducible. □

2.3.4 Exercises

Exercise 2.26 Verify Equation 2.26.

Exercise 2.27 **(a)** For $g \in O(n)$, use the chain rule to show that $\partial_{g \cdot x_i} f = g\left(\partial_{x_i} f\right)$ for smooth f on $\mathbb{R}^n$.
(b) For $g \in O(n)$, show that $\partial_{g \cdot p} f = g\left(\partial_p f\right)$ for $p \in V_m(\mathbb{R}^n)$.
(c) Show that $\langle \cdot, \cdot \rangle$ is $O(n)$-invariant on $V_m(\mathbb{R}^n)$.

Exercise 2.28 For $p \in V_m(\mathbb{R}^n)$ show that there exists a unique $h \in \bigoplus_k \mathcal{H}_{m-2k}(\mathbb{R}^n)$, so $p|_{S^{n-1}} = h|_{S^{n-1}}$.

Exercise 2.29 Show that Δ is an $O(n)$-map from $V_m(\mathbb{R}^n)$ onto $V_{m-2}(\mathbb{R}^n)$.

Exercise 2.30 Show that $\dim \mathcal{H}_0(\mathbb{R}^n) = 1$, $\dim \mathcal{H}_1(\mathbb{R}^n) = n$, and $\dim \mathcal{H}_m(\mathbb{R}^n) = \frac{(2m+n-2)(m+n-3)!}{m!(n-2)!}$ for $m \geq 2$.

Exercise 2.31 Show that $\mathcal{H}_m(\mathbb{R}^2)$ is $O(2)$-irreducible but not $SO(2)$-irreducible when $m \geq 2$.

Exercise 2.32 Exercises 2.32 through 2.34 outline an alternate method of proving irreducibility of $\mathcal{H}_m(\mathbb{R}^n)$ using reproducing kernels ([6]). Let $\mathcal{H}$ be a Hilbert space of functions on a space X that is closed under conjugation and such that evaluation at any $x \in X$ is a continuous operator on $\mathcal{H}$. Write $(\cdot, \cdot)$ for the inner product on $\mathcal{H}$. Then for $x \in X$, there exists a unique $\phi_x \in \mathcal{H}$, so $f(x) = (f, \phi_x)$ for $f \in \mathcal{H}$. The

function $\Phi : X \times X \to \mathbb{C}$, given by $(x, y) \to (\phi_y, \phi_x)$, is called the *reproducing kernel.*
(a) Show that $\Phi(x, y) = \phi_y(x)$ for $x, y \in X$ and $f(x) = (f, \Phi(\cdot, x))$ for $f \in \mathcal{H}$.
(b) Show that $\operatorname{span}\{\phi_x \mid x \in X\}$ is dense in $\mathcal{H}$.
(c) If $\{e_\alpha\}_{\alpha \in \mathcal{A}}$ is an orthonormal basis of $\mathcal{H}$, then $\Phi(x, y) = \sum_\alpha e_\alpha(x)\, \overline{e_\alpha(y)}$.
(d) If there exists a measure μ on X such that $\mathcal{H}$ is a closed subspace in $L^2(X, d\mu)$, then $f(x) = \int_X \overline{\Phi(y, x)}\, f(y)\, d\mu(y)$.

Exercise 2.33 Suppose there is a Lie group G acting transitively on X. Fix $x_0 \in X$ so that $X \cong G/H$ where $H = G^{x_0}$. Let G act on functions by $(gf)(x) = f(g^{-1}x)$ for $g \in G$ and $x \in X$. Assume this action preserves $\mathcal{H}$, is unitary, and that $\mathcal{H}^H = \{f \in \mathcal{H} \mid hf = f \text{ for } h \in H\}$ is one dimensional.
(a) Show that $g\phi_x = \phi_{gx}$ and $\Phi(gx, gy) = \Phi(x, y)$ for $g \in G$ and $x, y \in X$.
(b) Let W be a nonzero closed G-invariant subspace of $\mathcal{H}$ and write Φ_W for its reproducing kernel. Show that the function $x \to \Phi_W(x, x_0)$ lies in $\mathcal{H}^H$.
(c) Show that G acts irreducibly on $\mathcal{H}$.

Exercise 2.34 Let $\mathcal{H} = \mathcal{H}_m(\mathbb{R}^n) \subseteq V_m(\mathbb{R}^n)$, where $V_m(\mathbb{R}^n)$ is viewed as sitting in $L^2(S^{n-1})$ by restriction to S^{n-1}. Let $p_0 = (1, 0, \ldots, 0)$.
(a) Show that $V_m(\mathbb{R}^n)^{O(n-1)}$ consists of all functions of the form

$$x \to \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^j c_j (x, x)^j (x, p_0)^{k-2j}$$

for constants $c_j \in \mathbb{C}$.
(b) Find a linear recurrence formula on the c_j to show that $\dim \mathcal{H}_m(\mathbb{R}^n)^{O(n-1)} = 1$.
(c) Show that $\mathcal{H}_m(\mathbb{R}^n)$ is irreducible under $O(n)$.
(d) Show that $\mathcal{H}_m(\mathbb{R}^n)$ is still irreducible under restriction to $SO(n)$ for $n \geq 3$.

Exercise 2.35 Let $G = U(n)$, $V_{p,q}(\mathbb{C}^n)$ be the set of complex polynomials homogeneous of degree p in $z_1, \ldots, z_n$ and homogeneous of degree q in $\overline{z_1}, \ldots, \overline{z_n}$ equipped with the typical action of G, $\Delta_{p,q} = \sum_j \partial_{z_j} \partial_{\overline{z_j}}$, and $\mathcal{H}_{p,q}(\mathbb{C}^n) = V_{p,q}(\mathbb{C}^n) \cap \ker \Delta_{p,q}$. Use restriction to S^{2n-1} and techniques similar to those found in Exercises 2.32 through 2.34 to demonstrate the following.
(a) Show that $\Delta_{p,q}$ is a G-map from $V_{p,q}(\mathbb{C}^n)$ onto $V_{p-1,q-1}(\mathbb{C}^n)$.
(b) Show $\mathcal{H}_m(\mathbb{R}^{2n}) \cong \bigoplus_{p+q=m} \mathcal{H}_{p,q}(\mathbb{C}^n)$.
(c) Show that $\mathcal{H}_{p,q}(\mathbb{C}^n)$ is $U(n)$-irreducible.
(d) Show that $\mathcal{H}_{p,q}(\mathbb{C}^n)$ is still irreducible under restriction to $SU(n)$.

Exercise 2.36 Show that S^+ and S^- are inequivalent representations of $\operatorname{Spin}_n(\mathbb{R})$ for n even.