3

Harmonic Analysis

Throughout this chapter let G be a *compact* Lie group. This chapter studies a number of function spaces on G such as the set of continuous functions on G, $C(G)$, or the set of square integrable functions on G, $L^2(G)$, with respect to the Haar measure dg. These function spaces are examined in the light of their behavior under left and right translation by G.

3.1 Matrix Coefficients

3.1.1 Schur Orthogonality

Let (π, V) be a finite-dimensional unitary representation of a compact Lie group G with G-invariant inner product $(\cdot, \cdot)$. If $\{v_i\}$ is a basis for V, let $\{v_i^*\}$ be the *dual basis* for V, i.e., $(v_i, v_j^*) = \delta_{i,j}$ where $\delta_{i,j}$ is 1 when $i = j$ and 0 when $i \neq j$. With respect to this basis, the linear transformation $\pi(g) : V \to V$, $g \in G$, can be realized as matrix multiplication by the matrix whose entry in the $(i, j)^{\text{th}}$ position is

$$(gv_j, v_i^*).$$

The function $g \to (gv_j, v_i^*)$ is a smooth complex-valued function on G. The study of linear combinations of such functions turns out to be quite profitable.

Definition 3.1. Any function on a compact Lie group G of the form $f_{u,v}^V(g) = (gu, v)$ for a finite-dimensional unitary representation V of G with $u, v \in V$ and G-invariant inner product $(\cdot, \cdot)$ is called a *matrix coefficient* of G. The collection of all matrix coefficients is denoted $MC(G)$.

Lemma 3.2. *$MC(G)$ is a subalgebra of the set of smooth functions on G and contains the constant functions. If $\{v_i^\pi\}_{i=1}^{n_\pi}$ is a basis for E_π, $[\pi] \in \widehat{G}$, then $\{f_{v_i^\pi, v_j^\pi}^{E_\pi} \mid [\pi] \in \widehat{G}$ and $1 \leq i, j \leq n_\pi\}$ span $MC(G)$.*

Proof. By definition, a matrix coefficient is clearly a smooth function on G. If V, V' are unitary representations of G with G-invariant inner products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_{V'}$, then $U \oplus V$ is unitary with respect to the inner product $\big((u,v),\,(u',v')\big)_{V\oplus V'} = (u,u')_V + (v,v')_{V'}$ and $V \otimes V'$ is unitary with respect to the inner product

$$\left(\sum_i u_i \otimes v_i,\ \sum_j u_j' \otimes v_j'\right)_{V\otimes V'} = \sum_{i,j} (u_i, u_j')_V \,(v_i, v_j')_{V'}$$

(Exercise 3.1). Thus $cf^V_{u,u'} + f^{V'}_{v,v'} = f^{V\oplus V'}_{(cu,v),(u',v')}$, so $MC(G)$ is a subspace and $f^V_{u,u'} f^{V'}_{v,v'} = f^{V\otimes V'}_{u\otimes v, u'\otimes v'}$, so $MC(G)$ is an algebra. The constant functions are easily achieved as matrix coefficients of the trivial representation.

To verify the final statement of the lemma, first decompose V into irreducible mutually perpendicular summands (Exercise 3.2) as $V = \bigoplus_i V_i$ where each $V_i \cong E_{\pi_i}$. Any $v, v' \in V$ can be written $v = \sum_i v_i$ and $v' = \sum_i v_i'$ with $v_i, v_i' \in V_i$ so that $f^V_{v,v'} = \sum_i f^{V_i}_{v_i,v_i'}$. If $T_i : V_i \to E_{\pi_i}$ is an intertwining isomorphism, then $(T_i v_i, T_i v_i')_{E_{\pi_i}} = (v_i, v_i')_V$ defines a unitary structure on E_{π_i} so that $f^V_{v,v'} = \sum_i f^{E_{\pi_i}}_{T_i v_i, T_i v_i'}$. Expanding $T_i v_i$ and $T_i v_i'$ in terms of the basis for E_{π_i} finishes the proof. $\square$

The next theorem calculates the L^2 inner product of the matrix coefficients corresponding to irreducible representations.

Theorem 3.3 (Schur Orthogonality Relations). *Let U, V be irreducible finite-dimensional unitary representations of a compact Lie group G with G-invariant inner products $(\cdot,\cdot)_U$ and $(\cdot,\cdot)_V$. If $u_i \in U$ and $v_i \in V$,*

$$\int_G (gu_1, u_2)_U \,\overline{(gv_1, v_2)_V}\, dg = \begin{cases} 0 & \text{if } U \not\cong V \\ \frac{1}{\dim V}\,(u_1, v_1)_V\,\overline{(u_2, v_2)_V} & \text{if } U = V. \end{cases}$$

Proof. For $u \in U$ and $v \in V$, define $T_{u,v} : U \to V$ by $T_{u,v}(\cdot) = v\,(\cdot, u)_U$. For the sake of clarity, initially write the action of each representation as (π_U, U) and (π_V, V). Then the function $g \to \pi_U(g) \circ T_{u,v} \circ \pi_V^{-1}(g)$, $g \in G$, can be viewed, after choosing bases, as a matrix valued function. Integrating on each coordinate of the matrix (*c.f.* vector-valued integration in §3.2.2), define $\widetilde{T}_{u,v} : U \to V$ by

$$\widetilde{T}_{u,v} = \int_G \pi_U(g) \circ T_{u,v} \circ \pi_V^{-1}(g)\, dg.$$

For $h \in G$, the invariance of the measure implies that

$$\begin{aligned}\pi_U(h) \circ \widetilde{T}_{u,v} &= \int_G \pi_U(hg) \circ T_{u,v} \circ \pi_V^{-1}(g)\, dg = \int_G \pi_U(g) \circ T_{u,v} \circ \pi_V^{-1}(h^{-1}g)\, dg \\ &= \widetilde{T}_{u,v} \circ \pi_V(h),\end{aligned}$$

so that $\widetilde{T}_{u,v} \in \mathrm{Hom}_G(U, V)$. Irreducibility and Schur's Lemma (Theorem 2.12) show that $\widetilde{T}_{u,v} = cI$ where $c = c(u,v) \in \mathbb{C}$ with $c = 0$ when $U \not\cong V$. Unwinding the definitions and using the change of variables $g \to g^{-1}$, calculate

$$
\begin{aligned}
c\,(u_1, v_1)_V &= (\widetilde{T}_{u_2,v_2} u_1, v_1)_V = \int_G (g T_{u_2,v_2} g^{-1} u_1, v_1)_V \, dg \\
&= \int_G \left(\left(g^{-1}u_1, u_2\right)_U \; g v_2, v_1\right)_V \, dg = \int_G (g u_1, u_2)_U \; (g^{-1} v_2, v_1)_V \, dg \\
&= \int_G (g u_1, u_2)_U \; (v_2, g v_1)_V \, dg = \int_G (g u_1, u_2)_U \; \overline{(g v_1, v_2)}_V \, dg.
\end{aligned}
$$

Thus the theorem is finished when $U \not\cong V$. When $U = V$, it remains to calculate c. For this, take the trace of the identity $cI = \widetilde{T}_{u_2,v_2}$ to get

$$
\begin{aligned}
c \dim V = \operatorname{tr} \widetilde{T}_{u_2,v_2} &= \int_G \operatorname{tr}\left[g \circ T_{u_2,v_2} \circ g^{-1}\right] dg \\
&= \int_G \operatorname{tr} T_{u_2,v_2} \, dg = \operatorname{tr} T_{u_2,v_2}.
\end{aligned}
$$

To quickly calculate $\operatorname{tr} T_{u_2,v_2}$ for nonzero u_2, choose a basis for $U = V$ with v_2 as the first element. Since $T_{u_2,v_2}(\cdot) = v_2(\cdot, u_2)_V$, $\operatorname{tr} T_{u_2,v_2} = (v_2, u_2)_V$, so that $c = \frac{1}{\dim V} \overline{(u_2, v_2)}_V$ which finishes the proof. $\square$

If $U \cong V$ and $T : U \to V$ is a G-intertwining isomorphism, Theorem 2.20 implies there is a positive constant $c \in \mathbb{R}$, so that $(u_1, u_2)_U = c(Tu_1, Tu_2)_V$. In this case, the Schur orthogonality relation becomes

$$
\int_G (g u_1, u_2)_U \, \overline{(g v_1, v_2)}_V \, dg = \frac{c}{\dim V} \, (T u_1, v_1)_V \, \overline{(T u_2, v_2)}_V .
$$

Of course, T can be scaled so that $c = 1$ by replacing T with $\sqrt{c}T$.

3.1.2 Characters

Definition 3.4. The *character* of a finite-dimensional representation (π, V) of a compact Lie group G is the function on G defined by $\chi_V(g) = \operatorname{tr} \pi(g)$.

It turns out that character theory provides a powerful tool for studying representations. In fact, we will see in Theorem 3.7 below that, up to equivalence, a character completely determines the representation. Note for $\dim V > 1$, a character in the above sense is usually not a homomorphism.

Theorem 3.5. *Let V, V_i be finite-dimensional representations of a compact Lie group G.*
(1) $\chi_V \in MC(G)$.
(2) $\chi_V(e) = \dim V$.
(3) *If $V_1 \cong V_2$, then $\chi_{V_1} = \chi_{V_2}$.*
(4) $\chi_V(hgh^{-1}) = \chi_V(g)$ *for* $g, h \in G$.
(5) $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$.
(6) $\chi_{V_1 \otimes V_2} = \chi_{V_1} \chi_{V_2}$.
(7) $\chi_{V^*}(g) = \chi_{\overline{V}}(g) = \overline{\chi_V(g)} = \chi_V(g^{-1})$.
(8) $\chi_{\mathbb{C}}(g) = 1$ *for the trivial representation* $\mathbb{C}$.

Proof. Each statement of the theorem is straightforward to prove. We prove parts (1), (4), (5), and (7) and leave the rest as an exercise (Exercise 3.3). For part (1), let $\{v_i\}$ be an orthonormal basis for V with respect to a G-invariant inner product $(\cdot,\cdot)$. Then $\chi_V(g) = \sum_i (gv_i, v_i)$ so that $\chi_V \in MC(G)$. For part (4), calculate

$$\chi_V(hgh^{-1}) = \operatorname{tr}\left[\pi(h)\pi(g)\pi(h)^{-1}\right] = \operatorname{tr}\pi(g) = \chi_V(g).$$

For part (5), §2.2.1 shows that the action of G on $V_1 \oplus V_2$ can be realized by a matrix of the form $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ where the upper left block is given by the action of G on V_1 and the lower right block is given by the action of G on V_2. Taking traces finishes the assertion. For part (7), the equivalence $V^* \cong \overline{V}$ shows $\chi_{V^*}(g) = \chi_{\overline{V}}(g)$. From the discussion in §2.2.1 on $\overline{V}$, the matrix realizing the action of g on $\overline{V}$ is the conjugate of the matrix realizing the action of g on V. Taking traces shows $\chi_{\overline{V}}(g) = \overline{\chi_V(g)}$. Similarly, from the discussion in §2.2.1 on V^*, the matrix realizing the action of g on V^* is the inverse transpose of the matrix realizing the action of g on V. Taking traces shows $\chi_{V^*}(g) = \chi_V(g^{-1})$. □

Definition 3.6. If V is a finite-dimensional representation of a Lie group G, let $V^G = \{v \in V \mid gv = v \text{ for } g \in G\}$, i.e., V^G is the isotypic component of V corresponding to the trivial representation.

The next theorem calculates the L^2 inner product of characters corresponding to irreducible representations.

Theorem 3.7. *(1) Let V, W be finite-dimensional representations of a compact Lie group G. Then*

$$\int_G \chi_V(g)\,\overline{\chi_W(g)}\,dg = \dim \operatorname{Hom}_G(V, W).$$

In particular, $\int_G \chi_V(g)\,dg = \dim V^G$ and if V, W are irreducible, then

$$\int_G \chi_V(g)\,\overline{\chi_W(g)}\,dg = \begin{cases} 0 & \text{if } V \ncong W \\ 1 & \text{if } U \cong V. \end{cases}$$

(2) Up to equivalence, V is completely determined by its character, i.e., $\chi_V = \chi_W$ if and only if $V \cong W$. In particular, if V_i are representations of G, then $V \cong \bigoplus_i n_i V_i$ if and only if $\chi_V = \sum_i n_i \chi_{V_i}$.
(3) V is irreducible if and only if $\int_G |\chi_V(g)|^2\, dg = 1$.

Proof. Begin with the assumption that V, W are irreducible. Let $\{v_i\}$ and $\{w_j\}$ be an orthonormal bases for V and W with respect to the G-invariant inner products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_W$. Then

$$\chi_V(g)\,\overline{\chi_W(g)} = \sum_{i,j} (gv_i, v_i)_V\,\overline{(gw_j, w_j)_W},$$

so Schur orthogonality (Theorem 3.3) implies that $\int_G \chi_V(g)\,\overline{\chi_W(g)}\,dg$ is 0 when $V \not\cong W$. When $U \cong V$, $\chi_W = \chi_V$, so Schur orthogonality implies that

$$\int_G \chi_V(g)\,\overline{\chi_V(g)}\,dg = \frac{1}{\dim V} \sum_{i,j} \left|(v_i, v_j)_V\right|^2 = 1.$$

For arbitrary V, W, decompose V and W into irreducible summands as $V \cong \bigoplus_{[\pi]\in\widehat{G}} m_\pi E_\pi$ and $W = \bigoplus_{[\pi]\in\widehat{G}} n_\pi E_\pi$. Hence

$$\begin{aligned}
\int_G \chi_V(g)\,\overline{\chi_W(g)}\,dg &= \sum_{[\pi],[\pi']\in\widehat{G}} m_\pi n_{\pi'} \int_G \chi_{E_\pi}(g)\,\overline{\chi_{E_{\pi'}}(g)}\,dg \\
&= \sum_{[\pi]\in\widehat{G}} m_\pi n_\pi = \sum_{[\pi],[\pi']\in\widehat{G}} m_\pi n_{\pi'} \dim \operatorname{Hom}_G(E_\pi, E_{\pi'}) \\
&= \dim \operatorname{Hom}_G\Big(\bigoplus_{[\pi]\in\widehat{G}} m_\pi E_\pi, \bigoplus_{[\pi]\in\widehat{G}} n_\pi E_\pi\Big) = \dim \operatorname{Hom}_G(V, W).
\end{aligned}$$

The remaining statements follow easily from this result and the calculation of multiplicity in Theorem 2.24. In particular since V^G is the isotypic component of V corresponding to the trivial representation, $\dim \operatorname{Hom}_G(\mathbb{C}, V) = \dim V^G$ and thus $\dim V^G = \int_G \chi_{\mathbb{C}}(g)\,\overline{\chi_V(g)}\,dg = \int_G \overline{\chi_V(g)}\,dg$. Since $\dim V^G$ is a real number, the integrand may be conjugated with impunity and part (1) follows.

For part (2), V is completely determined by the multiplicities $m_\pi = \dim \operatorname{Hom}_G(E_\pi, V)$, $[\pi] \in \widehat{G}$. As this number is calculated by $\int_G \chi_{E_\pi}(g)\,\overline{\chi_V(g)}\,dg$, the representation is completely determined by χ_V. For part (3), V is irreducible if and only if $\dim \operatorname{Hom}_G(V, V) = 1$ by Corollary 2.19. In turn, this this is equivalent to $\int_G \chi_V(g)\,\overline{\chi_V(g)}\,dg = 1$. □

As an application of the power of character theory, we prove a theorem classifying irreducible representations of the direct product of two compact Lie groups, $G_1 \times G_2$, in terms of the irreducible representations of G_1 and G_2. This allows us to eventually focus our study on compact Lie groups that are as small as possible.

Definition 3.8. If V_i is a finite-dimensional representation of a Lie group G_i, $V_1 \otimes V_2$ is a representation of $G_1 \times G_2$ with action given by $(g_1, g_2) \sum_i v_{i_1} \otimes v_{i_2} = \sum_i \left(g_1 v_{i_1}\right) \otimes \left(g_2 v_{i_2}\right)$.

Theorem 3.9. *For compact Lie groups G_i, a finite-dimensional representation W of $G_1 \times G_2$ is irreducible if and only if $W \cong V_1 \otimes V_2$ for finite-dimensional irreducible representations V_i of G_i.*

Proof. If V_i are irreducible representations of G_i, then $\int_{G_i} \left|\chi_{V_i}(g)\right|^2 dg = 1$. Since $\chi_{V_1\otimes V_2}(g_1, g_2) = \chi_{V_1}(g_1)\,\chi_{V_2}(g_2)$ (Exercise 3.3) and since Haar measure on $G_1 \times G_2$ is given by $dg_1\,dg_2$ by uniqueness,

$$\int_{G_1\times G_2} \left|\chi_{\chi_{V_1\otimes V_2}}(g_1, g_2)\right|^2 \, dg_1\, dg_2$$

$$= \left(\int_{G_1} \left|\chi_{V_1}(g_1)\right|^2 \, dg_1\right)\left(\int_{G_2} \left|\chi_{V_2}(g_2)\right|^2 \, dg_2\right) \tag{3.10}$$

$$= 1,$$

so that $V_1 \otimes V_2$ is $G_1 \times G_2$-irreducible.

Conversely, suppose W is $G_1 \times G_2$-irreducible. Identifying G_1 with $G_1 \times \{e\}$ and G_2 with $\{e\} \times G_2$, decompose W with respect to G_2 as

$$\bigoplus_{[\pi]\in\widehat{G_2}} \mathrm{Hom}_{G_2}(E_\pi, W) \otimes E_\pi$$

under the G_2-map Φ induced by $\Phi(T \otimes v) = T(v)$. Recall that G_2 acts trivially on $\mathrm{Hom}_{G_2}(E_\pi, W)$ and view $\mathrm{Hom}_{G_2}(E_\pi, W)$ as a representation of G_1 by setting $(g_1 T)(v) = (g_1, e)T(v)$. Thus $\bigoplus_{[\pi]\in\widehat{G_2}} \mathrm{Hom}_{G_2}(E_\pi, W) \otimes E_\pi$ is a representation of $G_1 \times G_2$ and, in fact, Φ is now a $G_1 \times G_2$-intertwining isomorphism to W since

$$\begin{aligned}(g_1, g_2)\Phi(T \otimes v) &= (g_1, e)(e, g_2)\Phi(T \otimes v) = (g_1, e)\Phi(T \otimes g_2 v)\\ &= (g_1, e)T(g_2 v) = (g_1 T)(g_2 v) = \Phi((g_1 T) \otimes (g_2 v)).\end{aligned}$$

As W is irreducible, there exists exactly one $[\pi] \in \widehat{G_2}$ so that

$$W \cong \mathrm{Hom}_{G_2}(E_\pi, W) \otimes E_\pi.$$

Since E_π is G_2-irreducible, a calculation as in Equation 3.10 shows $\mathrm{Hom}_{G_2}(E_\pi, W)$ is G_1-irreducible as well. □

As a corollary of Theorem 3.9 (Exercise 3.10), it easily follows that $\widehat{G_1 \times G_2} \cong \widehat{G_1} \times \widehat{G_2}$.

3.1.3 Exercises

Exercise 3.1 If V, V' are finite-dimensional unitary representations of a Lie group G with G-invariant inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)'$, show the form $\big((u, u'),\ (v, v')\big) = (u, v) + (u', v')'$ on $V \oplus V'$ is a G-invariant inner product and the form

$$\left(\sum_i u_i \otimes u_i',\ \sum_j v_j \otimes v_j'\right) = \sum_{i,j} (u_i, v_j)(u_i', v_j')'$$

on $V \otimes V'$ is a G-invariant inner product.

Exercise 3.2 Show that any finite-dimensional unitary representation V of a compact Lie group G can be written as a direct sum of irreducible summands that are mutually perpendicular.

Exercise 3.3 Prove the remaining parts of Theorem 3.5. Also, if V_i are finite-dimensional representations of a compact Lie group G_i, show that $\chi_{V_1 \otimes V_2}(g_1, g_2) = \chi_{V_1}(g_1)\, \chi_{V_2}(g_2)$.

Exercise 3.4 Let G be a finite group acting on a finite set M. Define a representation of G on $C(M) = \{f : M \to \mathbb{C}\}$ by $(gf)(m) = f(g^{-1}m)$. Show that $\chi_{C(M)}(g) = |M^g|$ for $g \in G$ where $M^g = \{m \in M \mid gm = m\}$.

Exercise 3.5 **(a)** For the representation $V_n(\mathbb{C}^2)$ of $SU(2)$ from §2.1.2.2, calculate $\chi_{V_n(\mathbb{C}^2)}(g)$ for $g \in SU(2)$ in terms of the eigenvalues of g.
(b) Use a character computation to establish the *Clebsch–Gordan* formula:

$$V_n(\mathbb{C}^2) \otimes V_m(\mathbb{C}^2) \cong \bigoplus_{j=0}^{\min\{n,m\}} V_{n+m-2j}(\mathbb{C}^2).$$

Exercise 3.6 **(a)** For the representations $V_m(\mathbb{R}^3)$ and $\mathcal{H}_m(\mathbb{R}^3)$ of $SO(3)$ from §2.1.2.3, calculate $\chi_{V_m(\mathbb{R}^3)}(g)$ and $\chi_{\mathcal{H}_m(\mathbb{R}^3)}(g)$ for $g \in SO(3)$ of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

(b) For the half-spin representations $S^{\pm}$ of Spin(4) from §2.1.2.4, calculate $\chi_{S^{\pm}}(g)$ for $g \in \mathrm{Spin}(4)$ of the form $(\cos\theta_1 + e_1e_2\sin\theta_1)\,(\cos\theta_2 + e_3e_4\sin\theta_2)$.

Exercise 3.7 Let V be a finite-dimensional representation of G. Show $\chi_{\bigwedge^2 V}(g) = \frac{1}{2}\left(\chi_V(g)^2 - \chi_V(g^2)\right)$ and $\chi_{S^2V} = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2))$. Use this to show that $V \otimes V \cong S^2V \oplus \bigwedge^2 V$ (c.f., Exercise 2.15).

Exercise 3.8 A finite-dimensional representation (π, V) of a compact Lie group G is said to be of *real type* if there is a real vector space V_0 on which G acts that gives rise to the action on V by extension of scalars, i.e., by $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$. It is said to be of *quaternionic type* if there is a quaternionic vector space on which G acts that gives rise to the action on V by restriction of scalars. It is said to be of *complex type* if it is neither real nor quaternionic type.
(a) Show that V is of real type if and only if V possesses an invariant nondegenerate symmetric bilinear form. Show that V is of quaternionic type if and only if V possesses an invariant nondegenerate skew-symmetric bilinear form.
(b) Show that the set of G-invariant bilinear forms on V are given by $\mathrm{Hom}_G(V \otimes V, \mathbb{C}) \cong \mathrm{Hom}_G(V, V^*)$ (c.f., Exercise 2.15).
(c) For the remainder of the problem, let V be irreducible. Show that V is of complex type if and only if $V \ncong V^*$. When $V \cong V^*$, use Exercise 3.7 to conclude that V is of real or quaternionic type, but not both.
(d) Using Theorem 3.7 and the character formulas in Exercise 3.7, show that

$$\int_G \chi_V(g^2)\, dg = \begin{cases} 1 & \text{if } V \text{ is of real type} \\ 0 & \text{if } V \text{ is of complex type} \\ -1 & \text{if } V \text{ is of quaternionic type.} \end{cases}$$

(e) If χ_V is real valued, show that V is of real or quaternionic type.

Exercise 3.9 Let (π, V) be a finite-dimensional representation of a compact Lie group G. Use unitarity and an eigenspace decomposition to show $|\chi_V(g)| \leq \dim V$ with equality if and only if $\pi(g)$ is multiplication by a scalar.

Exercise 3.10 Let $[\pi_i] \in \widehat{G_i}$ for compact Lie groups G_i. Now show that the map $(E_{\pi_1}, E_{\pi_2}) \to E_{\pi_1} \otimes E_{\pi_2}$ induces an isomorphism $\widehat{G_1} \times \widehat{G_2} \cong \widehat{G_1 \times G_2}$.

3.2 Infinite-Dimensional Representations

In many applications it is important to remove the finite-dimensional restriction from the definition of a representation. As infinite-dimensional spaces are a bit more tricky than finite-dimensional ones, this requires a slight reworking of a few definitions. None of these modifications affect the finite-dimensional setting. Once these adjustments are made, it is perhaps a bit disappointing that the infinite-representation theory for compact Lie groups reduces to the finite-dimensional theory.

3.2.1 Basic Definitions and Schur's Lemma

Recall that a *topological vector space* is a vector space equipped with a topology so that vector addition and scalar multiplication are continuous. If V and V' are topological vector spaces, write $\text{Hom}(V, V')$ for the set of *continuous* linear transformations from V to V' and write $GL(V)$ for the set of invertible elements of $\text{Hom}(V, V)$.

The following definition (c.f. Definitions 2.1, 2.2, and 2.11) provides the necessary modifications to allow the study of infinite-dimensional representations. As usual in infinite dimensional settings, the main additions consist of explicitly requiring the action of the Lie group to be continuous in both variables and liberal use of the adjectives continuous and closed.

Definition 3.11. **(1)** A *representation* of a Lie group G on a topological vector space V is a pair (π, V), where $\pi : G \to GL(V)$ is a homomorphism and the map $G \times V \to V$ given by $(g, v) \to \pi(g)v$ is *continuous*.
(2) If (π, V) and (π', V') are representations on topological vector spaces, $T \in \text{Hom}(V, V')$ is called an *intertwining operator* or G*-map* if $T \circ \pi = \pi' \circ T$.
(3) The set of all G-maps is denoted by $\text{Hom}_G(V, V')$.
(4) The representations V and V' are *equivalent*, $V \cong V'$, if there exists a bijective G-map from V to V'.
(5) A subspace $U \subseteq V$ is G*-invariant* if $gU \subseteq U$ for $g \in G$. Thus when U is closed, U is a representation of G in its own right and is also called a *submodule* or a *subrepresentation*.
(6) A nonzero representation V is *irreducible* if the only *closed* G-invariant subspaces are $\{0\}$ and V. A nonzero representation is called *reducible* if there is a proper *closed* G-invariant subspace of V.

For the most part, the interesting topological vector space representations we will examine will be *unitary representations* on *Hilbert spaces*, i.e., representations on complete inner product spaces where the inner product is invariant under the Lie group (Definition 2.14). More generally, many of the results are applicable to Hausdorff *locally convex topological spaces* and especially to *Fréchet spaces* (see [37]). Recall that locally convex topological spaces are topological vector spaces whose topology is defined by a family of seminorms. A Fréchet space is a complete locally convex Hausdorff topological spaces whose topology is defined by a countable family of seminorms.

As a first example of an infinite-dimensional unitary representation on a Hilbert space, consider the action of S^1 on $L^2(S^1)$ given by $\left(\pi(e^{i\theta})f\right)(e^{i\alpha}) = f(e^{i(\alpha-\theta)})$ for $e^{i\theta} \in S^1$ and $f \in L^2(S^1)$. We will soon see (Lemma 3.20) that this example generalizes to any compact Lie group.

Next we upgrade Schur's Lemma (Theorem 2.12) to handle unitary representations on Hilbert spaces.

Theorem 3.12 (Schur's Lemma). *Let V and W be unitary representations of a Lie group G on Hilbert spaces. If V and W are irreducible, then*

$$\dim \operatorname{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W. \end{cases}$$

In general, the representation V is irreducible if and only if $\operatorname{Hom}_G(V, V) = \mathbb{C}I$.

Proof. Start with V and W irreducible. If $T \in \operatorname{Hom}_G(V, W)$ is nonzero, then $\ker T$ is closed, not all of V, and G-invariant, so irreducibility implies $\ker T = \{0\}$. Similarly, the image of T is nonzero and G-invariant, so continuity and irreducibility imply that $\overline{\text{range}\, T} = W$.

Using the definition of the adjoint map of T, $T^* : W \to V$, it immediately follows that $T^* \in \operatorname{Hom}_G(W, V)$ and that T^* is nonzero, injective, and has dense range (Exercise 3.11). Let $S = T^* \circ T \in \operatorname{Hom}_G(V, V)$ so that $S^* = S$. In the finite-dimensional case, we used the existence of an eigenvalue to finish the proof. In the infinite-dimensional setting however, eigenvalues (point spectrum) need not generally exist. To clear this hurdle, we invoke a standard theorem from a functional analysis course.

The Spectral Theorem for normal bounded operators (see [74] or [30] for details) says that there exists a projection valued measure E so that $S = \int_{\sigma(S)} \lambda \, dE$, where $\sigma(S)$ is the spectrum of S. It has the nice property that the only bounded endomorphisms of V commuting with S are the ones commuting with each self-adjoint projection $E(\Delta)$, Δ a Borel subset of $\sigma(S)$. In terms of understanding the notation $\int_{\sigma(S)} \lambda \, dE$, The Spectral Theorem also says that S is the limit, in the operator norm, of operators of the form $\sum_i \lambda_i E(\Delta_i)$ where $\{\Delta_i\}$ is a partition of $\sigma(S)$ and $\lambda_i \in \Delta_i$.

Since $S \in \operatorname{Hom}_G(V, V)$, $\pi(g)$ commutes with $E(\Delta)$ for each $g \in G$, so that $E(\Delta) \in \operatorname{Hom}_G(V, V)$. It has already been shown that nonzero elements of $\operatorname{Hom}_G(V, V)$ are injective. As $E(\Delta)$ is a projection, it must therefore be 0 or I. Thus $\sum_i \lambda_i E(\Delta_i) = kI$ for some (possibly zero) constant k. In particular, S is a multiple of the identity. Since S is injective, S is a nonzero multiple of the identity. It follows that T is invertible and that $V \cong W$.

Now suppose $T_i \in \mathrm{Hom}_G(V, W)$ are nonzero. Let $S = T_2^{-1} \circ T_1 \in \mathrm{Hom}_G(V, V)$ and write $S = \frac{1}{2}[(S + S^*) - i(iS - iS^*)]$. Using the same argument as above applied to the self-adjoint intertwining operators $S + S^*$ and $iS - iS^*$, it follows that S is a multiple of the identity. This proves the first statement of the theorem.

To prove the second statement, it only remains to show $\dim \mathrm{Hom}_G(V, V) \geq 2$ when V is not irreducible. If $U \subseteq V$ is a proper closed G-invariant subspace, then so is $U^\perp$ by unitarity. The two orthogonal projections onto U and $U^\perp$ do the trick. □

3.2.2 *G*-Finite Vectors

Throughout the rest of the book there will be numerous occasions where *vector-valued integration* on compact sets is required. In a finite-dimensional vector space, a basis can be chosen and then integration can be done coordinate-by-coordinate. For instance, vector-valued integration in this setting was already used in the proof of Theorem 3.7 for the definition of $\widetilde{T}_{u,v}$. Obvious generalizations can be made to Hilbert spaces by tossing in limits. In any case, functional analysis provides a general framework for this type of operation which we recall now (see [74] for details). Remember that G is still a compact Lie group throughout this chapter.

Let V be a Hausdorff locally convex topological space and $F : G \to V$ a continuous function. Then there exists a unique element in V, called

$$\int_G f(g)\, dg,$$

so that $T\left(\int_G f(g)\, dg\right) = \int_G T(f(g))\, dg$ for each $T \in \mathrm{Hom}(V, \mathbb{C})$. If V is a Fréchet space, $\int_G f(g)\, dg$ is the limit of elements of the form

$$\sum_{i=1}^{n} f(g_i)\, dg(\Delta_i),$$

where $\{\Delta_i\}_{i=1}^n$ is a finite Borel partition of G, $g_i \in \Delta_i$, and $dg(\Delta_i)$ is the measure of Δ_i with respect to the invariant measure.

Recall that a linear map T on V is *positive* if $(Tv, v) \geq 0$ for all $v \in V$ and strictly greater than zero for some v. The linear map T is *compact* if the closure of the image of the unit ball under T is compact. It is a standard fact from functional analysis that the set of compact operators is a closed left and right ideal under composition within the set of bounded operators (e.g., [74] or [30]).

We now turn our attention to finding a *canonical decomposition* (Theorem 2.24) suitable for unitary representations on Hilbert spaces. The hardest part is getting started. In fact, the heart of the matter is really contained in Lemma 3.13 below.

Lemma 3.13. *Let (π, V) be a unitary representation of a compact Lie group G on a Hilbert space. There exists a nonzero finite-dimensional G-invariant (closed) subspace of V.*

Proof. Begin with any self-adjoint positive compact operator $T_0 \in \operatorname{Hom}(V, V)$, e.g., any nonzero finite rank projection will work. Using vector-valued integration in $\operatorname{Hom}(V, V)$, define

$$T = \int_G \pi(g) \circ T_0 \circ \pi(g)^{-1}\, dg.$$

Since T is the limit in norm of operators of the form $\sum_i dg(\Delta_i)\, \pi(g_i) \circ T_0 \circ \pi(g_i)^{-1}$ with $g_i \in \Delta_i \subseteq G$, T is still a compact operator. T is G-invariant since dg is left invariant (e.g., see the proof of Theorem 3.7 and the operator $\widetilde{T}_{u,v}$). Using the positivity of T_0, T is seen to be nonzero by calculating

$$(Tv, v) = \int_G (\pi(g)T_0\pi(g)^{-1}v,\ v)\, dg = \int_G (T_0\pi(g)^{-1}v,\ \pi(g)^{-1}v)\, dg,$$

where $(\cdot, \cdot)$ is the invariant inner product on V. Since V is unitary, the adjoint of $\pi(g)$ is $\pi(g)^{-1}$. Using the fact that T_0 is self-adjoint, it therefore follows that T is also self-adjoint.

An additional bit of functional analysis is needed to finish the proof. Use the Spectral Theorem for compact self-adjoint operators (see [74] or [30] for details) to see that T possesses a nonzero eigenvalue λ whose corresponding (nonzero) eigenspace is finite dimensional. This eigenspace, i.e., $\ker(T - \lambda I)$, is the desired nonzero finite-dimensional G-invariant subspace of V. $\square$

If $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ are Hilbert spaces with inner products $(\cdot, \cdot)_\alpha$, recall that the *Hilbert space direct sum* is

$$\widehat{\bigoplus_{\alpha \in \mathcal{A}}} V_\alpha = \{(v_\alpha) \mid v_\alpha \in V_\alpha \text{ and } \sum_{\alpha \in \mathcal{A}} \|v_\alpha\|_\alpha^2 < \infty\}.$$

$\widehat{\bigoplus}_\alpha V_\alpha$ is a Hilbert space with inner product $\left((v_\alpha), (v'_\alpha)\right) = \sum_\alpha (v_\alpha, v'_\alpha)_\alpha$ and contains $\bigoplus_\alpha V_\alpha$ as a dense subspace with $V_\alpha \perp V_\beta$ for distinct $\alpha, \beta \in \mathcal{A}$.

Definition 3.14. If V is a representation of a Lie group G on a topological vector space, the set of *G-finite vectors* is the set of all $v \in V$ so that Gv generates a finite-dimensional subspace, i.e.,

$$V_{G\text{-fin}} = \{v \in V \mid \dim(\operatorname{span}\{gv \mid g \in G\}) < \infty\}.$$

The next corollary shows that we do not really get anything new by allowing infinite-dimensional unitary Hilbert space representations.

Corollary 3.15. *Let (π, V) be a unitary representation of a compact Lie group G on a Hilbert space. There exists finite-dimensional irreducible G-submodules $V_\alpha \subseteq V$ so that*

$$V = \widehat{\bigoplus_\alpha} V_\alpha.$$

In particular, the irreducible unitary representations of G are all finite dimensional. Moreover, the set of G-finite vectors is dense in V.

Proof. Zorn's Lemma says that any partially ordered set has a maximal element if every linearly ordered subset has an upper bound. With this in mind, consider the collection of all sets $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ satisfying the properties: (1) each V_α is finite-dimensional, G-invariant, and irreducible; and (2) $V_\alpha \perp V_\beta$ for distinct $\alpha, \beta \in \mathcal{A}$. Partially order this collection by inclusion. By taking a union, every linearly ordered subset clearly has an upper bound. Let $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ be a maximal element. If $\widehat{\bigoplus}_\alpha V_\alpha \neq V$, then $\left(\widehat{\bigoplus}_\alpha V_\alpha\right)^\perp$ is closed, nonempty, and G-invariant, and so a unitary Hilbert space representation in its own right. In particular, Lemma 3.13 and Corollary 2.17 imply that there exists a finite dimensional, G-invariant, irreducible submodule $V_\gamma \subseteq \left(\widehat{\bigoplus}_\alpha V_\alpha\right)^\perp$. This, however, violates maximality and the corollary is finished. □

As was the case in §2.2.4, the above decomposition is not canonical. This situation will be remedied next in §3.2.3 below.

3.2.3 Canonical Decomposition

First, we update the notion of isotypic component from Definition 2.22 in order to handle infinite-dimensional unitary representations. The only real change replaces direct sums with Hilbert space direct sums.

Definition 3.16. Let V be a unitary representation of a compact Lie group G on a Hilbert space. For $[\pi] \in \widehat{G}$, let $V_{[\pi]}$ be the largest subspace of V that is a Hilbert space direct sum of irreducible submodules equivalent to E_π. The submodule $V_{[\pi]}$ is called the *π-isotypic component* of V.

As in the finite-dimensional case, the above definition of the isotypic component $V_{[\pi]}$ is well defined and $V_{[\pi]}$ is the closure of the sum of all submodules of V equivalent

to E_π These statements are verified using Zorn's Lemma in a fashion similar to the proof of Corollary 3.15 (Exercise 3.12).

Lemma 3.17. *Let V be a unitary representation of a compact Lie group G on a Hilbert space with invariant inner product $(\cdot,\cdot)_V$ and let E_π, $[\pi] \in \widehat{G}$, be an irreducible representation of G with invariant inner product $(\cdot,\cdot)_{E_\pi}$. Then* $\mathrm{Hom}_G(E_\pi, V)$ *is a Hilbert space with a G-invariant inner product $(\cdot,\cdot)_{\mathrm{Hom}}$ defined by $(T_1, T_2)_{\mathrm{Hom}} I = T_2^* \circ T_1$. It satisfies*

$$(T_1, T_2)_{\mathrm{Hom}}\,(x_1, x_2)_{E_\pi} = (T_1 x_1, T_2 x_2)_V \tag{3.18}$$

for $T_i \in \mathrm{Hom}_G(E_\pi, V)$ and $x_i \in E_\pi$. Moreover, $\|T\|_{\mathrm{Hom}}$ is the same as the operator norm of T.

Proof. The adjoint of T_2, $T_2^* \in \mathrm{Hom}(V, E_\pi)$, is still a G-map since

$$
\begin{aligned}
(T_2^*(gv), x)_{E_\pi} &= (gv, T_2 x)_V = (v, T_2\left(g^{-1}x\right))_V = (T_2^* v, g^{-1}x)_{E_\pi} \\
&= (gT_2^* v, x)_{E_\pi}
\end{aligned}
$$

for $x \in E_\pi$ and $v \in V$. Thus $T_2^* \circ T_1 \in \mathrm{Hom}(E_\pi, E_\pi)$. Schur's Lemma implies that there is a scalar $(T_1, T_2)_{\mathrm{Hom}} \in \mathbb{C}$, so that $(T_1, T_2)_{\mathrm{Hom}} I = T_2^* \circ T_1$.

By definition, $(\cdot, \cdot)_{\mathrm{Hom}}$ is clearly a Hermitian form on $\mathrm{Hom}_G(E_\pi, V)$ and

$$
\begin{aligned}
(T_1 x_1, T_2 x_2)_V &= (T_2^* \left(T_1 x_1\right), x_2)_{E_\pi} = ((T_1, T_2)_{\mathrm{Hom}}\, x_1, x_2)_{E_\pi} \\
&= (T_1, T_2)_{\mathrm{Hom}}\, (x_1, x_2)_{E_\pi}.
\end{aligned}
$$

In particular, for $T \in \mathrm{Hom}_G(E_\pi, V)$, $\|T\|_{\mathrm{Hom}}$ is the quotient of $\|Tx\|_V$ and $\|x\|_{E_\pi}$ for any nonzero $x \in E_\pi$. Thus $\|T\|_{\mathrm{Hom}}$ is the same as the operator norm of T viewed as an element of $\mathrm{Hom}(E_\pi, V)$. Hence $(\cdot, \cdot)_{\mathrm{Hom}}$ is an inner product making $\mathrm{Hom}_G(E_\pi, V)$ into a Hilbert space. □

Note Equation 3.18 is independent of the choice of invariant inner product on E_π. To see this directly, observe that scaling $(\cdot, \cdot)_{E_\pi}$ scales T_2^*, and therefore $(\cdot, \cdot)_{\mathrm{Hom}_G(E_\pi, V)}$, by the inverse scalar so that the product of $(\cdot, \cdot)_{E_\pi}$ and $(\cdot, \cdot)_{\mathrm{Hom}_G(E_\pi, V)}$ remains unchanged.

If V_i are Hilbert spaces with inner products $(\cdot, \cdot)_i$, recall that the *Hilbert space tensor product*, $V_1 \widehat{\otimes} V_2$, is the completion of $V_1 \otimes V_2$ with respect to the inner product generated by $(v_1 \otimes v_2, v_1' \otimes v_2') = (v_1, v_1')\,(v_2, v_2')$ (c.f. Exercise 3.1).

Theorem 3.19 (Canonical Decomposition). *Let V be a unitary representation of a compact Lie group G on a Hilbert space.*
(1) There is a G-intertwining unitary isomorphism ι_π

$$
\mathrm{Hom}_G(E_\pi, V) \widehat{\otimes} E_\pi \xrightarrow{\cong} V_{[\pi]}
$$

induced by $\iota_\pi\,(T \otimes v) = T(v)$ for $T \in \mathrm{Hom}_G(E_\pi, V)$ and $v \in V$.
(2) There is a G-intertwining unitary isomorphism

$$
\widehat{\bigoplus_{[\pi] \in \widehat{G}}} \mathrm{Hom}_G(E_\pi, V) \widehat{\otimes} E_\pi \xrightarrow{\cong} V = \widehat{\bigoplus_{[\pi] \in \widehat{G}}} V_{[\pi]}.
$$

Proof. As in the proof of Theorem 2.24, ι_π is a well-defined G-map from $\mathrm{Hom}_G(E_\pi, V) \otimes E_\pi$ to $V_{[\pi]}$ with dense range (since $V_{[\pi]}$ is a Hilbert space direct sum of irreducible submodules instead of finite direct sum as in Theorem 2.24). As Lemma 3.17 implies ι_π is unitary on $\mathrm{Hom}_G(E_\pi, V) \otimes E_\pi$, it follows that ι_π is injective and uniquely extends by continuity to a G-intertwining unitary isomorphism from $\mathrm{Hom}_G(E_\pi, V) \widehat{\otimes} E_\pi$ to $V_{[\pi]}$. Finally, V is the closure of $\sum_{[\pi] \in \widehat{G}} V_{[\pi]}$ by Corollary 3.15 and the sum is orthogonal by Corollary 2.21. □

3.2.4 Exercises

Exercise 3.11 Let V and W be unitary representations of a compact Lie group G on Hilbert spaces and let $T \in \mathrm{Hom}_G(V, W)$ be injective with dense range. Show that $T^* \in \mathrm{Hom}_G(W, V)$, T^* is injective, and that T^* has dense range.

Exercise 3.12 Let V be a unitary representation of a compact Lie group G on a Hilbert space and let $[\pi] \in \widehat{G}$.
(a) Consider the collection of all sets $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ satisfying the properties: (1) each V_α is a submodule of V isomorphic to E_π and (2) $V_\alpha \perp V_\beta$ for distinct $\alpha, \beta \in \mathcal{A}$. Partially order this collection by inclusion and use Zorn's Lemma to show that there is a maximal element.
(b) Write $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ for the maximal element. Show that the orthogonal projection $P : V \to \left(\widehat{\bigoplus}_{\alpha \in \mathcal{A}} V_\alpha\right)^\perp$ is a G-map. If $V_\gamma \subseteq V$ is any submodule equivalent to E_π, use irreducibility and maximality to show that $PV_\gamma = \{0\}$.
(c) Show that the definition of the isotypic component $V_{[\pi]}$ in Definition 3.16 is well defined and that $V_{[\pi]}$ is the closure of the sum of all submodules of V equivalent to E_π.

Exercise 3.13 Recall that $\widehat{S^1} \cong \mathbb{Z}$ via the one-dimensional representations $\pi_n(e^{i\theta}) = e^{in\theta}$ for $n \in \mathbb{Z}$ (Exercise 2.21). View $L^2(S^1)$ as a unitary representation of S^1 under the action $(e^{i\theta} \cdot f)(e^{i\alpha}) = f(e^{i(\alpha-\theta)})$ for $f \in L^2(S^1)$. Calculate $\mathrm{Hom}_{S^1}(\pi_n, L^2(S^1))$ and conclude that $L^2(S^1) = \widehat{\bigoplus}_{n\in\mathbb{Z}} \mathbb{C}e^{in\theta}$.

Exercise 3.14 Use Exercise 2.28 and Theorem 2.33 to show that

$$L^2(S^{n-1}) = \widehat{\bigoplus}_{m\in\mathbb{N}} \mathcal{H}_m(\mathbb{R}^n)|_{S^{n-1}}, \quad n \geq 2,$$

is the canonical decomposition of $L^2(S^{n-1})$ under $O(n)$ (or $SO(n)$ for $n \geq 3$) with respect to usual action $(gf)(v) = f(g^{-1}v)$.

Exercise 3.15 Recall that the irreducible unitary representations of $\mathbb{R}$ are given by the one-dimensional representations $\pi_r(x) = e^{irx}$ for $r \in \mathbb{R}$ (Exercise 2.21) and consider the unitary representation of $\mathbb{R}$ on $L^2(\mathbb{R})$ under the action $(x \cdot f)(y) = f(x-y)$ for $f \in L^2(\mathbb{R})$. Show $L^2(\mathbb{R}) \neq \widehat{\bigoplus}_{r\in\mathbb{R}} L^2(\mathbb{R})_{\pi_r}$ by showing $L^2(\mathbb{R})_{\pi_r} = \{0\}$.

3.3 The Peter–Weyl Theorem

Let G be a compact Lie group. In this section we decompose $L^2(G)$ under left and right translation of functions. The canonical decomposition reduces the work to calculating $\mathrm{Hom}_G(E_\pi, L^2(G))$. Instead of attacking this problem directly, it turns out to be easy (Lemma 3.23) to calculate that $\mathrm{Hom}_G(E_\pi, C(G)_{G\text{-fin}})$. Using the Stone–Weierstrass Theorem (Theorem 3.25), it is shown that $C(G)_{G\text{-fin}}$ is dense in $L^2(G)$. In turn, this density result allows the calculation of $\mathrm{Hom}_G(E_\pi, L^2(G))$.

3.3.1 The Left and Right Regular Representation

The set of continuous functions on a compact Lie group G, $C(G)$, is a Banach space with respect to the norm $\|f\|_{C(G)} = \sup_{g\in G} |f(g)|$ and the set of square integrable functions, $L^2(G)$, is a Hilbert space with respect to the norm $\|f\|_{L^2(G)} = \int_G |f(g)|^2 \, dg$. Both spaces carry a left and right action l_g and r_g of G given by

$$\left(l_g f\right)(h) = f(g^{-1}h)$$
$$\left(r_g f\right)(h) = f(hg)$$

which, as the next theorem shows, are representations. They are called the *left* and *right regular representations*.

Lemma 3.20. *The left and right actions of a compact Lie group G on $C(G)$ and $L^2(G)$ are representations and norm preserving.*

Proof. The only statement from Definition 3.11 that still requires checking is continuity of the map $(g, f) \to l_g f$ (since r_g is handled similarly). Working in $C(G)$ first, calculate

$$\begin{aligned}\left|f_1(g_1^{-1}h) - f_2(g_2^{-1}h)\right| &\le \left|f_1(g_1^{-1}h) - f_1(g_2^{-1}h)\right| + \left|f_1(g_2^{-1}h) - f_2(g_2^{-1}h)\right| \\ &\le \left|f_1(g_1^{-1}h) - f_1(g_2^{-1}h)\right| + \|f_1 - f_2\|_{C(G)}\,.\end{aligned}$$

Since f_1 is continuous on compact G and since the map $g \to g^{-1}h$ is continuous, it follows that $\left\|l_{g_1} f_1 - l_{g_2} f_2\right\|_{C(G)}$ can be made arbitrarily small by choosing (g_1, f_1) sufficiently close to (g_2, f_2).

Next, working with $f_i \in L^2(G)$, choose $f \in C(G)$ and calculate the following:

$$\begin{aligned}\left\|l_{g_1} f_1 - l_{g_2} f_2\right\|_{L^2(G)} &= \left\|f_1 - l_{g_1^{-1}g_2} f_2\right\|_{L^2(G)} \\ &\le \|f_1 - f_2\|_{L^2(G)} + \left\|f_2 - l_{g_1^{-1}g_2} f_2\right\|_{L^2(G)} \\ &= \|f_1 - f_2\|_{L^2(G)} + \left\|l_{g_1} f_2 - l_{g_2} f_2\right\|_{L^2(G)} \\ &\le \|f_1 - f_2\|_{L^2(G)} + \left\|l_{g_1} f_2 - l_{g_1} f\right\|_{L^2(G)} \\ &\quad + \left\|l_{g_1} f - l_{g_2} f\right\|_{L^2(G)} + \left\|l_{g_2} f - l_{g_2} f_2\right\|_{L^2(G)} \\ &= \|f_1 - f_2\|_{L^2(G)} + 2\,\|f_2 - f\|_{L^2(G)} + \left\|l_{g_1} f - l_{g_2} f\right\|_{L^2(G)} \\ &\le \|f_1 - f_2\|_{L^2(G)} + 2\,\|f_2 - f\|_{L^2(G)} + \left\|l_{g_1} f - l_{g_2} f\right\|_{C(G)}\,.\end{aligned}$$

Since f may be chosen arbitrarily close to f_2 in the L^2 norm and since G already acts continuously on $C(G)$, the result follows. □

The first important theorem identifies the G-finite vectors of $C(G)$ with the set of matrix coefficients, $MC(G)$. Even though there are two actions of G on $C(G)$, i.e., l_g and r_g, it turns out that both actions produce the same set of G-finite vectors (Theorem 3.21). As a result, write $C(G)_{G\text{-fin}}$ unambiguously for the set of G-finite vectors with respect to either action.

Theorem 3.21. *(1) For a compact Lie group G, the set of G-finite vectors of $C(G)$ with respect to left action, l_g, coincides with set of G-finite vectors of $C(G)$ with respect to right action, r_g.*
(2) $C(G)_{G\text{-fin}} = MC(G)$.

Proof. We first show that $C(G)_{G\text{-fin}}$, with respect to left action, is the set of matrix coefficients. Let $f_{u,v}^V(g) = (gu, v)$ be a matrix coefficient for a finite-dimensional unitary representation V of G with $u, v \in V$ and G-invariant inner product $(\cdot,\cdot)$. Then $(l_g f_{u,v}^V)(h) = (g^{-1}hu, v) = (hu, gv)$ so that $l_g f_{u,v}^V = f_{u,gv}^V$. Hence $\{l_g f_{u,v}^V \mid g \in G\} \subseteq \{f_{u,v'}^V \mid v' \in V\}$. Since V is finite dimensional, $f_{u,v}^V \in C(G)_{G\text{-fin}}$, and thus $MC(G) \subseteq C(G)_{G\text{-fin}}$.

Conversely, let $f \in C(G)_{G\text{-fin}}$. By definition, there is a finite-dimensional submodule, $V \subseteq C(G)$, with respect to the left action so that $f \in V$. Since $\overline{gf} = g\overline{f}$, $\overline{V} = \{\overline{v} \mid v \in V\}$ is also a finite-dimensional submodule of $C(G)$. Write $(\cdot,\cdot)$ for the L^2 norm restricted to $\overline{V}$. The linear functional on $\overline{V}$ that evaluates functions at e is continuous, so there exists $\overline{v}_0 \in \overline{V}$ so that $\overline{v}(e) = (\overline{v}, \overline{v}_0)$ for $\overline{v} \in \overline{V}$. In particular, $\overline{f}(g) = l_{g^{-1}}\overline{f}(e) = (l_{g^{-1}}\overline{f}, \overline{v}_0) = (\overline{f}, l_g\overline{v}_0)$. In particular, $f = f_{\overline{v}_0,\overline{f}}^{\overline{V}} \in MC(G)$. Thus $C(G)_{G\text{-fin}} \subseteq MC(G)$ and part (2) is done (with respect to the left action).

For part (1), let f be a left G-finite vector in $C(G)$. By the above paragraph, there is a matrix coefficient, so $f = f_{u,v}^V$. Thus $(r_g f)(h) = (hgu, v)$ so that $r_g f = f_{gu,v}^V$. Since $\{gu \mid g \in G\}$ is contained in the finite-dimensional space V, it follows that the set of left G-finite vectors are contained in the set of right G-finite vectors.

Conversely, let f be a right G-finite vector. As before, pick a finite-dimensional submodule, $V \subseteq C(G)$, with respect to the left action so that $f \in V$. Write $(\cdot,\cdot)$ for the L^2 norm restricted to V. The linear functional on V that evaluates functions at e is continuous so there exists $v_0 \in V$, so that $v(e) = (v, v_0)$ for $v \in V$. In particular, $f(g) = r_g f(e) = (r_g f, v_0)$. In particular, $f = f_{f,v_0}^V \in MC(G)$, so that the set of right G-finite vectors is contained in the set of left G-finite vectors. □

Based on our experience with the canonical decomposition, we hope $C(G)_{G\text{-fin}}$ decomposes under the *left* action into terms isomorphic to

$$\mathrm{Hom}_G(E_\pi, C(G)_{G\text{-fin}}) \otimes E_\pi$$

for $[\pi] \in \widehat{G}$. In this case, l_g acts trivially on $\mathrm{Hom}_G(E_\pi, C(G)_{G\text{-fin}})$ so that E_π carries the entire left action. However, Theorem 3.21 says that $C(G)_{G\text{-fin}}$ is actually a $G \times G$-module under the action $((g_1, g_2)f)(g) = \left(r_{g_1} l_{g_2} f\right)(g) = f(g_2^{-1} g g_1)$. In light of Theorem 3.9, it is therefore reasonable to hope $\mathrm{Hom}_G(E_\pi, C(G)_{G\text{-fin}})$ will carry the *right* action. This, of course, requires a different action on $\mathrm{Hom}_G(E_\pi, C(G)_{G\text{-fin}})$ than the trivial action defined in §2.2.1. Towards this end and with respect to the left action on $C(G)_{G\text{-fin}}$, define a second action of G on $\mathrm{Hom}_G(E_\pi, C(G)_{G\text{-fin}})$ and $\mathrm{Hom}_G(E_\pi, C(G))$ by

$$(gT)(x) = r_g(Tx) \tag{3.22}$$

for $g \in G$, $x \in E_\pi$, and $T \in \mathrm{Hom}_G(E_\pi, C(G))$. To verify this is well defined, calculate

$$l_{g_1}((g_2T)(x)) = l_{g_1} r_{g_2}(Tx) = r_{g_2} l_{g_1}(Tx) = r_{g_2}(T(g_1x)) = ((g_2T)(g_1x)),$$

so that $g_2T \in \mathrm{Hom}_G(E_\pi, C(G))$. If $T \in \mathrm{Hom}_G(E_\pi, C(G)_{G\text{-fin}})$, then $g_2T \in \mathrm{Hom}_G(E_\pi, C(G)_{G\text{-fin}})$ as well by Theorem 3.21.

The next lemma is a special case of *Frobenius Reciprocity* in §7.4.1. It does not depend on the fact that E_π is irreducible.

Lemma 3.23. *With respect to the left action of a compact Lie group G on $C(G)_{G\text{-fin}}$ and the action on* $\mathrm{Hom}_G(E_\pi, C(G)_{G\text{-fin}})$ *given by Equation 3.22,*

$$\mathrm{Hom}_G(E_\pi, C(G)) = \mathrm{Hom}_G(E_\pi, C(G)_{G\text{-fin}}) \cong E_\pi^*$$

as G-modules. The intertwining map is induced by mapping $T \in \mathrm{Hom}_G(E_\pi, C(G)_{G\text{-fin}})$ to $\lambda_T \in E_\pi^*$ where

$$\lambda_T(x) = (T(x))(e)$$

for $x \in E_\pi$.

Proof. Let $T \in \mathrm{Hom}_G(E_\pi, C(G)_{G\text{-fin}})$ and define λ_T as in the statement of the lemma. This is a G-map since

$$\begin{aligned}(g\lambda_T)(x) &= \lambda_T(g^{-1}x) = (T(g^{-1}x))(e) = (l_{g^{-1}}(Tx))(e) = (Tx)(g) \\ &= (r_g(Tx))(e) = ((gT)(x))(e) = \lambda_{gT}(x),\end{aligned}$$

so $g\lambda_T = \lambda_{gT}$ for $g \in G$.

We claim that the inverse map is obtained by mapping $\lambda \in E_\pi^*$ to

$$T_\lambda \in \mathrm{Hom}_G(E_\pi, C(G)_{G\text{-fin}})$$

by

$$(T_\lambda(x))(h) = \lambda(h^{-1}x)$$

for $h \in G$. To see that this is well defined, calculate

$$(l_g(T_\lambda(x)))(h) = (T_\lambda(x))(g^{-1}h) = \lambda(h^{-1}gx) = (T_\lambda(gx))(h)$$

so that $l_g(T_\lambda(x)) = T_\lambda(gx)$. This shows that T_λ is a G-map and, since E_π is finite dimensional, $T_\lambda(x) \in C(G)_{G\text{-fin}}$. To see that this operation is the desired inverse, calculate

$$\lambda_{T_\lambda}(x) = (T_\lambda(x))(e) = \lambda(x)$$

and

$$\left(T_{\lambda_T}(x)\right)(h) = \lambda_T(h^{-1}x) = (T(h^{-1}x))(e) = (l_{h^{-1}}(Tx))(e) = (Tx)(h).$$

Hence $\mathrm{Hom}_G(E_\pi, C(G)_{G\text{-fin}}) \cong E_\pi^*$.

To see $\mathrm{Hom}_G(E_\pi, C(G)_{G\text{-fin}}) = \mathrm{Hom}_G(E_\pi, C(G))$, observe that the map $T \to \lambda_T$ is actually a well-defined map from $\mathrm{Hom}_G(E_\pi, C(G))$ to E_π^*. Since the inverse is still given by $\lambda \to T_\lambda$ and $T_\lambda \in \mathrm{Hom}_G(E_\pi, C(G)_{G\text{-fin}})$, the proof is finished. □

Note that if E_π^* inherits an invariant inner product from E_π in the usual fashion, the above isomorphism need not be unitary with respect to the inner product on $\mathrm{Hom}_G(E_\pi, C(G))$ given in Lemma 3.17. In fact, they can be off by a scalar multiple determined by $\dim E_\pi$. The exact relationship will be made clear in §3.4.

3.3.2 Main Result

For $n \in \mathbb{Z}$, consider the representation (π_n, E_{π_n}) of S^1 where $E_{\pi_n} = \mathbb{C}$ and $\pi_n : S^1 \to GL(1,\mathbb{C})$ is given by $(\pi_n(g))(x) = g^n x$ for $g \in S^1$ and $x \in E_{\pi_n}$. In so doing, we realize the isomorphism $\mathbb{Z} \cong \widehat{S^1}$ (c.f. Exercise 3.13). Define the function $f_n : S^1 \to \mathbb{C}$ by $f_n(g) = g^n$. Standard results from Fourier analysis show that $\{f_n \mid n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(S^1)$. By mapping $1 \in E_{\pi_n} \to f_n$, we could say that there is an is an induced isomorphism $\widehat{\bigoplus}_{n\in\mathbb{Z}} E_{\pi_n} \cong L^2(S^1)$. This map even intertwines with the right regular action of $L^2(S^1)$.

In order to generalize to groups that are not Abelian and to accommodate both the left and right regular actions, we will phrase the result a bit differently. Consider the map from $E^*_{\pi_n} \otimes E_{\pi_n}$ to $L^2(S^1)$ induced by mapping $\lambda \otimes x \in E^*_{\pi_n} \otimes E_{\pi_n}$ to the function $f_{\lambda\otimes x}$ where $f_{\lambda\otimes x}(g) = \lambda(\pi_n(g^{-1})x)$ for $g \in S^1$. If $1^* \in E^*_{\pi_n}$ maps 1 to 1, notice $f_{1^*\otimes 1} = f_{-n}$, so there is still an induced isomorphism

$$\widehat{\bigoplus}_{\pi_n\in\widehat{S^1}} E^*_{\pi_n} \otimes E_{\pi_n} \cong L^2(S^1).$$

Moreover, it is easy to check that this isomorphism is an $S^1 \times S^1$-intertwining map with $(g_1, g_2) \in S^1 \times S^1$ acting on on $L^2(S^1)$ by $r_{g_1} \circ l_{g_2}$. Thus the results of Fourier analysis on S^1 can be thought of as arising directly from the representation theory of S^1. This result will generalize to all compact Lie groups.

Theorem 3.24. *Let G be a compact Lie group. As a $G \times G$-module with $(g_1, g_2) \in G \times G$ acting as $r_{g_1} \circ l_{g_2} = l_{g_2} \circ r_{g_1}$ on $C(G)_{G\text{-fin}}$,*

$$C(G)_{G\text{-fin}} \cong \bigoplus_{[\pi]\in\widehat{G}} E^*_\pi \otimes E_\pi.$$

The intertwining isomorphism is induced by mapping $\lambda \otimes x \in E^*_\pi \otimes E_\pi$ to $f_{\lambda\otimes x} \in C(G)_{G\text{-fin}}$ where $f_{\lambda\otimes x}(g) = \lambda(g^{-1}x)$ for $g \in G$.

Proof. The proof of this theorem is really not much more than the proof of Theorem 2.24 coupled with Lemma 3.23 and Theorem 3.21. To see that the given map is a G-map, calculate

$$((g_1, g_2) f_{\lambda\otimes x})(g) = \lambda(g_1^{-1} g^{-1} g_2 x) = (g_1\lambda)(g^{-1} g_2 x) = f_{g_1\lambda\otimes g_2 x}.$$

To see that the map is surjective, Lemma 3.2 shows that it suffices to verify that each matrix coefficient of the form $f^{E_\pi}_{u,v}(g) = (gu, v)$ is achieved where $[\pi] \in \widehat{G}$, $(\cdot,\cdot)$ is a G-invariant inner product on E_π, and $u, v \in E_\pi$. Since $C(G)_{G\text{-fin}}$ is closed under complex conjugation, it suffices to show $\overline{f^{E_\pi}_{u,v}(g)} = (v, gu) = (g^{-1}v, u)$ is achieved. For this, take $\lambda = (\cdot, u)$ so that $f_{\lambda\otimes v} = \overline{f^{E_\pi}_{u,v}}$.

It remains to see that the map is injective. Any element of the kernel lies in a finite sum of $W = \bigoplus_{i=1}^N E^*_{\pi_i} \otimes E_{\pi_i}$. Restricted to W, the kernel is $G \times G$-invariant. Since the kernel's isotypic components are contained in the isotypic components of W, it follows that the kernel is either $\{0\}$ or a direct sum of certain of the $E^*_{\pi_i} \otimes E_{\pi_i}$. As $f_{\lambda\otimes x}(g)$ is clearly nonzero for nonzero $\lambda \otimes x \in E^*_{\pi_i} \otimes E_{\pi_i}$, the kernel must be $\{0\}$. $\square$

Theorem 3.25 (Peter–Weyl). *Let G be a compact Lie group. $C(G)_{G\text{-fin}}$ is dense in $C(G)$ and in $L^2(G)$.*

Proof. Since $C(G)$ is dense in $L^2(G)$, it suffices to prove the first statement. For this, recall that $C(G)_{G\text{-fin}}$ is an algebra that is closed under complex conjugation and contains 1. By the Stone–Weierstrass Theorem, it only remains to show that $C(G)_{G\text{-fin}}$ separates points. For this, using left translation, it is enough to show that for any $g_0 \in G$, $g_0 \neq e$, there exists $f \in C(G)_{G\text{-fin}}$ so that $f(g_0) \neq f(e)$.

By the Hausdorff condition and continuity of left translation, choose an open neighborhood U of e so that $U \cap (g_0 U) = \emptyset$. The characteristic function for U, χ_U, is a nonzero function in $L^2(G)$. Since $l_{g_0}\chi_U = \chi_{g_0 U}$, $(l_{g_0}\chi_U, \chi_U) = 0$. Because $(\chi_U, \chi_U) > 0$, l_{g_0} cannot be the identity operator on $L^2(G)$. By Corollary 3.15 and with respect to the left action of G on $L^2(G)$, there exist finite-dimensional irreducible G-submodules $V_\alpha \subseteq L^2(G)$ so that $L^2(G) = \widehat{\bigoplus}_\alpha V_\alpha$. In particular, there is an α_0 so that l_{g_0} does not act by the identity on V_{α_0}. Thus there exists $x \in V_{\alpha_0}$ so that $l_{g_0}x \neq x$, and so there is a $y \in V_{\alpha_0}$, so that $(l_{g_0}x, y) \neq (x, y)$. The matrix coefficient $f = f_{x,y}^{V_\alpha}$ is therefore the desired function. □

Coupling this density result with the canonical decomposition and the version of Frobenius reciprocity contained in Lemma 3.23, it is now possible to decompose $L^2(G)$. Since the two results are so linked, the following corollary is also often referred to as the Peter–Weyl Theorem.

Corollary 3.26. *Let G be a compact Lie group. As a $G \times G$-module with $(g_1, g_2) \in G \times G$ acting as $r_{g_1} \circ l_{g_2}$ on $L^2(G)$,*

$$L^2(G) \cong \widehat{\bigoplus_{[\pi]\in\widehat{G}}} E_\pi^* \otimes E_\pi.$$

The intertwining isomorphism is induced by mapping $\lambda \otimes v \in E_\pi^* \otimes E_\pi$ to $f_{\lambda\otimes v}$ where $f_{\lambda\otimes v}(g) = \lambda(g^{-1}v)$ for $g \in G$. With respect to the same conventions as in Lemma 3.23, $\mathrm{Hom}_G(E_\pi, L^2(G)) = \mathrm{Hom}_G(E_\pi, C(G)) \cong E_\pi^*$ *as G-modules.*

Proof. With respect to the left action, the canonical decomposition says that there is an intertwining isomorphism

$$\iota : \widehat{\bigoplus_{[\pi]\in\widehat{G}}} \mathrm{Hom}_G(E_\pi, L^2(G)) \mathbin{\widehat{\otimes}} E_\pi \to L^2(G)$$

induced by $\iota(T \otimes v) = T(v)$ for $T \in \mathrm{Hom}_G(E_\pi, L^2(G))$ and $v \in L^2(G)$. Using the natural inclusion $\mathrm{Hom}_G(E_\pi, C(G)) \hookrightarrow \mathrm{Hom}_G(E_\pi, L^2(G))$ and Lemma 3.23, there is an injective map $\kappa : E_\pi^* \hookrightarrow \mathrm{Hom}_G(E_\pi, L^2(G))$ induced by mapping $\lambda \in E_\pi^*$ to $T_\lambda \in \mathrm{Hom}_G(E_\pi, L^2(G))$ via $(T_\lambda(v))(g) = \lambda(g^{-1}v)$. We first show that κ is an isomorphism.

Argue by contradiction. Suppose $\kappa(E_\pi^*)$ is a proper subset of $\mathrm{Hom}_G(E_\pi, L^2(G))$. Then, since ι is an isomorphism and E_π^* is finite dimensional, $\iota(\kappa(E_\pi^*) \otimes E_\pi)$ is a proper closed subset of $\iota(\mathrm{Hom}_G(E_\pi, L^2(G)) \otimes E_\pi)$. Choose a nonzero $f \in \iota(\mathrm{Hom}_G(E_\pi, L^2(G)) \otimes E_\pi)$ that is perpendicular to $\iota(\kappa(E_\pi^*) \otimes E_\pi)$. By virtue of the fact that $\iota(\mathrm{Hom}_G(E_\pi, L^2(G)) \otimes E_\pi)$ is the π-isotypic component of $L^2(G)$ for the left action and by Corollary 2.21, it follows that f is perpendicular to $\iota(\bigoplus_{[\pi]\in\widehat{G}} \kappa(E_\pi^*) \otimes E_\pi)$. Since $\iota(T_\lambda \otimes v) = T_\lambda(v) = f_{\lambda\otimes v}$, Theorem 3.24 shows f is perpendicular to $C(G)_{G\text{-fin}}$. By the Peter–Weyl Theorem, this is a contradiction, and so $E_\pi^* \cong \mathrm{Hom}_G(E_\pi, L^2(G))$.

Hence there is an isomorphism $\widehat{\bigoplus}_{[\pi]\in\widehat{G}} E_\pi^* \otimes E_\pi \to L^2(G)$ induced by mapping $\lambda \otimes v$ to $f_{\lambda\otimes v}$. The calculation given in the proof of Theorem 3.24 shows that this map is a $G \times G$-map when restricted to the subspace $\bigoplus_{[\pi]\in\widehat{G}} E_\pi^* \otimes E_\pi$. Since this subspace is dense, continuity finishes the proof. □

By Lemma 3.17, $E_\pi^* \cong \mathrm{Hom}_G(E_\pi, L^2(G))$ is equipped with a natural inner product. In §3.4 we will see how to rescale the above isomorphism on each component $E_\pi^* \otimes E_\pi$, so that the resulting map is unitary.

3.3.3 Applications

3.3.3.1 Orthonormal Basis for $L^2(G)$ and Faithful Representations

Corollary 3.27. *Let G be a compact Lie group. If $\{v_i^\pi\}_{i=1}^{n_\pi}$ is an orthonormal basis for E_π, $[\pi] \in \widehat{G}$, then $\{(\dim E_\pi)^{\frac{1}{2}} f^{E_\pi}_{v_i^\pi, v_j^\pi} \mid [\pi] \in \widehat{G}$ and $1 \le i, j \le n_\pi\}$ is an orthonormal basis for $L^2(G)$.*

Proof. This follows immediately from Lemma 3.2, Theorem 3.21, the Schur orthogonality relations, and the Peter–Weyl Theorem. □

Theorem 3.28. *A compact Lie group G possesses a* faithful representation*, i.e., there exists a (finite-dimensional representation) (π, V) of G for which π is injective.*

Proof. By the proof of the Peter–Weyl Theorem, for $g_1 \in G^0$, $g_1 \neq e$, there exists a finite-dimensional representation (π_1, V_1) of G, so that $\pi_1(g_1)$ is not the identity operator. Thus $\ker \pi_1$ is a closed proper Lie subgroup of G, and so a compact Lie group in its own right. Since $\ker \pi_1$ is a regular submanifold that does not contain a neighborhood of e, it follows that $\dim \ker \pi_1 < \dim G$. If $\dim \ker \pi_1 > 0$, choose $g_2 \in (\ker \pi_1)^0$, $g_2 \neq e$, and let (π_2, V_2) be a representation of G, so that $\pi_2(g_2)$ is not the identity. Then $\ker(\pi_1 \oplus \pi_2)$ is a compact Lie group with $\ker(\pi_1 \oplus \pi_2) < \dim \ker \pi_1$.

Continuing in this manner, there are representations (π_i, V_i), $1 \le i \le N$, of G, so that $\dim \ker(\pi_1 \oplus \cdots \oplus \pi_N) = 0$. Since G is compact, $\ker(\pi_1 \oplus \cdots \oplus \pi_N) = \{h_1, h_2, \ldots, h_M\}$ for $h_i \in G$. Choose representations (π_{N+i}, V_{N+i}), $1 \le i \le M$, of G, so that $\pi_{N+i}(h_i)$ is not the identity. The representation $\pi_1 \oplus \cdots \oplus \pi_{N+M}$ does the trick. □

Thus compact groups fall in the category of *linear groups* since each is now seen to be isomorphic to a closed subgroup of $GL(n, \mathbb{C})$. Even better, since compact, each is isomorphic to a closed subgroup of $U(n)$ by Theorem 2.15.

3.3.3.2 Class Functions

Definition 3.29. Let G be a Lie group. A function $f \in C(G)$ is called a *continuous class function* if $f(ghg^{-1}) = f(h)$ for all $g, h \in G$. Similarly, a function $f \in L^2(G)$ is called an L^2 *class function* if for each $g \in G$, $f(ghg^{-1}) = f(h)$ for almost all $h \in G$.

Theorem 3.30. *Let G be a compact Lie group and let χ be the set of irreducible characters, i.e., $\chi = \{\chi_{E_\pi} \mid [\pi] \in \widehat{G}\}$.*
(1) The span of χ equals the set of continuous class functions in $C(G)_{G\text{-fin}}$.
(2) The span of χ is dense in the set of continuous class functions.
(3) The set χ is an orthonormal basis for the set of L^2 class functions. In particular, if f is an L^2 class function, then

$$f = \sum_{[\pi]\in\widehat{G}} (f, \chi_{E_\pi})_{L^2(G)}\, \chi_{E_\pi}$$

as an L^2 function with respect to L^2 convergence and

$$\|f\|_{L^2(G)}^2 = \sum_{[\pi]\in\widehat{G}} \left|(f, \chi_{E_\pi})_{L^2(G)}\right|^2.$$

Proof. For part (1), recall from Theorem 3.24 that $C(G)_{G\text{-fin}} \cong \bigoplus_{[\pi]\in\widehat{G}} E_\pi^* \otimes E_\pi$ as a $G \times G$-module. View $C(G)_{G\text{-fin}}$ and $E_\pi^* \otimes E_\pi$ as G-modules via the diagonal embedding $G \hookrightarrow G \times G$ given by $g \to (g, g)$. In particular, $(gf)(h) = f(g^{-1}hg)$ for $f \in C(G)_{G\text{-fin}}$, so that f is a class function if and only if $gf = f$ for all $g \in G$.

Also recall that the isomorphism of G-modules $E_\pi^* \otimes E_\pi \cong \mathrm{Hom}(E_\pi, E_\pi)$ from Exercise 2.15 is induced by mapping $\lambda \otimes v$ to the linear map $v\lambda(\cdot)$ for $\lambda \in E_\pi^*$ and $v \in E_\pi$. Using this isomorphism,

$$C(G)_{G\text{-fin}} \cong \bigoplus_{[\pi]\in\widehat{G}} \mathrm{Hom}(E_\pi, E_\pi) \tag{3.31}$$

as a G-module under the diagonal action. For $T \in \mathrm{Hom}(E_\pi, E_\pi)$, T satisfies $gT = T$ for all $g \in G$ if and only if $T \in \mathrm{Hom}_G(E_\pi, E_\pi)$. By Schur's Lemma, this is if and only if $T = \mathbb{C}I_{E_\pi}$ where I_{E_π} is the identity operator. Thus the set of class functions in $C(G)_{G\text{-fin}}$ is isomorphic to $\bigoplus_{[\pi]\in\widehat{G}} \mathbb{C}I_{E_\pi}$.

If $\{x_i\}$ is an orthonormal basis for E_π and $(\cdot,\cdot)$ is a G-invariant inner product, then $I_{E_\pi} = \sum_i (\cdot, x_i)x_i$. Tracing the definitions back, the corresponding element in $E_\pi^* \otimes E_\pi$ is $\sum_i (\cdot, x_i) \otimes x_i$ and the corresponding function in $C(G)_{G\text{-fin}}$ is $g \to \sum_i (g^{-1}x_i, x_i)$. Since $(g^{-1}x_i, x_i) = \overline{(gx_i, x_i)}$, this means the class function corresponding to I_{E_π} under Equation 3.31 is exactly $\overline{\chi_{E_\pi}}$. In light of Lemma 3.2 and Theorem 3.21, part (1) is finished .

For part (2), let f be a continuous class function. By the Peter–Weyl Theorem, for $\epsilon > 0$ choose $\varphi \in C(G)_{G\text{-fin}}$, so that $\|f - \varphi\|_{C(G)} < \epsilon$. Define $\widetilde{\varphi}(h) = \int_G \varphi(g^{-1}hg)\, dg$, so that $\widetilde{\varphi}$ is a continuous class function. Using the fact that f is a class function,

$$\begin{aligned}\|f - \widetilde{\varphi}\|_{C(G)} &= \sup_{h\in G} |f(h) - \widetilde{\varphi}(h)| = \sup_{h\in G} \left| \int_G \left(f(g^{-1}hg) - \varphi(g^{-1}hg)\right) dg \right| \\ &\le \sup_{h\in G} \int_G \left|f(g^{-1}hg) - \varphi(g^{-1}hg)\right| dg \le \|f - \varphi\|_{C(G)} < \epsilon.\end{aligned}$$

It therefore suffices to show that $\widetilde{\varphi} \in \operatorname{span} \chi$.

For this, use Theorem 3.24 to write $\varphi(g) = \sum_i (gx_i, y_i)$ for $x_i, y_i \in E_{\pi_i}$. Thus $\widetilde{\varphi}(h) = \sum_i (\int_G g^{-1}hgx_i\, dg,\, y_i)$. However, on E_{π_i}, the operator $\int_G g^{-1}hg\, dg$ is a G-map and therefore acts as a scalar c_i by Schur's Lemma. Taking traces on E_{π_i},

$$\chi_{E_{\pi_i}}(h) = \operatorname{tr}\left(\int_G g^{-1}hg\, dg\right) = \operatorname{tr}\left(c_i I_{E_{\pi_i}}\right) = c_i \dim E_{\pi_i},$$

so that $\widetilde{\varphi}(h) = \sum_i \frac{(x_i, y_i)}{\dim E_{\pi_i}} \chi_{E_{\pi_i}}(h)$ which finishes (2).

For part (3), let f be an L^2 class function. By the Peter–Weyl Theorem, choose $\varphi \in C(G)_{G\text{-fin}}$ so that $\|f - \varphi\|_{L^2(G)} < \epsilon$. Then $\widetilde{\varphi} \in \operatorname{span} \chi$. Using the integral form of the Minkowski integral inequality and invariant integration,

$$\begin{aligned}\|f - \widetilde{\varphi}\|_{L^2(G)} &= \left(\int_G |f(h) - \widetilde{\varphi}(h)|^2\, dh\right)^{\frac{1}{2}} \\ &= \left(\int_G \left|\int_G \left(f(g^{-1}hg) - \varphi(g^{-1}hg)\right) dg\right|^2 dh\right)^{\frac{1}{2}} \\ &\le \int_G \left(\int_G \left|f(g^{-1}hg) - \varphi(g^{-1}hg)\right|^2 dh\right)^{\frac{1}{2}} dg \\ &= \int_G \left(\int_G |f(h) - \varphi(h)|^2\, dh\right)^{\frac{1}{2}} dg = \|f - \varphi\|_{L^2(G)} < \epsilon.\end{aligned}$$

The proof is finished by the Schur orthogonality relations and elementary Hilbert space theory. □

3.3.3.3 Classification of Irreducible Representation of $SU(2)$**.** From §2.1.2.2, recall that the representations $V_n(\mathbb{C}^2)$ of $SU(2)$ were shown to be irreducible in §2.3.1. By dimension, each is obviously inequivalent to the others. In fact, they are the only irreducible representations up to isomorphism (c.f. Exercise 6.8 for a purely algebraic proof).

Theorem 3.32. *The map* $n \to V_n(\mathbb{C}^2)$ *establishes an isomorphism* $\mathbb{N} \cong \widehat{SU(2)}$.

Proof. Viewing S^1 as a subgroup of $SU(2)$ via the inclusion $e^{i\theta} \to \operatorname{diag}(e^{i\theta}, e^{-i\theta})$, Equation 2.25 calculates the character of $V_n(\mathbb{C}^2)$ restricted to S^1 to be

$$\chi_{V_n(\mathbb{C}^2)}(e^{i\theta}) = \sum_{k=0}^{n} e^{i(n-2k)\theta}. \tag{3.33}$$

A simple inductive argument (Exercise 3.21) using Equation 3.33 shows that $\operatorname{span}\{\chi_{V_n(\mathbb{C}^2)}(e^{i\theta}) \mid n \in \mathbb{N}\}$ equals $\operatorname{span}\{\cos n\theta \mid n \in \mathbb{N}\}$.

Since every element of $SU(2)$ is uniquely diagonalizable to elements of the form $e^{\pm i\theta} \in S^1$, it is easy to see (Exercise 3.21) that restriction to S^1 establishes a norm preserving bijection from the set of continuous class functions on $SU(2)$ to the set of even continuous functions on S^1.

From elementary Fourier analysis, $\operatorname{span}\{\cos n\theta \mid n \in \mathbb{N}\}$ is dense in the set of even continuous functions on S^1. Thus $\operatorname{span}\{\chi_{V_n(\mathbb{C}^2)}(e^{i\theta}) \mid n \in \mathbb{N}\}$ is dense within the set of continuous class functions on $SU(2)$ and therefore dense within the set of L^2 class functions. Part (3) of Theorem 3.30 therefore shows that there are no other irreducible characters. Since a representation is determined by its character, Theorem 3.7, the proof is finished. □

Notice $\dim V_n(\mathbb{C}^2) = n+1$, so that the dimension is a complete invariant for irreducible representations of $SU(2)$.

3.3.4 Exercises

Exercise 3.16 Recall that $\widehat{S^1} \cong \mathbb{Z}$ (c.f. Exercise 3.13). Use the theorems of this chapter to recover the standard results of Fourier analysis on S^1. Namely, show that the trigonometric polynomials, $\operatorname{span}\{e^{in\theta} \mid n \in \mathbb{Z}\}$, are dense in $C(S^1)$ and that $\{e^{in\theta} \mid n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(S^1)$.

Exercise 3.17 **(a)** Let G be a compact Lie group. Use the fact that $\widehat{G \times G} \cong \widehat{G} \times \widehat{G}$ (Exercise 3.10) and the nature of G-finite vectors to show that any $G \times G$-submodule of $C(G)_{G\text{-fin}}$ corresponds to $\bigoplus_{[\pi]\in\mathcal{A}} E_\pi^* \otimes E_\pi$ for some $\mathcal{A} \subseteq \widehat{G}$ under the correspondence $C(G)_{G\text{-fin}} \cong \bigoplus_{[\pi]\in\widehat{G}} E_\pi^* \otimes E_\pi$.
(b) Let $\pi : G \to GL(n, \mathbb{C})$ be a faithful representation of G with $\pi_{i,j}(g)$, denoting the $(i, j)^{\text{th}}$ entry of the matrix $\pi(g)$ for $g \in G$. Show that the set of functions $\{\pi_{i,j}, \overline{\pi_{i,j}} \mid 1 \leq i, j, \leq n\}$ generate $MC(G) = C(G)_{G\text{-fin}}$ as an algebra over $\mathbb{C}$. In particular, $C(G)_{G\text{-fin}}$ is finitely generated.
(c) Let V be a faithful representation of G. Show that each irreducible representation of G is a submodule of $\left(\bigotimes^n V\right) \oplus \left(\bigotimes^m \overline{V}\right)$ for some $n, m \in \mathbb{N}$.

Exercise 3.18 Let G be a compact Lie group. The *commutator subgroup* of G, G', is the subgroup generated by $\{g_1 g_2 g_1^{-1} g_2^{-1} \mid g_i \in G\}$ and G is Abelian if and only if $G' = \{e\}$. Use the fact that G' acts trivially on 1-dimensional representations to show that all irreducible representations of a compact Lie group are one-dimensional if and only if G is Abelian (c.f. Exercise 2.21).

Exercise 3.19 Let G be a finite group.
(a) Show that $\int_G f(g)\,dg = \frac{1}{|G|} \sum_{g\in G} f(g)$.
(b) Use character theory to show that the number of inequivalent irreducible representations is the number of conjugacy classes in G.
(c) Show that $|G|$ equals the sum of the squares of the dimensions of its irreducible representations.

Exercise 3.20 If a compact Lie group G is not finite, show that $\widehat{G}$ is countably infinite.

Exercise 3.21 **(a)** Viewing $S^1 \hookrightarrow SU(2)$ via the inclusion $e^{i\theta} \to \mathrm{diag}(e^{i\theta}, e^{-i\theta})$, show that the span$\{\chi_{V_n(\mathbb{C}^2)}(e^{i\theta}) \mid n \in \mathbb{N}\}$ equals the span$\{\cos n\theta \mid n \in \mathbb{N}\}$.
(b) Show restriction to S^1 establishes a norm preserving bijection from the set of continuous class functions on $SU(2)$ to the set of even continuous functions on S^1 (c.f. §7.3.1 for a general statement).

Exercise 3.22 **(a)** Continue to view $S^1 \hookrightarrow SU(2)$. For the representations $V_n(\mathbb{C}^2)$ of $SU(2)$, show $\chi_{V_n(\mathbb{C}^2)}(e^{i\theta}) = \frac{\sin(n+1)\theta}{\sin\theta}$ when $\theta \notin \pi\mathbb{Z}$.
(b) Let f be a continuous class function on $SU(2)$. Show that

$$\int_{SU(2)} f(g)\, dg = \frac{2}{\pi} \int_0^{\pi} f(\mathrm{diag}(e^{i\theta}, e^{-i\theta})) \sin^2\theta \, d\theta$$

by first showing the above integral equation holds when $f = \chi_{V_n(\mathbb{C}^2)}$, c.f. Exercise 7.9.

Exercise 3.23 **(a)** Let V be an irreducible representation of a compact Lie group G. Show that

$$\dim V \int_G \chi_V(g^{-1}hgk)\, dg = \chi_V(h)\chi_V(k)$$

for $h, k \in G$.
(b) Conversely, if $f \in C(G)$ satisfies $\int_G f(g^{-1}hgk)\, dg = f(h)f(k)$ for all $h, k \in G$, show that there is an irreducible representation V of G, so that $f = (\dim V)^{-1}\chi_V$.

Exercise 3.24 **(a)** Use the isomorphism $SO(3) \cong SU(2)/\{\pm I\}$, Lemma 1.23, to show that the set of inequivalent irreducible representations of $SO(3)$ can be indexed by $\{V_{2n}(\mathbb{C}^2) \mid n \in \mathbb{N}\}$.
(b) Using a dimension count, Theorem 2.33, and Exercise 2.30, show that $V_{2n}(\mathbb{C}^2) \cong \mathcal{H}_n(\mathbb{R}^3)$ as $SO(3)$-modules. Conclude that $\{\mathcal{H}_n(\mathbb{R}^3) \mid n \in \mathbb{N}\}$ comprises a complete set of inequivalent irreducible representations for $SO(3)$.
(c) Use Exercise 3.5 to show that

$$\mathcal{H}_n(\mathbb{R}^3) \otimes \mathcal{H}_m(\mathbb{R}^3) \cong \bigoplus_{j=0}^{\min\{n,m\}} \mathcal{H}_{n+m-j}(\mathbb{R}^3).$$

3.4 Fourier Theory

Recall that the *Fourier transform* on S^1 can be thought of as an isomorphism $\wedge : L^2(S^1) \to l^2(\mathbb{Z})$, where

$$\widehat{f}(n) = \int_{S^1} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}$$

with $\|f\| = \|\widehat{f}\|$. The inverse is given by the *Fourier series*

$$f(\theta) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{in\theta},$$

where convergence is as $L^2(S^1)$ functions. It is well known that even when $f \in C(S^1)$, the Fourier series may not converge pointwise to f. However continuity and any positive Lipschitz condition will guarantee uniform convergence.

Since we recognize $\frac{d\theta}{2\pi}$ as the invariant measure on S^1 and $\mathbb{Z}$ as parametrizing $\widehat{S^1}$ with n corresponding to the (one-dimensional) representation $e^{i\theta} \to e^{in\theta}$, it seems likely this result can be generalized to any compact Lie group G. In fact, the scalar valued Fourier transform in Theorem 3.43 will establish a unitary isomorphism

$$\{L^2(G) \text{ class functions}\} \cong l^2(\widehat{G}).$$

Note in the case of $G = S^1$, the class function assumption is vacuous since S^1 is Abelian.

In order to handle all L^2 functions when G is not Abelian, the operator valued Fourier transform in the Plancherel Theorem (Theorem 3.38) will establish a unitary isomorphism

$$L^2(G) \cong \widehat{\bigoplus}_{[\pi]\in\widehat{G}} \operatorname{End}(E_\pi).$$

Remarkably, this isomorphism will also preserve the natural algebra structure of both sides. Note that for $G = S^1$, the right-hand side in the above equation reduces to $l^2(\widehat{G})$ since $\operatorname{End}(E_\pi) \cong \mathbb{C}$.

In terms of proofs, most of the work needed for the general case is already done in Corollary 3.26. In essence, only some bookkeeping and definition chasing is required to appropriately rescale existing maps.

3.4.1 Convolution

Let G be a compact Lie group. Write $\operatorname{End}(V) = \operatorname{Hom}(V, V)$ for the set of endomorphisms on a vector space V. Since G has finite volume, $L^2(G) \subseteq L^1(G)$, so that the following definition makes sense.

Definition 3.34. (1) For $[\pi] \in \widehat{G}$, define $\pi : L^2(G) \to \operatorname{End}(E_\pi)$ by

$$(\pi(f))(v) = \int_G f(g)gv\,dg$$

for $f \in L^2(G)$ and $v \in E_\pi$.
(2) Define $\widetilde{f} \in L^2(G)$ by $\widetilde{f}(g) = \overline{f(g^{-1})}$.

From a standard analysis course (Exercise 3.25 or see [37] or [73]), recall that the *convolution* operator $* : L^2(G) \times L^2(G) \to C(G)$ is given by

$$(f_1 * f_2)(g) = \int_G f_1(gh^{-1})f_2(h)\,dh$$

for $f_i \in L^2(G)$ and $g \in G$.

Lemma 3.35. *Let G be a compact Lie group, $[\pi] \in \widehat{G}$ with G-invariant inner product $(\cdot, \cdot)$ on E_π, $f_i, f \in L^2(G)$, and $v_i \in E_\pi$.*
(1) $\pi(f_1 * f_2) = \pi(f_1) \circ \pi(f_2)$.
(2) $(\pi(f)v_1, v_2) = \big(v_1, \pi(\widetilde{f})v_2\big)$, *i.e.,* $\pi(f)^* = \pi(\widetilde{f})$.

Proof. For part (1) with $v \in E_\pi$, use Fubini's Theorem and a change of variables $g \to gh$ to calculate

$$\begin{aligned}
\pi(f_1 * f_2)(v) &= \int_G \int_G f_1(gh^{-1}) f_2(h) gv \, dh \, dg \\
&= \int_G \int_G f_1(g) f_2(h) ghv \, dg \, dh \\
&= \int_G f_1(g) g \left(\int_G f_2(h) hv \, dh \right) dg = \pi(f_1) \left(\pi(f_2)(v)\right).
\end{aligned}$$

For part (2), calculate the following:

$$\begin{aligned}
(\pi(f)v_1, v_2) &= \int_G f(g)(gv_1, v_2) \, dg = \int_G (v_1, \overline{f(g)} g^{-1} v_2) \, dg \\
&= \int_G (v_1, \widetilde{f}(g) g v_2) \, dg = \big(v_1, \pi(\widetilde{f}) v_2\big). \qquad \square
\end{aligned}$$

3.4.2 Plancherel Theorem

The motivation for the next definition comes from Corollary 3.26 and the decomposition $L^2(G) \cong \widehat{\bigoplus}_{[\pi] \in \widehat{G}} E_\pi^* \otimes E_\pi$ coupled with the isomorphism $E_\pi^* \otimes E_\pi \cong \mathrm{End}(E_\pi)$.

Definition 3.36. (1) Let G be a compact Lie group and $[\pi] \in \widehat{G}$ with a G-invariant inner product $(\cdot, \cdot)$ on E_π. Then $\mathrm{End}(E_\pi)$ is a Hilbert space with respect to the *Hilbert–Schmidt* inner product

$$(T, S)_{HS} = \mathrm{tr}(S^* \circ T) = \sum_i (Tv_i, Sv_i)$$

with $T, S \in \mathrm{End}(E_\pi)$, S^* the adjoint of S with respect to $(\cdot, \cdot)$, and $\{v_i\}$ an orthonormal basis for E_π. The corresponding *Hilbert–Schmidt* norm is

$$\|T\|_{HS} = \mathrm{tr}(T^*T)^{\frac{1}{2}} = \left(\sum_i \|Tv_i\|^2 \right)^{\frac{1}{2}}.$$

Write $\mathrm{End}(E_\pi)_{HS}$ when viewing $\mathrm{End}(E_\pi)$ as a Hilbert space equipped with the Hilbert–Schmidt inner product.
(2) Let $\mathrm{Op}(\widehat{G})$ be the Hilbert space

$$\mathrm{Op}(\widehat{G}) = \widehat{\bigoplus_{[\pi] \in \widehat{G}}} \mathrm{End}(E_\pi)_{HS}.$$

Equip $\mathrm{Op}(\widehat{G})$ with the algebra structure

$$(T_\pi)_{[\pi]\in\widehat{G}}(S_\pi)_{[\pi]\in\widehat{G}} = ((\dim E_\pi)^{-\frac{1}{2}}\ T_\pi \circ S_\pi)_{[\pi]\in\widehat{G}}$$

and the $G \times G$-module structure

$$(g_1, g_2)(T_\pi)_{[\pi]\in\widehat{G}} = \left(\pi(g_2) \circ T \circ \pi(g_1^{-1})\right)_{[\pi]\in\widehat{G}}$$

for $g_i \in G$ and $T \in \mathrm{End}(E_\pi)$.

Some comments are in order. First, note that the inner product on $\mathrm{End}(E_\pi)_{HS}$ is independent of the choice of invariant inner product on E_π since scaling the inner product on E_π does not change S^*. Secondly, it must be verified that the algebra structure and $G \times G$-module structure on $\mathrm{Op}(\widehat{G})$ are well defined. Since these are straightforward exercises, they are left to the reader (Exercise 3.26).

Definition 3.37. **(1)** Let G be a compact Lie group. The *operator valued Fourier transform*, $\mathcal{F} : L^2(G) \to \mathrm{Op}(\widehat{G})$, is defined by

$$\mathcal{F}f = \left((\dim E_\pi)^{\frac{1}{2}}\ \pi(f)\right)_{[\pi]\in\widehat{G}}.$$

(2) For $T_\pi \in \mathrm{End}(E_\pi)$, write $\mathrm{tr}(T_\pi \circ g^{-1})$ for the smooth function on G defined by $g \to \mathrm{tr}(T_\pi \circ \pi(g^{-1}))$. The *inverse operator valued Fourier transform*, $\mathcal{I} : \mathrm{Op}(\widehat{G}) \to L^2(G)$, is given by

$$\mathcal{I}(T_\pi)_{[\pi]\in\widehat{G}} = \sum_{[\pi]\in\widehat{G}} (\dim E_\pi)^{\frac{1}{2}}\ \mathrm{tr}(T_\pi \circ g^{-1})$$

with respect to L^2 convergence.

It is necessary to check that $\mathcal{F}$ and $\mathcal{I}$ are well defined and inverses of each other. These details will be checked in the proof below. In the following theorem, view $L^2(G)$ as an algebra with respect to convolution and remember that $L^2(G)$ is a $G \times G$-module with $(g_1, g_2) \in G \times G$ acting as $r_{g_1} \circ l_{g_2}$ so $((g_1, g_2)f)(g) = f(g_2^{-1} g g_1)$ for $f \in L^2(G)$ and $g_i, g \in G$.

Theorem 3.38 (Plancherel Theorem). *Let G be a compact Lie group. The maps $\mathcal{F}$ and $\mathcal{I}$ are well defined unitary, algebra, $G \times G$-intertwining isomorphisms and inverse to each other so that*

$$\mathcal{F} : L^2(G) \overset{\cong}{\to} \mathrm{Op}(\widehat{G})$$

with $\|f\|_{L^2(G)} = \|\mathcal{F}f\|_{\mathrm{Op}(\widehat{G})}$, $\mathcal{F}(f_1 * f_2) = (\mathcal{F}f_1)(\mathcal{F}f_2)$, $\mathcal{F}((g_1, g_2)f) = (g_1, g_2)(\mathcal{F}f)$, and $\mathcal{F}^{-1} = \mathcal{I}$ for $f \in L^2(G)$ and $g_i \in G$.

Proof. Recall the decomposition $L^2(G) \cong \widehat{\bigoplus}_{[\pi]\in\widehat{G}} E_\pi^* \otimes E_\pi$ from Corollary 3.26 that maps $\lambda \otimes v \in E_\pi^* \otimes E_\pi$ to $f_{\lambda\otimes v}$ where $f_{\lambda\otimes v}(g) = \lambda(g^{-1}v)$ for $g \in G$. Since $\mathrm{Op}(\widehat{G}) = \widehat{\bigoplus}_{[\pi]\in\widehat{G}} \mathrm{End}(E_\pi)_{HS}$ and since isometries on dense sets uniquely extend by continuity, it suffices to check that $\mathcal{F}$ restricts to a unitary, algebra, $G \times G$-intertwining isomorphism from $\mathrm{span}\{f_{\lambda\otimes v} \mid \lambda \otimes v \in E_\pi^* \otimes E_\pi\}$ to $\mathrm{End}(E_\pi)$ with inverse $\mathcal{I}$. Here $\mathrm{End}(E_\pi)$ is viewed as a subspace of $\mathrm{Op}(\widehat{G})$ under the natural inclusion $\mathrm{End}(E_\pi) \hookrightarrow \mathrm{Op}(\widehat{G})$.

Write $(\cdot,\cdot)$ for a G-invariant inner product on E_π. Any $\lambda \in E_\pi^*$ may be uniquely written as $\lambda = (\cdot, v)$ for some $v \in E_\pi$. Thus the main problem revolves around evaluating $\pi'(f_{(\cdot,v_1)\otimes v_2})$ for $[\pi'] \in \widehat{G}$ and $v_i \in E_\pi$. Therefore choose $w_i \in E_{\pi'}$ and a G-invariant inner product $(\cdot,\cdot)'$ on $E_{\pi'}$ and calculate

$$\begin{aligned}(\pi'(f_{(\cdot,v_1)\otimes v_2})(w_1), w_2)' &= \int_G (\pi(g^{-1})v_2, v_1)(\pi'(g)w_1, w_2)'\,dg \\ &= \int_G (\pi'(g)w_1, w_2)'\overline{(\pi(g)v_1, v_2)}\,dg.\end{aligned}$$

If $\pi' \ncong \pi$, the Schur orthogonality relations imply that $(\pi'(f_{(\cdot,v_1)\otimes v_2})(w_1), w_2)' = 0$, so that $\pi'(f_{(\cdot,v_1)\otimes v_2}) = 0$. Thus $\mathcal{F}$ maps $\mathrm{span}\{f_{\lambda\otimes v} \mid \lambda \otimes v \in E_\pi^* \otimes E_\pi\}$ to $\mathrm{End}(E_\pi)$. On the other hand, if $\pi' = \pi$, the Schur orthogonality relations imply that

$$(\pi(f_{(\cdot,v_1)\otimes v_2})(w_1), w_2) = (\dim E_\pi)^{-1}\,(w_1, v_1)\overline{(w_2, v_2)}.$$

In particular, $\pi(f_{(\cdot,v_1)\otimes v_2}) = (\dim E_\pi)^{-1}\,(\cdot, v_1)v_2$, so

$$\mathcal{F} f_{(\cdot,v_1)\otimes v_2} = (\dim E_\pi)^{-\frac{1}{2}}\,(\cdot, v_1)v_2.$$

Viewed as a map from $\mathrm{span}\{f_{\lambda\otimes v} \mid \lambda \otimes v \in E_\pi^* \otimes E_\pi\}$ to $\mathrm{End}(E_\pi)$, this shows that $\mathcal{F}$ is surjective and, by dimension count, an isomorphism.

To see that $\mathcal{I}$ is the inverse of $\mathcal{F}$, calculate the trace using an orthonormal basis that starts with $\|v_2\|^{-1}\,v_2$:

$$\begin{aligned}\mathrm{tr}([(\cdot, v_1)v_2] \circ \pi(g^{-1})) &= \left(\left[[(\cdot, v_1)v_2] \circ \pi(g^{-1})\right](\frac{v_2}{\|v_2\|}), \frac{v_2}{\|v_2\|}\right) \\ &= \left(\pi(g^{-1})v_2, v_1\right) = f_{(\cdot,v_1)\otimes v_2}(g).\end{aligned}$$

Thus

$$\mathcal{I}\left((\dim E_\pi)^{-\frac{1}{2}}\,(\cdot, v_1)v_2\right) = f_{(\cdot,v_1)\otimes v_2} \tag{3.39}$$

and $\mathcal{I} = \mathcal{F}^{-1}$.

To check unitarity, use the Schur orthogonality relations to calculate

$$\begin{aligned}\left(f_{(\cdot,v_1)\otimes v_2}, f_{(\cdot,v_3)\otimes v_4}\right)_{L^2(G)} &= \int_G \left(g^{-1}v_2, v_1\right)\overline{\left(g^{-1}v_4, v_3\right)}\,dg \\ &= (\dim E_\pi)^{-1}\,(v_2, v_4)\overline{(v_1, v_3)}.\end{aligned}$$

To calculate a Hilbert–Schmidt norm, first observe that the adjoint of $(\cdot, v_3)v_4 \in \mathrm{End}(E_\pi)_{HS}$ is $(\cdot, v_4)v_3$ since

$$((v_5, v_3)v_4, v_6) = (v_5, v_3)\,(v_4, v_6) = (v_5, (v_6, v_4)\,v_3).$$

Hence

$$\begin{aligned}
\left(\mathcal{F} f_{(\cdot,v_1)\otimes v_2}, \mathcal{F} f_{(\cdot,v_3)\otimes v_4}\right)_{HS} &= (\dim E_\pi)^{-1}\,((\cdot, v_1)v_2, (\cdot, v_3)v_4)_{HS} \\
&= (\dim E_\pi)^{-1}\,\mathrm{tr}\,[((\cdot, v_1)v_2, v_4)v_3] \\
&= (\dim E_\pi)^{-1}\,(v_2, v_4)\,\mathrm{tr}\,[(\cdot, v_1)v_3] \\
&= (\dim E_\pi)^{-1}\,(v_2, v_4)\left((\frac{v_3}{\|v_3\|}, v_1)v_3, \frac{v_3}{\|v_3\|}\right) \\
&= (\dim E_\pi)^{-1}\,(v_2, v_4)(v_3, v_1) \\
&= \left(f_{(\cdot,v_1)\otimes v_2}, f_{(\cdot,v_3)\otimes v_4}\right)_{L^2(G)},
\end{aligned}$$

and so $\mathcal{F}$ is unitary.

To check that the algebra structures are preserved, simply use Lemma 3.35 to observe that $\pi(f_1 * f_2) = \pi(f_1) \circ \pi(f_2)$. Thus

$$\begin{aligned}
\mathcal{F}(f_1 * f_2) &= (\dim E_\pi)^{\frac{1}{2}}\,\pi(f_1) \circ \pi(f_2) \\
&= (\dim E_\pi)^{-\frac{1}{2}}\,\mathcal{F} f_1 \circ \mathcal{F} f_2 = (\mathcal{F} f_1)\,(\mathcal{F} f_2)\,,
\end{aligned}$$

as desired.

Finally, to see $\mathcal{F}$ is a $G \times G$-map, first observe that

$$\begin{aligned}
\left((g_1, g_2) f_{(\cdot,v_1)\otimes v_2}\right)(g) &= f_{(\cdot,v_1)\otimes v_2}(g_2^{-1} g g_1) = (g_1^{-1} g^{-1} g_2 v_2, v_1) \\
&= (g^{-1} g_2 v_2, g_1 v_1) = f_{(\cdot,g_1 v_1)\otimes g_2 v_2}(g).
\end{aligned}$$

Thus

$$\begin{aligned}
\mathcal{F}((g_1, g_2) f_{(\cdot,v_1)\otimes v_2}) = \mathcal{F} f_{(\cdot,g_1 v_1)\otimes g_2 v_2} &= (\dim E_\pi)^{-\frac{1}{2}}\,(\cdot, g_1 v_1) g_2 v_2 \\
&= \pi(g_2) \circ (\cdot, v_1) v_2 \circ \pi(g_1^{-1}) = (g_1, g_2)(\mathcal{F} f),
\end{aligned}$$

which finishes the proof. ◻

Corollary 3.40. *Let G be a compact Lie group and $f, f_i \in L^2(G)$.*
(1) *Then the* Parseval–Plancherel formula *holds:*

$$\|f\|_{L^2(G)}^2 = \sum_{[\pi]\in\widehat{G}} \dim E_\pi\,\|\pi(f)\|_{HS}^2\,.$$

(2) *Under the natural inclusion* $\mathrm{End}(E_\pi) \hookrightarrow \mathrm{Op}(\widehat{G})$, $\mathcal{I} I_{E_\pi} = (\dim E_\pi)^{\frac{1}{2}}\,\chi_{\overline{E_\pi}}$ *where* $I_{E_\pi} \in \mathrm{End}(E_\pi)$ *is the identity operator. Moreover,*

$$f = \sum_{[\pi]\in\widehat{G}} (\dim E_\pi)\;\; f * \chi_{E_\pi}$$

with respect to L^2 convergence.
(3)

$$(f_1, f_2)_{L^2(G)} = \sum_{[\pi]\in\widehat{G}} (\dim E_\pi) \operatorname{tr} \pi(\widetilde{f_2} * f_1).$$

Proof. Part (1) follows immediately from the Plancherel Theorem. Similarly, part (2) will also follow from the Plancherel Theorem once we show $\mathcal{I} I_{E_\pi} = (\dim E_\pi)^{\frac{1}{2}} \chi_{\overline{E_\pi}}$ since $(\dim E_\pi)^{\frac{1}{2}} I_{E_\pi}$ acts on $\mathrm{Op}(\widehat{G})$ by projecting to $\mathrm{End}(E_\pi)$. Although this result is implicitly contained in the proof of Theorem 3.30, it is simple to verify directly. Let $\{x_i\}$ be an orthonormal basis for E_π where $(\cdot,\cdot)$ is a G-invariant inner product. Hence $I_{E_{\overline{\gamma}}} = \sum_i (\cdot, x_i)x_i$. Equation 3.39 shows $\mathcal{I} I_{E_\pi} = (\dim E_\pi)^{\frac{1}{2}} \sum_i f_{(\cdot,x_i)\otimes x_i}$ where $f_{(\cdot,x_i)\otimes x_i}(g) = (g^{-1}x_i, x_i)$. Thus $\mathcal{I} I_{E_\pi} = (\dim E_\pi)^{\frac{1}{2}} \chi_{\overline{E_\pi}}$ by Theorem 3.5.

For part (3), the Plancherel Theorem and Lemma 3.35 imply that

$$\begin{aligned}(f_1, f_2)_{L^2(G)} = (\mathcal{F} f_1, \mathcal{F} f_2)_{HS} &= \sum_{[\pi]\in\widehat{G}} (\dim E_\pi) \operatorname{tr}\left(\pi\,(f_2)^* \circ \pi\,(f_1)\right) \\ &= \sum_{[\pi]\in\widehat{G}} (\dim E_\pi) \operatorname{tr} \pi(\widetilde{f_2} * f_1).\end{aligned}$$ □

Definition 3.41. Let G be a compact Lie group and $f \in L^2(G)$. Define the *scalar valued Fourier transform* by

$$\widehat{f}(\pi) = \operatorname{tr} \pi(f)$$

for $[\pi] \in \widehat{G}$.

Note that $\widehat{f}$ can also be computed by the formula

$$\widehat{f}(\pi) = \int_G f(g)\chi_{E_\pi}(g)\, dg = (f, \chi_{\overline{E_\pi}})_{L^2(G)}$$

since $\widehat{f}(\pi) = \sum_i (\pi(f)v_i, v_i) = \int_G f(g) \sum_i (gv_i, v_i)\, dg$, where $\{v_i\}$ is an orthonormal basis for E_π.

Theorem 3.42 (Scalar Fourier Inversion). *Let G be a compact Lie group and $f \in$ $\operatorname{span}\left(L^2(G) * L^2(G)\right) \subseteq C(G)$. Then*

$$f(e) = \sum_{[\pi]\in\widehat{G}} (\dim E_\pi)\, \widehat{f}(\pi).$$

Proof. If $f = f_1 * f_2$ for $f_i \in L^2(G)$, then by Corollary 3.40,

$$\begin{aligned}f(e) &= \int_G f_1(g^{-1}) f_2(g)\, dg = \int_G f_2(g)\overline{\widetilde{f_1}(g)}\, dg \\ &= \left(f_2, \widetilde{f_1}\right)_{L^2(G)} = \sum_{[\pi]\in\widehat{G}} (\dim E_\pi) \operatorname{tr} \pi(f_1 * f_2).\end{aligned}$$ □

As already noted, even for $G = S^1$ the Scalar Fourier Inversion Theorem can fail if f is only assumed to be continuous. However, using Lie algebra techniques and the Plancherel Theorem, it is possible to show that the Scalar Fourier Inversion Theorem holds when f is continuously differentiable. In particular, the Scalar Fourier Inversion Theorem holds for smooth f.

Theorem 3.43. *Let G be a compact Lie group. The map $f \to (\widehat{f}(\pi))_{[\pi]\in\widehat{G}}$ establishes a unitary isomorphism*

$$\{L^2(G) \text{ class functions}\} \cong l^2(\widehat{G}).$$

For $[\gamma] \in \widehat{G}$, the image of χ_{E_γ} under this map is $(\delta_{\pi,\overline{\gamma}})_{[\pi]\in\widehat{G}}$, where $\delta_{\pi,\overline{\gamma}}$ is 1 when $\pi \cong \overline{\gamma}$ and 0 when $\pi \ncong \overline{\gamma}$.

Proof. This result is implicitly embedded in the proof of Theorem 3.30. However it is trivial to check directly. Observe that

$$\widehat{\chi_{E_\gamma}}(\pi) = \int_G \chi_{E_\gamma}(g)\chi_{E_\pi}(g)\,dg,$$

so that Theorem 3.7 implies χ_{E_π} is mapped to $(\delta_{\pi,\overline{\gamma}})_{[\pi]\in\widehat{G}}$. Since $\{\chi_{E_\pi} \mid [\pi] \in \widehat{G}\}$ is an orthonormal basis for $\{L^2(G)$ class functions$\}$, the result follows. □

3.4.3 Projection Operators and More General Spaces

Let G be a compact Lie group and (γ, V) a unitary representation of G on a Hilbert space. For $[\pi] \in \widehat{G}$, it will turn out that the operator $(\dim E_\pi)\,\gamma(\chi_{\overline{E_\pi}})$ is the orthogonal G-intertwining projection of V onto $V_{[\pi]}$. In fact, the main part of this result is true in a much more general setting than Hilbert space representations.

Now only assume V is a Hausdorff complete locally convex topological space. The notions of G-finite vector and isotypic component carry over from §3.2.2 and §3.2.3 in the obvious fashion.

Definition 3.44. Let V be a representation of a compact Lie group G on a Hausdorff complete locally convex topological space.
(1) The set of *G-finite vectors*, $V_{G\text{-fin}}$, is the set of $v \in V$ where $\operatorname{span}\{\pi(G)v\}$ is finite dimensional.
(2) For $[\pi] \in \widehat{G}$, let $V^0_{[\pi]}$ be the sum of all irreducible submodules equivalent to E_π.
(3) The closure $V_{[\pi]} = \overline{V^0_{[\pi]}}$ is called the *π-isotypic component* of V.

Theorem 3.45. *Let (γ, V) be a representation of a compact Lie group G on a complete Hausdorff locally convex topological space.*
(1) *For $[\pi], [\pi'] \in \widehat{G}$, the operator $(\dim E_\pi)\,\gamma(\chi_{\overline{E_\pi}})$ is a G-intertwining projection of V onto $V_{[\pi]}$ that acts as the identity on $V_{[\pi]}$ and acts as zero on $V_{[\pi']}$ for $\pi' \ncong \pi$.*
(2) *If (γ, V) is a unitary representation on a Hilbert space, then $(\dim E_\pi)\,\gamma(\chi_{\overline{E_\pi}})$ is also self-adjoint, i.e., orthogonal.*

Proof. For part (1), let $g \in G$ and $v \in V$, and observe that

$$\begin{aligned}\left(g\gamma(\chi_{E\pi})g^{-1}\right)v &= \int_G \chi_{E\pi}(h)ghg^{-1}v\,dh = \int_G \chi_{E\pi}(g^{-1}hg)hv\,dh \\ &= \int_G \chi_{E\pi}(h)hv\,dh = \gamma(\chi_{E\pi})v,\end{aligned}$$

so that $\gamma(\chi_{E\pi})$ is a G-map. Applying this to the representation $E_{\pi'}$, Schur's Lemma shows that $\pi'(\chi_{\overline{E\pi}}) = c_{\pi',\pi} I_{E_{\pi'}}$ for $c_{\pi',\pi} \in \mathbb{C}$. Taking traces,

$$(\dim E_{\pi'})c_{\pi',\pi} = \int_G \overline{\chi_{E\pi}(g)}\,\mathrm{tr}\,\pi'(g)\,dg = \int_G \chi_{E\pi'}(g)\overline{\chi_{E\pi}(g)}\,dg.$$

By Theorem 3.7, $c_{\pi',\pi}$ is 0 when $\pi' \not\cong \pi$ and $(\dim E_\pi)^{-1}$ when $\pi' = \pi$. Since any $v \in V^0_{[\pi']}$ lies in a submodule of V that is isomorphic to $E_{\pi'}$, $(\dim E_\pi)\,\pi'(\chi_{\overline{E\pi}})$ acts on $V^0_{[\pi']}$ as the identity when $\pi' = \pi$ and by zero when $\pi' \not\cong \pi$. Continuity finishes part (1).

For part (2), Lemma 3.35 implies that $\gamma(\chi_{E\pi})^* = \gamma(\widetilde{\chi_{E\pi}})$. But Theorem 3.5 shows $\widetilde{\chi_{E\pi}} = \chi_{E\pi}$. □

Theorem 3.46. *Let (γ, V) be a representation of a compact Lie group G on a Hausdorff complete locally convex topological space.*
(1) $V_{G\text{-fin}} = \bigoplus_{[\pi]\in\widehat{G}} V^0_{[\pi]}$.
(2) $V_{G\text{-fin}}$ *is dense in* V.
(3) If V is irreducible, then V is finite dimensional.

Proof. For part (1), the definitions and Corollary 2.17 imply $V_{G\text{-fin}} = \sum_{[\pi]\in\widehat{G}} V^0_{[\pi]}$, so it only remains to see the sum is direct. However, the existence of the projections in Theorem 3.45 trivially establish this result.

For part (2), suppose $\lambda \in V^*$ vanishes on $V_{G\text{-fin}}$. By the Hahn–Banach Theorem, it suffices to show $\lambda = 0$. For $x \in V$, define $f_x \in C(G)$ by $f_x(g) = \lambda(gx)$. Clearly $\lambda = 0$ if and only if $f_x = 0$ for all x. Looking to use Corollary 3.40, calculate

$$\begin{aligned}\left(f_x * \chi_{E_\pi}\right)(g) &= \int_G \lambda(ghx)\chi_{E_\pi}(h^{-1})\,dh = \lambda\Big(\int_G \overline{\chi_{E_\pi}(h)}ghx\,dh\Big) \\ &= \lambda(g\pi(\chi_{\overline{E_\pi}})x) = f_{\pi(\chi_{\overline{E_\pi}})x}(g).\end{aligned}$$

Since Theorem 3.45 shows that $\pi(\chi_{\overline{E_\pi}})x \in V_{[\pi]}$ and since λ vanishes on each $V_{[\pi]}$ by continuity, $f_x * \chi_{E_\pi} = 0$. Thus $f_x = 0$ and part (2) is finished.

For part (3), observe that part (2) shows V contains a finite-dimensional irreducible submodule W. Since finite-dimensional subspaces are closed, irreducibility implies that $V = W$. □

In particular, notice that even allowing the greater generality of representations on complete locally convex topological spaces still leaves us with the same set of irreducible representations, $\widehat{G}$.

The following corollary will be needed in §7.4.

Corollary 3.47. *Let G be a compact Lie group. Suppose $\mathcal{S} \subseteq C(G)$ is a subspace equipped with a topology so that:*
(a) *$\mathcal{S}$ is dense in $C(G)$,*
(b) *$\mathcal{S}$ is a Hausdorff complete locally convex topological space,*
(c) *the topology on $\mathcal{S}$ is stronger than uniform convergence, i.e., convergence in $\mathcal{S}$ implies convergence in $C(G)$, and*
(d) *$\mathcal{S}$ is invariant under l_g and r_g and, with these actions, $\mathcal{S}$ is a $G \times G$-module.*
Then $\mathcal{S}_{G\text{-fin}} = C(G)_{G\text{-fin}}$.

Proof. Clearly $\mathcal{S}_{[\pi]} \subseteq C(G)_{[\pi]}$ for $[\pi] \in \widehat{G}$. Note $C(G)_{[\pi]} \cong E_\pi^* \otimes E_\pi$ by Theorem 3.24. Arguing by contradiction, suppose $\mathcal{S}_{[\pi]} \subsetneq C(G)_{[\pi]}$ for some $[\pi] \in \widehat{G}$. Then there exists a nonzero $f \in C(G)_{[\pi]}$ that is perpendicular to $\mathcal{S}_{[\pi]}$ with respect to the L^2 norm. By Corollary 3.26 and Theorem 3.46, it follows that f is perpendicular to all of $\mathcal{S}_{G\text{-fin}}$. However, this is a contradiction to the fact that $\mathcal{S}_{G\text{-fin}}$ is dense in $L^2(G)$ by (a) and (c). □

As an example, $\mathcal{S}$ could be the set of smooth functions on G or the set of real analytic functions on G. One interpretation of Corollary 3.47 says that $C(G)_{G\text{-fin}}$ is the smallest reasonable class of test functions on G that are usually useful for representation theory. Thus the topological dual $C(G)^*_{G\text{-fin}}$ of distributions is the largest class of useful generalized functions on G.

3.4.4 Exercises

Exercise 3.25 Let G be a compact Lie group. For $f_i \in L^2(G)$ and $g \in G$, examine $\sup_{h\in G} \left|\left(l_g\,(f_1 * f_2)\right)(h) - (f_1 * f_2)\,(h)\right|$ to show $f_1 * f_2 \in C(G)$.

Exercise 3.26 **(a)** If V is a (finite-dimensional) vector space and $\|\cdot\|$ is the operator norm on $\operatorname{End}(V)$, show that $\|T \circ S\|_{HS} \leq \|T\|\,\|S\|_{HS}$ and $\|T\| \leq \|T\|_{HS}$ for $T, S \in \operatorname{End}(V)$.
(b) Let G be a compact Lie group. Show that $((\dim E_\pi)^{-\frac{1}{2}}\, T_\pi \circ S_\pi)_{[\pi]\in\widehat{G}} \in \operatorname{Op}(\widehat{G})$ when $(T_\pi)_{[\pi]\in\widehat{G}}, (S_\pi)_{[\pi]\in\widehat{G}} \in \operatorname{Op}(\widehat{G})$. Is the factor $(\dim E_\pi)^{-\frac{1}{2}}$ even needed for this statement?
(c) Show $(g_2 \circ T_\pi \circ g_1^{-1})_{[\pi]\in\widehat{G}} \in \operatorname{Op}(\widehat{G})$ for $g_i \in G$ and that this action defines a representation of $G \times G$ on $\operatorname{Op}(\widehat{G})$.

Exercise 3.27 Let G be a compact Lie group and $f \in \operatorname{span}\left(L^2(G) * L^2(G)\right) \subseteq C(G)$. Show that

$$f(g) = \sum_{[\pi]\in\widehat{G}} (\dim E_\pi)\, \widehat{\left(r_g f\right)}(\pi).$$

Exercise 3.28 With respect to convolution, show that $C(G)_{G\text{-fin}}$ is an algebra with center spanned by the set of irreducible characters, i.e., by $\{\chi_{E_\pi} \mid [\pi] \in \widehat{G}\}$.

Exercise 3.29 For this problem, recall Exercise 3.22. Let f be a smooth class function on $SU(2)$. Show that

$$f(I) = \frac{2}{\pi} \sum_{n=0}^{\infty} (n+1) \int_0^{\pi} f(\mathrm{diag}(e^{i\theta}, e^{-i\theta})) \sin\theta \sin(n+1)\theta \, d\theta.$$

Exercise 3.30 Let G be a compact Lie group. Show that G is Abelian if and only if the convolution on $C(G)$ is commutative (c.f. Exercise 3.18).

Exercise 3.31 Let V be a representation of a compact Lie group G on a Hausdorff complete locally convex topological space. For $[\pi] \in \widehat{G}$, show that $V^0_{[\pi]}$ is the largest subspace of V that is a direct sum of irreducible submodules equivalent to E_π.

Exercise 3.32 **(a)** Let (π, V) be a representation of a compact Lie group G on a Hausdorff complete locally convex topological space and f a continuous class function on G. Show that $\pi(f)$ commutes with $\pi(g)$ for $g \in G$.
(b) Show that $\pi(f)$ acts on $V_{[\pi]}$, $[\pi] \in \widehat{G}$, by $(\dim E_\pi)^{-1}(f, \chi_{\overline{E_\pi}})_{L^2(G)}$.

Exercise 3.33 Let (π, V) be a representation of a compact Lie group G on a Hausdorff complete locally convex topological space, $v \in V^0_{[\pi]}$ for $[\pi] \in \widehat{G}$, and $S = \mathrm{span}\{\pi(G)v\}$. For $\lambda \in S^*$, define $f_\lambda \in C(G)$ by $f_\lambda(g) = \lambda(g^{-1}v)$. Show that $\dim S \le (\dim E_\pi)^2$.