4

Lie Algebras

By their nature, Lie groups are usually nonlinear objects. However, it turns out there is a way to linearize their study by looking at the tangent space to the identity. The resulting object is called a Lie algebra. Simply by virtue of the fact that vector spaces are simpler than groups, the Lie algebra provides a powerful tool for studying Lie groups and their representations.

4.1 Basic Definitions

4.1.1 Lie Algebras of Linear Lie Groups

Let M be a manifold. Recall that a *vector field* on M is a smooth section of the *tangent bundle*, $T(M) = \cup_{m \in M} T_m(M)$. If G is a Lie group and $g \in G$, then the map $l_g : G \to G$ defined by $l_g h = gh$ for $g \in G$ is a diffeomorphism. A vector field X on G is called *left invariant* if $dl_g X = X$ for all $g \in G$. Since G acts on itself simply transitively under left multiplication, the tangent space of G at e, $T_e(G)$, is clearly in bijection with the space of left invariant vector fields. The correspondence maps $v \in T_e(G)$ to the vector field X where $X_g = dl_g v$, $g \in G$, and conversely maps a left invariant vector field X to $v = X_e \in T_e(G)$.

Elementary differential geometry shows that the set of left invariant vector fields is an algebra under the Lie bracket of vector fields (see [8] or [88]). Using the bijection of left invariant vector fields with $T_e(G)$, it follows that $T_e(G)$ has a natural algebra structure which is called the *Lie algebra* of G.

Since we are interested in compact groups, there is a way to bypass much of this differential geometry. Recall from Theorem 3.28 that a compact group G is a linear group, i.e., G is isomorphic to a closed Lie subgroup of $GL(n, \mathbb{C})$. In the setting of Lie subgroups of $GL(n, \mathbb{C})$, the Lie algebra has an explicit matrix realization which we develop in this chapter. It should be remarked, however, that the theorems in this chapter easily generalize to any Lie group.

Taking our cue from the above discussion, we will define an algebra structure on $T_e(G)$ viewed as a subspace of $T_I(GL(n, \mathbb{C}))$. Since $GL(n, \mathbb{C})$ is an open (dense) set in $M_{n,n}(\mathbb{C}) \cong \mathbb{R}^{2n^2}$, we will identify $T_I(GL(n,\mathbb{C}))$ with $\mathfrak{gl}(n,\mathbb{C})$ where

$$\mathfrak{gl}(n,\mathbb{F}) = M_{n,n}(\mathbb{F}).$$

The identification of $T_I(GL(n,\mathbb{C}))$ with $\mathfrak{gl}(n,\mathbb{C})$ is the standard one for open sets in $\mathbb{R}^{2n^2}$. Namely, to any $X \in T_I(GL(n,\mathbb{C}))$, find a smooth curve $\gamma : (-\epsilon,\epsilon) \to GL(n,\mathbb{C})$, $\epsilon > 0$, so that $\gamma(0) = I$, and so $X(f) = \frac{d}{dt}(f \circ \gamma)|_{t=0}$ for smooth functions f on $GL(n,\mathbb{C})$. The map sending X to $\gamma'(0)$ is a bijection from $T_I(GL(n,\mathbb{C}))$ to $\mathfrak{gl}(n,\mathbb{C})$.

Definition 4.1. Let G be a Lie subgroup of $GL(n,\mathbb{C})$.
(a) The *Lie algebra* of G is

$$\mathfrak{g} = \{\gamma'(0) \mid \gamma(0) = I \text{ and } \gamma : (-\epsilon,\epsilon) \to G, \epsilon > 0, \text{ is smooth}\} \subseteq \mathfrak{gl}(n,\mathbb{C}).$$

(b) The *Lie bracket* on $\mathfrak{g}$ is given by

$$[X,Y] = XY - YX.$$

Given a compact group G, Theorem 3.28 says that there is a faithful representation $\pi : G \to GL(n,\mathbb{C})$. Identifying G with its image under π, G may be viewed as a closed Lie subgroup of $GL(n,\mathbb{C})$. Using this identification, we use Definition 4.1 to define the *Lie algebra* of G. We will see in §4.2.1 that this construction is well defined up to isomorphism.

Theorem 4.2. *Let G be a Lie subgroup of $GL(n,\mathbb{C})$.*
(a) *Then $\mathfrak{g}$ is a real vector space.*
(b) *The Lie bracket is linear in each variable, skew symmetric, i.e., $[X,Y] = -[Y,X]$, and satisfies the* Jacobi identity

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$$

for $X, Y, Z \in \mathfrak{g}$.
(c) *Finally, $\mathfrak{g}$ is closed under the Lie bracket and therefore an algebra.*

Proof. Let $X_i = \gamma_i'(0) \in \mathfrak{g}$. For $r \in \mathbb{R}$, consider the smooth curve γ that maps a neighborhood of $0 \in \mathbb{R}$ to G defined by $\gamma(t) = \gamma_1(rt)\gamma_2(t)$. Then

$$\gamma'(0) = \left(r\gamma_1'(rt)\gamma_2(t) + \gamma_1(rt)\gamma_2'(t)\right)|_{t=0} = rX_1 + X_2$$

so that $\mathfrak{g}$ is a real vector space.

The statements regarding the basic properties of the Lie bracket in part (b) are elementary and left as an exercise (Exercise 4.1). To see that $\mathfrak{g}$ is closed under the bracket, consider the smooth curve σ_s that maps a neighborhood of 0 to G defined by $\sigma_s(t) = \gamma_1(s)\gamma_2(t)\,(\gamma_1(s))^{-1}$. In particular, $\sigma_s'(0) = \gamma_1(s)X_2\,(\gamma_1(s))^{-1} \in \mathfrak{g}$. Since the map $s \to \sigma_s'(0)$ is a smooth curve in a finite-dimensional vector space, tangent vectors to this curve also lie in $\mathfrak{g}$. Applying $\frac{d}{ds}|_{s=0}$, we calculate

$$\frac{d}{ds}\left(\gamma_1(s)X_2\,(\gamma_1(s))^{-1}\right)|_{s=0} = X_1X_2 - X_2X_1 = [X_1,X_2],$$

so that $[X_1,X_2] \in \mathfrak{g}$. □

4.1.2 Exponential Map

Let G be a Lie subgroup of $GL(n, \mathbb{C})$ and $g \in G$. Since G is a submanifold of $GL(n, \mathbb{C})$, $T_g(G)$ can be identified with

$$\{\gamma'(0) \mid \gamma(0) = g \text{ and } \gamma : (-\epsilon, \epsilon) \to G, \epsilon > 0, \text{ is smooth}\} \tag{4.3}$$

in the usual manner by mapping $\gamma'(0)$ to the element of $T_g(G)$ that acts on a smooth function f by $\frac{d}{dt}(f \circ \gamma)|_{t=0}$. Now if $\gamma(0) = I$ and $\gamma : (-\epsilon, \epsilon) \to G$, $\epsilon > 0$, is smooth, then $\sigma(t) = g\gamma(t)$ satisfies $\sigma(0) = g$ and $\sigma'(0) = g\gamma'(0)$. Since left multiplication is a diffeomorphism, Equation 4.3 identifies $T_g(G)$ with the set

$$g\mathfrak{g} = \{gX \mid X \in \mathfrak{g}\}.$$

We make use of this identification without further comment.

Definition 4.4. Let G be a Lie subgroup of $GL(n, \mathbb{C})$ and $X \in \mathfrak{g}$.
(a) Let $\widetilde{X}$ be the vector field on G defined by $\widetilde{X}_g = gX$, $g \in G$.
(b) Let γ_X be the *integral curve* of $\widetilde{X}$ through I, i.e., γ_X is the unique maximally defined smooth curve in G satisfying

$$\gamma_X(0) = I$$

and

$$\gamma_X'(t) = \widetilde{X}_{\gamma_X(t)} = \gamma_X(t)X.$$

It is well known from the theory of differential equations that integral curves exist and are unique (see [8] or [88]).

Theorem 4.5. *Let G be a Lie subgroup of $GL(n, \mathbb{C})$ and $X \in \mathfrak{g}$.*
(a) *Then*

$$\gamma_X(t) = \exp(tX) = e^{tX} = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n.$$

(b) *Moreover γ_X is a homomorphism and complete, i.e., it is defined for all $t \in \mathbb{R}$, so that $e^{tX} \in G$ for all $t \in \mathbb{R}$.*

Proof. It is a familiar fact that the map $t \to e^{tX}$ is a well-defined smooth homomorphism of $\mathbb{R}$ into $GL(n, \mathbb{C})$ (Exercise 4.3). Hence, first extend $\widetilde{X}$ to a vector field on $GL(n, \mathbb{C})$ by $\widetilde{X}_g = gX$, $g \in GL(n, \mathbb{C})$. Since $e^{0X} = I$ and $\frac{d}{dt}e^{tX} = e^{tX}X$, $t \to e^{tX}$ is the unique integral curve for $\widetilde{X}$ passing through I as a vector field on $GL(n, \mathbb{C})$. It is obviously complete. On the other hand, since G is a submanifold of $GL(n, \mathbb{C})$, γ_X may be viewed as a curve in $GL(n, \mathbb{C})$. It is still an integral curve for $\widetilde{X}$ passing through I as a vector field on $GL(n, \mathbb{C})$. By uniqueness, $\gamma_X(t) = e^{tX}$ on the domain of γ_X. In particular, there is an $\epsilon > 0$, so that $\gamma_X(t) = e^{tX}$ for $t \in (-\epsilon, \epsilon)$. Thus $e^{tX} \in G$ for $t \in (-\epsilon, \epsilon)$. But since $e^{ntX} = (e^{tX})^n$ for $n \in \mathbb{N}$, $e^{tX} \in G$ for all $t \in \mathbb{R}$, which finishes the proof. ∎

Note that Theorem 4.5 shows that the map $t \to e^{tX}$ is actually a smooth map from $\mathbb{R}$ to G for $X \in \mathfrak{g}$.

Theorem 4.6. *Let G be a Lie subgroup of $GL(n, \mathbb{C})$.*
(a) $\mathfrak{g} = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid e^{tX} \in G$ *for* $t \in \mathbb{R}\}$.
(b) *The map* $\exp\colon \mathfrak{g} \to G$ *is a local diffeomorphism near* 0*, i.e., there is a neighborhood of* 0 *in* $\mathfrak{g}$ *on which* $\exp$ *restricts to a diffeomorphism onto a neighborhood of* I *in* G.
(c) *When G is connected,* $\exp \mathfrak{g}$ *generates* G.

Proof. To see $\mathfrak{g}$ is contained in $\{X \in \mathfrak{gl}(n, \mathbb{C}) \mid e^{tX} \in G \text{ for } t \in \mathbb{R}\}$, use Theorem 4.5. Conversely, if $e^{tX} \in G$ for $t \in \mathbb{R}$ for all $X \in \mathfrak{gl}(n, \mathbb{C})$, apply $\frac{d}{dt}|_{t=0}$ and use the definition to see $X \in \mathfrak{g}$.

For part (b), by the Inverse Mapping theorem, it suffices to show the differential of $\exp : \mathfrak{g} \to G$ is invertible at I. In fact, we will see that the differential of exp at I is the identity map on all of $GL(n, \mathbb{C})$. Let $X \in \mathfrak{gl}(n, \mathbb{C})$. Then, under our tangent space identifications, the differential of exp maps X to $\frac{d}{dt} e^{tX}|_{t=0} = X$, as claimed. Part (c) follows from Theorem 1.15. □

Note from the proof of Theorem 4.6 that $X \in \mathfrak{gl}(n, \mathbb{C})$ is an element of $\mathfrak{g}$ if $e^{tX} \in G$ for all t on a neighborhood of 0. However, it is not sufficient to merely verify that $e^X \in G$ (Exercise 4.9). Also in general, exp need not be onto (Exercise 4.7). However, when G is compact and connected, we will in fact see in §5.1.4 that $G = \exp \mathfrak{g}$.

4.1.3 Lie Algebras for the Compact Classical Lie Groups

We already know that the Lie algebra of $GL(n, \mathbb{F})$ is $\mathfrak{gl}(n, \mathbb{F})$. The Lie algebra of $SL(n, \mathbb{F})$ turns out to be

$$\mathfrak{sl}(n, \mathbb{F}) = \{X \in \mathfrak{gl}(n, \mathbb{F}) \mid \operatorname{tr} X = 0\}.$$

To check this, use Theorem 4.6. Suppose X is in the Lie algebra of $SL(n, \mathbb{F})$. Then $1 = \det e^{tX} = e^{t \operatorname{tr} X}$ for $t \in \mathbb{R}$ (Exercise 4.3). Applying $\frac{d}{dt}|_{t=0}$ implies $0 = \operatorname{tr} X$. On the other hand, if $\operatorname{tr} X = 0$, then $\det e^{tX} = e^{t \operatorname{tr} X} = 1$, so that X is in the Lie algebra of $SL(n, \mathbb{F})$.

It remains to calculate the Lie algebras for the compact classical Lie groups.

4.1.3.1 $SU(n)$ First, we show that the Lie algebra of $U(n)$ is

$$\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X^* = -X\}.$$

Again, this follows from Theorem 4.6. Suppose X is in the Lie algebra of $U(n)$. Then $I = e^{tX} \left(e^{tX}\right)^* = e^{tX} e^{tX^*}$ for $t \in \mathbb{R}$ (Exercise 4.3). Applying $\frac{d}{dt}|_{t=0}$ implies that $0 = X + X^*$. On the other hand, if $X^* = -X$, then $e^{tX} e^{tX^*} = e^{tX} e^{-tX} = I$, so that X is in the Lie algebra of $U(n)$.

To calculate the Lie algebra of $SU(n)$, simply toss the determinant condition into the mix. It is handled as in the case of $SL(n, \mathbb{F})$. Thus the Lie algebra of $SU(n)$ is

$$\mathfrak{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X^* = -X, \ \operatorname{tr} X = 0\}.$$

Using the fact that the tangent space has the same dimension as the manifold, we now have a simple way to calculate the dimension of $U(n)$ and $SU(n)$. In particular, since $\dim \mathfrak{u}(n) = 2\frac{n(n-1)}{2} + n$, $\dim U(n) = n^2$ and, since $\dim \mathfrak{su}(n) = 2\frac{n(n-1)}{2} + n - 1$, $\dim SU(n) = n^2 - 1$.

4.1.3.2 $SO(n)$ Working with X^t instead of X^* for $X \in \mathfrak{gl}(n, \mathbb{R})$, $O(n)$ and $SO(n)$ are handled in the same way as $U(n)$ and $SU(n)$. Thus the Lie algebras for $O(n)$ and $SO(n)$ are, respectively,

$$\mathfrak{o}(n) = \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid X^t = -X\}$$
$$\mathfrak{so}(n) = \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid X^t = -X, \ \operatorname{tr} X = 0\} = \mathfrak{o}(n).$$

Both groups have the same tangent space at I since $SO(n) = O(n)^0$. In particular, both groups also have the same dimension and, since $\dim \mathfrak{o}(n) = \frac{n(n-1)}{2}$, $\dim O(n) = \dim SO(n) = \frac{n(n-1)}{2}$.

4.1.3.3 $Sp(n)$ Recall from §1.1.4.3 that two realizations were given for $Sp(n)$. We give the corresponding Lie algebra for each.

The first realization was $Sp(n) = \{g \in GL(n, \mathbb{H}) \mid g^* g = I\}$. Since $GL(n, \mathbb{H})$ is an open dense set in $\mathfrak{gl}(n, \mathbb{H}) = M_{n,n}(\mathbb{H}) \cong \mathbb{R}^{4n^2}$, $\mathfrak{gl}(n, \mathbb{H})$ can be identified with the tangent space $T_I(GL(n, \mathbb{H}))$. It is therefore clear that Definition 4.1 generalizes in the obvious fashion so as to realize the Lie algebra of $Sp(n)$ inside $\mathfrak{gl}(n, \mathbb{H})$. Working within this scheme and mimicking the case of $U(n)$, it follows that the Lie algebra of this realization of $Sp(n)$ is

$$\mathfrak{sp}(n) = \{X \in \mathfrak{gl}(n, \mathbb{H}) \mid X^* = -X\}.$$

Since $\dim \mathfrak{sp}(n) = 4\frac{n(n-1)}{2} + 3n$, we see that $\dim Sp(n) = 2n^2 + n$.

The second realization of $Sp(n)$ was as $Sp(n) \cong U(2n) \cap Sp(n, \mathbb{C})$, where $Sp(n, \mathbb{C}) = \{g \in GL(2n, \mathbb{C}) \mid g^t J g = J\}$ and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

The Lie algebra of this realization of $Sp(n)$ is

$$\mathfrak{sp}(n) \cong \mathfrak{u}(2n) \cap \mathfrak{sp}(n, \mathbb{C}),$$

where the Lie algebra of $Sp(n, \mathbb{C})$ is

$$\mathfrak{sp}(n, \mathbb{C}) = \{X \in \mathfrak{gl}(2n, \mathbb{C}) \mid X^t J = -JX\}.$$

The only statement that needs checking is the identification of $\mathfrak{sp}(n, \mathbb{C})$. As usual, this follows from Theorem 4.6. Suppose X is in the Lie algebra of $Sp(n, \mathbb{C})$. Then $e^{tX^t} J e^{tX} = J$ for $t \in \mathbb{R}$. Applying $\frac{d}{dt}|_{t=0}$ implies $0 = X^t J + JX$. On the other hand, if $X^t J = -JX$, then $JXJ^{-1} = -X^t$, so $e^{tX^t} J e^{tX} J^{-1} = e^{tX^t} e^{tJXJ^{-1}} = e^{tX^t} e^{-tX^t} = I$, so that X is in the Lie algebra of $Sp(n, \mathbb{C})$.

4.1.4 Exercises

Exercise 4.1 Let G be a Lie subgroup of $GL(n, \mathbb{C})$. Show that the Lie bracket is linear in each variable, skew-symmetric, and satisfies the Jacobi identity.

Exercise 4.2 **(1)** Show that the map $\widetilde{\vartheta} : \mathbb{H} \to M_{2,2}(\mathbb{C})$ from Equation 1.13 in §1.1.4.3 induces an isomorphism $Sp(1) \cong SU(2)$ of Lie groups.
(2) Show that

$$\widetilde{\vartheta} i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \widetilde{\vartheta} j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \widetilde{\vartheta} k = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

(3) Let $\mathrm{Im}(\mathbb{H}) = \mathrm{span}_{\mathbb{R}}\{i, j, k\}$ and equip $\mathrm{Im}(\mathbb{H})$ with the algebra structure $[u, v] = 2\,\mathrm{Im}(uv) = uv - \overline{uv} = uv - vu$ for $u, v \in \mathrm{Im}(\mathbb{H})$. Show $\widetilde{\vartheta}$ induces an isomorphism $\mathrm{Im}(\mathbb{H}) \cong \mathfrak{su}(2)$ as (Lie) algebras.

Exercise 4.3 **(1)** Let $X, Y \in \mathfrak{gl}(n, \mathbb{C})$. Show that the map $t \to e^{tX}$ is a well-defined smooth homomorphism of $\mathbb{R}$ into $GL(n, \mathbb{C})$.
(2) If X and Y commute, show that $e^{X+Y} = e^X e^Y$. Show by example that this need not be true when X and do not Y commute.
(3) Show that $\det e^X = e^{\mathrm{tr}\,X}$, $\left(e^X\right)^* = e^{X^*}$, $\left(e^X\right)^{-1} = e^{-X}$, $\frac{d}{dt} e^{tX} = e^{tX} X = X e^{tX}$, and $A e^X A^{-1} = e^{AXA^{-1}}$ for $A \in GL(n, \mathbb{C})$.

Exercise 4.4 **(1)** For $x, y \in \mathbb{R}$, show that

$$\exp \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = e^x \begin{pmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{pmatrix}$$
$$\exp \begin{pmatrix} x & y \\ y & x \end{pmatrix} = e^x \begin{pmatrix} \cosh y & \sinh y \\ \sinh y & \cosh y \end{pmatrix}$$
$$\exp \begin{pmatrix} x & 0 \\ y & x \end{pmatrix} = e^x \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}.$$

(2) Show every matrix in $\mathfrak{gl}(2, \mathbb{R})$ is conjugate to one of the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, $\begin{pmatrix} x & y \\ y & x \end{pmatrix}$, or $\begin{pmatrix} x & 0 \\ y & x \end{pmatrix}$.

Exercise 4.5 **(1)** The *Euclidean motion group* on $\mathbb{R}^n$ consists of the set of transformations of $\mathbb{R}^n$ of the form $x \to Ax + b$, where $A \in GL(n, \mathbb{R})$ and $b \in \mathbb{R}^n$ for $x \in \mathbb{R}^n$. Show that the Euclidean motion group can be realized as a linear group of the form

$$\{\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \mid A \in GL(n, \mathbb{R}), b \in \mathbb{R}^n\}.$$

(2) Use power series to make sense of $\frac{(e^A - I)}{A}$.
(3) Show that the exponential map is given in this case by

$$\exp\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^A & \frac{(e^A - I)}{A} b \\ 0 & 1 \end{pmatrix}.$$

Exercise 4.6 **(1)** Let $X \in \mathfrak{sl}(2, \mathbb{C})$ be given by

$$X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

with $a, b, c \in \mathbb{C}$ and let $\lambda \in \mathbb{C}$ so that $\lambda^2 = a^2 + bc$. Show $e^X = (\cosh \lambda)\, I + \frac{\sinh \lambda}{\lambda} X$.
(2) Let $X \in \mathfrak{so}(3)$ be given by

$$X = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

for $a, b, c \in \mathbb{R}$ and let $\theta = \sqrt{a^2 + b^2 + c^2}$. Show $e^X = I + \frac{\sin\theta}{\theta} X + \frac{1-\cos\theta}{\theta^2} X^2$. Also show that e^X is the rotation about $(c, -b, a)$ through an angle θ.

Exercise 4.7 **(1)** Show that the map $\exp\colon \mathfrak{sl}(2, \mathbb{R}) \to SL(2, \mathbb{R})$ is not onto by showing that the complement of the image of exp consists of all $g \in SL(2, \mathbb{R})$ that are conjugate to

$$\begin{pmatrix} -1 & \pm 1 \\ 0 & -1 \end{pmatrix}$$

(i.e., all $g \neq -I$ with both eigenvalues equal to -1).
(2) Calculate the image of $\mathfrak{gl}(2, \mathbb{R})$ under exp.

Exercise 4.8 **(1)** Use the Jordan canonical form to show $\exp\colon \mathfrak{gl}(n, \mathbb{C}) \to GL(n, \mathbb{C})$ is surjective.
(2) Show that $\exp\colon \mathfrak{u}(n) \to U(n)$ and $\exp\colon \mathfrak{su}(n) \to SU(n)$ are surjective maps.
(3) Show that $\exp\colon \mathfrak{so}(n) \to SO(n)$ is surjective.
(4) Show that $\exp\colon \mathfrak{sp}(n) \to Sp(n)$ is surjective.

Exercise 4.9 Find an $X \in \mathfrak{gl}(2, \mathbb{C})$ so that $e^X \in SL(2, \mathbb{C})$, but $X \notin \mathfrak{sl}(2, \mathbb{C})$.

Exercise 4.10 Let G be a Lie subgroup of $GL(n, \mathbb{C})$. Show $G^0 = \{I\}$ if and only if $\mathfrak{g} = \{0\}$.

Exercise 4.11 Let G be a Lie subgroup of $GL(n, \mathbb{C})$ and $\varphi : \mathbb{R} \to G$ a continuous homomorphism.
(1) Show that φ is smooth if and only if φ is smooth at 0.
(2) Let U be a neighborhood of 0 in $\mathfrak{g}$ on which exp is injective. Show it is possible to linearly reparametrize φ, i.e., replace $\varphi(t)$ by $\varphi(st)$ for some nonzero $s \in \mathbb{R}$, so that $\varphi([-1, 1]) \subseteq \exp U$.
(3) Let $X \in U$ so that $\exp X = 1$. Show that $\varphi(t) = e^{tX}$ for $t \in \mathbb{Q}$.
(4) Show that $\varphi(t) = e^{tX}$ for $t \in \mathbb{R}$ and conclude that φ is actually real analytic and, in particular, smooth.

Exercise 4.12 **(1)** Let G be a Lie subgroup of $GL(n, \mathbb{C})$. Let $\{X_i\}_{i=1}^n$ be a basis for $\mathfrak{g}$. By calculating the differential on each standard basis vector, show that the map

$$(t_1, \ldots, t_n) \to e^{t_1 X_1} \cdots e^{t_n X_n}$$

is a local diffeomorphism near 0 from $\mathbb{R}^n$ to G. The coordinates $(t_1, \ldots, t_n)$ are called *coordinates of the second kind.*
(2) Show that the map

$$(t_1, \ldots, t_n) \to e^{t_1 X_1 + \cdots + t_n X_n}$$

is a local diffeomorphism near 0 from $\mathbb{R}^n$ to G. The coordinates $(t_1, \ldots, t_n)$ are called *coordinates of the first kind.*

Exercise 4.13 Suppose G and H are Lie subgroups of general linear groups and $\varphi : H \to G$ is a continuous homomorphism. Using Exercises 4.11 and 4.12, show that φ is actually a real analytic and therefore smooth map.

Exercise 4.14 Suppose $B(\cdot, \cdot)$ is a bilinear form on $\mathbb{F}^n$. Let

$$\begin{aligned}\operatorname{Aut}(B) &= \{g \in GL(n, \mathbb{F}) \mid (gv, gw) = (v, w),\, v, w \in \mathbb{F}^n\}\\ \operatorname{Der}(B) &= \{X \in \mathfrak{gl}(n, \mathbb{F}) \mid (Xv, w) = -(v, Xw),\, v, w \in \mathbb{F}^n\}.\end{aligned}$$

Show that $\operatorname{Aut}(B)$ is a closed Lie subgroup of $GL(n, \mathbb{F})$ with Lie algebra $\operatorname{Der}(B)$.

Exercise 4.15 Suppose $\mathbb{F}^n$ is equipped with an algebra structure $\cdot$. Let

$$\begin{aligned}\operatorname{Aut}(\cdot) &= \{g \in GL(n, \mathbb{F}) \mid g(v \cdot w) = gv \cdot gw,\, v, w \in \mathbb{F}^n\}\\ \operatorname{Der}(\cdot) &= \{X \in \mathfrak{gl}(n, \mathbb{F}) \mid X(v \cdot w) = Xv \cdot w + v \cdot Xw,\, v, w \in \mathbb{F}^n\}.\end{aligned}$$

Show that $\operatorname{Aut}(\cdot)$ is a closed Lie subgroup of $GL(n, \mathbb{F})$ with Lie algebra $\operatorname{Der}(\cdot)$.

Exercise 4.16 Let G be a Lie subgroup of $GL(n, \mathbb{C})$. Use the exponential map to show G has a neighborhood of I that contains no subgroup of G other than $\{e\}$.

Exercise 4.17 For $X, Y \in \mathfrak{gl}(n, \mathbb{C})$, show that $e^{X+Y} = \lim_{n\to\infty}(e^{\frac{X}{n}} e^{\frac{Y}{n}})^n$.

4.2 Further Constructions

4.2.1 Lie Algebra Homomorphisms

Definition 4.7. Suppose $\varphi : H \to G$ is a homomorphism of Lie subgroups of general linear groups. Let the *differential* of φ, $d\varphi : \mathfrak{h} \to \mathfrak{g}$, be given by

$$d\varphi(X) = \frac{d}{dt}\varphi(e^{tX})|_{t=0}.$$

This is well defined by Theorem 4.6 and Definition 4.1. Note by the chain rule that if $\gamma : \mathbb{R} \to G$ is any smooth map with $\gamma'(0) = X$, then $d\varphi$ can be alternately computed as $d\varphi(X) = \frac{d}{dt}\varphi(\gamma(t))|_{t=0}$. If one examines the identifications of Lie algebras with tangent spaces, then it is straightforward to see that the above definition of $d\varphi$ corresponds to the usual differential geometry definition of the differential $d\varphi : T_I(H) \to T_I(G)$. In particular, $d\varphi$ is a linear map and $d(\varphi_1 \circ \varphi_2) = d\varphi_1 \circ d\varphi_2$. Alternatively, this can be verified directly with the chain and product rules (Exercise 4.18).

Theorem 4.8. *Suppose $\varphi, \varphi_i : H \to G$ are homomorphisms of Lie subgroups of general linear groups.*
(a) *The following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{d\varphi} & \mathfrak{g} \\ \exp \downarrow & & \downarrow \exp \\ H & \xrightarrow{\varphi} & G \end{array}$$

so that $\exp \circ d\varphi = \varphi \circ \exp$*,* i.e.*,* $e^{d\varphi X} = \varphi(e^X)$ *for* $X \in \mathfrak{h}$.
(b) *The differential $d\varphi$ is a* homomorphism of Lie algebras*,* i.e.*,*

$$d\varphi[X, Y] = [d\varphi X, d\varphi Y]$$

for $X, Y \in \mathfrak{h}$.
(c) *If H is connected and $d\varphi_1 = d\varphi_2$, then $\varphi_1 = \varphi_2$.*

Proof. For part (a), observe that since φ is a homomorphism that

$$\frac{d}{dt}\varphi(e^{tX}) = \frac{d}{ds}\varphi(e^{(t+s)X})|_{s=0} = \varphi(e^{tX})\frac{d}{ds}\varphi(e^{sX})|_{s=0} = \varphi(e^{tX})d\varphi X.$$

Thus $t \to \varphi(e^{tX})$ is the integral curve of $\widetilde{d\varphi X}$ through I. Theorem 4.5 therefore implies $\varphi(e^{tX}) = e^{td\varphi X}$.

For part (b), start with the equality $\varphi(e^{tX}e^{sY}e^{-tX}) = e^{td\varphi X}e^{sd\varphi Y}e^{-td\varphi X}$ that follows from the fact that φ is a homomorphism and part (a). Apply $\frac{\partial}{\partial s}|_{s=0}$ and rewrite $e^{tX}e^{sY}e^{-tX}$ as $e^{se^{tX}Ye^{-tX}}$ (Exercise 4.3) to get

$$d\varphi(e^{tX}Ye^{-tX}) = e^{td\varphi X}\, d\varphi Y\, e^{-td\varphi X}.$$

Next apply $\frac{d}{dt}|_{t=0}$ to get

$$\frac{d}{dt}\left(d\varphi(e^{tX}Ye^{-tX})\right)|_{t=0} = d\varphi X d\varphi Y - d\varphi Y d\varphi X = [d\varphi X, d\varphi Y]$$

and use the fact that $d\varphi$ is linear to get

$$d\varphi([X, Y]) = d\varphi(XY - YX) = \frac{d}{dt}d\varphi(e^{tX}Ye^{-tX})|_{t=0} = [d\varphi X, d\varphi Y].$$

For part (c), use part (a) to show that φ_1 and φ_2 agree on $\exp \mathfrak{h}$. By Theorem 4.6 and since φ_i is a homomorphism, the proof is finished. □

As a corollary of Theorem 4.8, we can check that the Lie algebra of a compact group is well defined up to isomorphism. To see this, suppose G_i are Lie subgroups of general linear groups with $\varphi : G_1 \to G_2$ an isomorphism. Since $\varphi \circ \varphi^{-1}$ and $\varphi^{-1} \circ \varphi$ are the identity maps, taking differentials shows $d\varphi$ is a Lie algebra isomorphism from $\mathfrak{g}_1$ to $\mathfrak{g}_2$.

A smooth homomorphism of the additive group $\mathbb{R}$ into a Lie group G is called a *one-parameter subgroup*. The next corollary shows that all one-parameter subgroups are of the form $t \to e^{tX}$ for $X \in \mathfrak{g}$.

Corollary 4.9. *Let G be a Lie subgroup of $GL(n, \mathbb{C})$ and let $\gamma : \mathbb{R} \to G$ be a smooth homomorphism, i.e., $\gamma(s+t) = \gamma(s)\gamma(t)$ for $s, t \in \mathbb{R}$. If $\gamma'(0) = X$, then $\gamma(t) = e^{tX}$.*

Proof. View the multiplicative group $\mathbb{R}^+$ as a Lie subgroup of $GL(1, \mathbb{C})$. Let $\widetilde{\gamma}, \sigma : \mathbb{R}^+ \to G$ be the two homomorphisms defined by $\widetilde{\gamma} = \gamma \circ \ln$ and $\sigma(x) = e^{(\ln x)X}$. Then

$$d\widetilde{\gamma}(x) = \frac{d}{dt}\widetilde{\gamma}(e^{tx})|_{t=0} = \frac{d}{dt}\gamma(tx)|_{t=0} = xX$$
$$d\sigma(x) = \frac{d}{dt}\sigma(e^{tx})|_{t=0} = \frac{d}{dt}e^{txX}|_{t=0} = xX.$$

Theorem 4.8 thus shows that $\widetilde{\gamma} = \sigma$ so that $\gamma(t) = e^{tX}$. □

In the definition below any choice of basis can be used to identify $GL(\mathfrak{g})$ and $\mathrm{End}(\mathfrak{g})$ with $GL(\dim \mathfrak{g}, \mathbb{R})$ and $\mathfrak{gl}(\dim \mathfrak{g}, \mathbb{R})$, respectively. Under this identification, exp corresponds to the map $\exp : \mathrm{End}(\mathfrak{g}) \to GL(\mathfrak{g})$ with $e^T = \sum_{k=0}^{\infty} \frac{1}{k!} T^k$ for $T \in \mathrm{End}(\mathfrak{g})$ where $T^k X = (T \circ \cdots \circ T)X$ (k copies) for $T \in \mathrm{End}(\mathfrak{g})$ and $X \in \mathfrak{g}$.

Definition 4.10. Let G be a Lie subgroup of $GL(n, \mathbb{C})$.
(a) For $g \in G$, let *conjugation*, $c_g : G \to G$, be the Lie group homomorphism given by $c_g(h) = ghg^{-1}$ for $h \in G$
(b) The *Adjoint representation* of G on $\mathfrak{g}$, $\mathrm{Ad} : G \to GL(\mathfrak{g})$, is given by $\mathrm{Ad}(g) = d\left(c_g\right)$.
(c) The *adjoint representation* of $\mathfrak{g}$ on $\mathfrak{g}$, $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, is given by $\mathrm{ad} = d\,\mathrm{Ad}$, i.e., $(\mathrm{ad}\, X)\, Y = \frac{d}{dt}(\mathrm{Ad}(e^{tX})Y)|_{t=0}$ for $X, Y \in \mathfrak{g}$.

Some notes are in order. Except for the fact that $\mathfrak{g}$ is a real vector space instead of a complex one, Ad is seen to satisfy the key property of a representation,

$$\mathrm{Ad}(g_1 g_2) = \mathrm{Ad}(g_1)\, \mathrm{Ad}(g_2),$$

by taking the differential of the relation $c_{g_1 g_2} = c_{g_1} \circ c_{g_2}$ for $g_i \in G$. More explicitly, however, $dc_g(X) = \frac{d}{dt}(ge^{tX}g^{-1})|_{t=0}$ so that

$$\mathrm{Ad}(g)X = gXg^{-1}.$$

Applying Theorem 4.8, we see that

$$c_g e^X = e^{\mathrm{Ad}(g)X}.$$

Since this is simply the statement $ge^X g^{-1} = e^{gXg^{-1}}$, the equality is already well known from linear algebra.

Secondly, $((d\,\mathrm{Ad})\,(X))\,Y = \frac{d}{dt}e^{tX}Ye^{-tX}|_{t=0}$ so that

$$(\mathrm{ad}\,X)\,Y = XY - YX = [X, Y].$$

Applying Theorem 4.8, we see that

$$\mathrm{Ad}(e^X) = e^{\mathrm{ad}\,X}. \tag{4.11}$$

The notion of a representation of a Lie algebra will be developed in §6.1. When that is done, ad will, in fact, be a representation of $\mathfrak{g}$ on itself.

4.2.2 Lie Subgroups and Subalgebras

If M is a manifold and ξ_i are vector fields, the *Lie bracket of vector fields* is defined as $[\xi_1, \xi_2] = \xi_1\xi_2 - \xi_2\xi_1$. For $M = \mathbb{R}^n$, it is easy to see (Exercise 4.19) that the Lie bracket of the vector fields $\xi = \sum_i \xi_i(x)\frac{\partial}{\partial x_i}$ and $\eta = \sum_i \eta_i(x)\frac{\partial}{\partial x_i}$ is given by

$$[\xi, \eta] = \sum_i \sum_j \left(\xi_j \frac{\partial \eta_i}{\partial x_j} - \eta_j \frac{\partial \xi_i}{\partial x_j}\right) \frac{\partial}{\partial x_i}. \tag{4.12}$$

For $M = GL(n, \mathbb{C})$, recall that $GL(n, \mathbb{C})$ is viewed as an open set in $M_{n,n}(\mathbb{C}) \cong \mathbb{R}^{2n^2} \cong \mathbb{R}^{n^2} \times \mathbb{R}^{n^2}$ by writing $Z \in M_{n,n}(\mathbb{C})$ as $Z = X + iY$, $X, Y \in M_{n,n}(\mathbb{R})$, and mapping Z to (X, Y). For $A \in \mathfrak{gl}(n, \mathbb{C})$, the value of the vector field $\widetilde{A}$ at the point $Z \in GL(n, \mathbb{C})$ is defined as ZA. Unraveling our identifications (see the discussion around Equation 4.3 for the usual identification of $T_g(G)$ with $g\mathfrak{g}$), this means that the vector field $\widetilde{A}$ on $GL(n, \mathbb{C})$ corresponds to the vector field

$$\partial_A = \sum_{i,j} \sum_k \mathrm{Re}(z_{ik}A_{kj}) \frac{\partial}{\partial x_{ij}} + \sum_{i,j} \sum_k \mathrm{Im}(z_{ik}A_{kj}) \frac{\partial}{\partial y_{ij}}$$

on the open set of $\mathbb{R}^{2n^2}$ cut out by the determinant.

Lemma 4.13. *For $A, B \in M_n(\mathbb{C})$, $[\partial_A, \partial_B] = \partial_{[A,B]}$.*

Proof. For the sake of clarity of exposition, we will verify this lemma for $M_n(\mathbb{R})$ and leave the general case of $M_n(\mathbb{C})$ to the reader. In this setting and with $A \in M_n(\mathbb{R})$, ∂_A is simply $\sum_{i,j}\sum_k x_{ik}A_{kj}\frac{\partial}{\partial x_{ij}}$. Writing $\delta_{i,p}$ for 0 when $i \neq p$ and for 1 when $i = p$, Equation 4.12 shows that

$$\begin{aligned}
[\partial_A, \partial_B] &= \sum_{i,j} \sum_{p,q} \left(\sum_k x_{pk}A_{kq}\delta_{i,p}B_{qj} - \sum_k x_{pk}B_{kq}\delta_{i,p}A_{qj}\right) \frac{\partial}{\partial x_{ij}} \\
&= \sum_{i,j} \sum_{q,k} x_{ik}\left(A_{kq}B_{qj} - B_{kq}A_{qj}\right) \frac{\partial}{\partial x_{ij}} \\
&= \sum_{i,j} \sum_k x_{ik}[A, B]_{kj} \frac{\partial}{\partial x_{ij}} = \partial_{[A,B]}.
\end{aligned}$$

□

For us, the importance of Lemma 4.13 is that if $\mathfrak{h}$ is a k-dimensional subalgebra of $\mathfrak{gl}(n, \mathbb{C})$, then the vector fields $\{\partial_X \mid X \in \mathfrak{h}\}$ form a subalgebra under the Lie bracket of vector fields. Moreover, on $GL(n, \mathbb{C})$, their value at each point determines a smooth rank k subbundle of the tangent bundle. Thus Frobenius' theorem from differential geometry (see [8] or [88]) says this subbundle foliates $GL(n, \mathbb{C})$ into *integral submanifolds*. In particular, there is a unique maximal connected k-dimensional submanifold H of $GL(n, \mathbb{C})$ so that $I \in H$ and $T_h(H) = \{(\partial_X)_h \mid X \in \mathfrak{h}\}$, $h \in H$, where $(\partial_X)_h$ is the value of ∂_X at h. Under our usual identification, this means that the tangent space of H at h corresponds to $h\mathfrak{h}$, i.e., that $\{\gamma'(0) \mid \gamma(0) = h$ and $\gamma : (-\epsilon, \epsilon) \to H, \epsilon > 0$, is smooth$\} = h\mathfrak{h}$. Finally, it is an important fact that integral submanifolds such as H, as was the case for regular submanifolds, satisfy the property that when $f : M \to G$ is a smooth map of manifolds with $f(M) \subseteq H$, then $f : M \to H$ is also a smooth map (see [88]).

Theorem 4.14. *Let G be a Lie subgroup of $GL(n, \mathbb{C})$. There is a bijection between the set of connected Lie subgroups of G and the set of subalgebras of $\mathfrak{g}$. If H is a connected Lie subgroup of G, the correspondence maps H to its Lie algebra $\mathfrak{h}$.*

Proof. Suppose $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. Let H be the unique maximal connected submanifold of G so that $I \in H$, and so the tangent space of H at h corresponds to $h\mathfrak{h}$ for $h \in H$. Now the connected submanifold $h_0^{-1}H$, $h_0 \in H$, contains I. Moreover, since $\frac{d}{dt}(h_0^{-1}\gamma(t))|_{t=0} = h_0^{-1}\gamma'(0)$, the tangent space of $h_0^{-1}H$ at $h_0^{-1}h$ corresponds to $h_0^{-1}h\mathfrak{h}$. Uniqueness of the integral submanifold therefore shows $h_0^{-1}H = H$. A similar argument shows that $h_0H = H$, so that H is a subgroup of G. By the remark above the statement of this theorem, the multiplication and inverse operations are smooth as maps on H, so that H is a Lie subgroup of G. Hence the correspondence is surjective.

To see it is injective, suppose H and H' are connected Lie subgroups of G, so that $\mathfrak{h} = \mathfrak{h}'$. Using the exponential map and Theorem 4.6, H and H' share a neighborhood of I. Since they are both connected, this forces $H = H'$. □

4.2.3 Covering Homomorphisms

Theorem 4.15. *Let H and G be connected Lie subgroups of general linear groups and $\varphi : H \to G$ a homomorphism of Lie groups. Then φ is a covering map if and only if $d\varphi$ is an isomorphism.*

Proof. If φ is a covering, then there is a neighborhood U of I in H and a neighborhood V of I in G, so that φ restricts to a diffeomorphism $\varphi : U \to V$. Thus the differential at I, $d\varphi$, is an isomorphism.

Suppose now that $d\varphi$ is an isomorphism. By the Inverse Mapping theorem, there is a neighborhood U_0 of I in H and a neighborhood V_0 of I in G so that φ restricts to a diffeomorphism $\varphi : U_0 \to V_0$. In particular, $\ker\varphi \cap U_0 = \{I\}$. Let V be a connected neighborhood of I in V_0 so that $VV^{-1} \subseteq V_0$ (Exercise 1.4) and let $U = \varphi^{-1}V \cap U_0$ so that U is connected, $UU^{-1} \subseteq U_0$, and $\varphi : U \to V$ is still a diffeomorphism.

As φ is a homomorphism, $\varphi^{-1}V = U \ker\varphi$. To see that φ satisfies the covering condition at $I \in G$, we show that the set of connected components of $\varphi^{-1}V$ is $\{U\gamma \mid \gamma \in \ker\varphi\}$. For this, it suffices to show that $U\gamma_1 \cap U\gamma_2 = \emptyset$ for distinct $\gamma_i \in \ker\varphi$. Suppose $u_1\gamma_1 = u_2\gamma_2$ for $u_i \in U$ and $\gamma_i \in \ker\varphi$. Then $\gamma_2\gamma_1^{-1} = u_2^{-1}u_1$ is in $U_0 \cap \ker\varphi$ and so $\gamma_2\gamma_1^{-1} = I$, as desired.

It remains to see that φ satisfies the covering condition at any $g \in G$. For this, first note that φ is surjective since G is connected, φ is a homomorphism, and the image of φ contains a neighborhood of I (Theorem 1.15). Choose $h \in H$ so that $\varphi(h) = g$. Then gV is a connected neighborhood of g in G and $\varphi^{-1}(gV) = hU \ker\varphi$. The set of connected components of $hU \ker\varphi$ is clearly $\{hU\gamma \mid \gamma \in \ker\varphi\}$. Since φ restricted to $hU\gamma$ is obviously a diffeomorphism to gV, φ is a covering map. □

Theorem 4.16. *Let H and G be connected Lie subgroups of general linear groups with H simply connected. If $\psi : \mathfrak{h} \to \mathfrak{g}$ is a homomorphism of Lie algebras, then there exists a unique homomorphism of Lie groups $\varphi : H \to G$ so that $d\varphi = \psi$.*

Proof. Uniqueness follows from Theorem 4.8. For existence, suppose H is a Lie subgroup of $GL(n, \mathbb{C})$ and G is a subgroup of $GL(m, \mathbb{C})$. Then we may view $H \times G$ as a block diagonal Lie subgroup of $GL(n+m, \mathbb{C})$. When this is done, the Lie algebra of $H \times G$ is clearly the direct sum of $\mathfrak{h}$ and $\mathfrak{g}$ in $\mathfrak{gl}(n+m, \mathbb{C})$. More importantly, note $\mathfrak{h}$ and $\mathfrak{g}$ commute and define

$$\mathfrak{a} = \{X + \psi X \mid X \in \mathfrak{h}\} \subseteq \mathfrak{h} \oplus \mathfrak{g}.$$

Using the fact that ψ is a homomorphism of Lie algebras, if follows that $\mathfrak{a}$ is a subalgebra of $\mathfrak{h} \oplus \mathfrak{g}$ since

$$[X + \psi X, Y + \psi Y] = [X, Y] + [\psi X, \psi Y] = [X, Y] + \psi[X, Y]$$

for $X, Y \in \mathfrak{h}$.

Let A be the connected Lie subgroup of $H \times G$ with Lie algebra $\mathfrak{a}$ (Theorem 4.14) and let π_H and π_G be the Lie group homomorphisms projecting A to H and G, respectively. By the definitions, $d\pi_H(X + \psi X) = X$ and $d\pi_G(X + \psi X) = \psi X$. Then $d\pi_H$ is a Lie algebra isomorphism of $\mathfrak{a}$ and $\mathfrak{h}$, so that Theorem 4.15 implies π_H is a covering map from A to H. Since H is simply connected, this means that $\pi_H : A \to H$ is an isomorphism. Define the Lie group homomorphism $\varphi : H \to G$ by $\varphi = \pi_G \circ \pi_H^{-1}$ to finish the proof. □

Note Theorem 4.16 can easily fail when H is not simply connected (Exercise 4.20).

4.2.4 Exercises

Exercise 4.18 **(1)** Let $\varphi : H \to G$ be a homomorphism of linear Lie groups. Use the fact that $\frac{d}{dt}\left(e^{trX}e^{tY}\right)|_{t=0} = rX + Y$, $X, Y \in \mathfrak{h}$, to directly show that $d\varphi : \mathfrak{h} \to \mathfrak{g}$ is a linear map.
(2) Let $\varphi' : K \to H$ be a homomorphism of linear Lie groups. Show that $d(\varphi \circ \varphi') = d\varphi \circ d\varphi'$.

Exercise 4.19 Verify that Equation 4.12 holds.

Exercise 4.20 Use the spin representations to show that Theorem 4.16 can fail when H is not simply connected.

Exercise 4.21 **(1)** Let

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ and } F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Show that $[H, E] = 2E$, $[H, F] = -2F$, and $[E, F] = H$.
(2) Up to the Ad action of $SL(2, \mathbb{R})$, find all Lie subalgebras of $\mathfrak{sl}(2, \mathbb{R})$.

Exercise 4.22 **(1)** Let G be a Lie subgroup of a linear group and $H \subseteq G$. Show that the *centralizer of* H *in* G,

$$Z_G(H) = \{g \in G \mid gh = hg, h \in H\},$$

is a Lie subgroup of G with Lie algebra the *centralizer of* H *in* $\mathfrak{g}$,

$$\mathfrak{z}_\mathfrak{g}(H) = \{X \in \mathfrak{g} \mid \operatorname{Ad}(h)X = X, h \in H\}.$$

(2) If $\mathfrak{h} \subseteq \mathfrak{g}$, show that the *centralizer of* $\mathfrak{h}$ *in* G,

$$Z_G(\mathfrak{h}) = \{g \in G \mid \operatorname{Ad}(g)X = X, X \in \mathfrak{h}\},$$

is a Lie subgroup of G with Lie algebra the *centralizer of* $\mathfrak{h}$ *in* $\mathfrak{g}$,

$$\mathfrak{z}_\mathfrak{g}(\mathfrak{h}) = \{Y \in \mathfrak{g} \mid [Y, X] = 0, X \in \mathfrak{h}\}.$$

(3) If H is a connected Lie subgroup of G, show $Z_G(H) = Z_G(\mathfrak{h})$ and $\mathfrak{z}_\mathfrak{g}(H) = \mathfrak{z}_\mathfrak{g}(\mathfrak{h})$.

Exercise 4.23 **(1)** Let G be a Lie subgroup of a linear group and let H be a connected Lie subgroup of G. Show that the *normalizer of* H *in* G,

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\},$$

is a Lie subgroup of G with Lie algebra the *normalizer of* $\mathfrak{h}$ *in* $\mathfrak{g}$,

$$\mathfrak{n}_\mathfrak{g}(\mathfrak{h}) = \{Y \in \mathfrak{g} \mid [Y, \mathfrak{h}] \subseteq \mathfrak{h}\}.$$

(2) Show H is normal in G if and only if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

Exercise 4.24 **(1)** Let $\varphi : H \to G$ be a homomorphism of Lie subgroups of linear groups. Show that $\ker \varphi$ is a closed Lie subgroup of H with Lie algebra $\ker d\varphi$.
(2) Show that the Lie subgroup $\varphi(H)$ of G has Lie algebra $d\varphi\mathfrak{h}$.

Exercise 4.25 If G is a Lie subgroup of a linear group satisfying $\operatorname{span}[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, show that $\operatorname{tr}(\operatorname{ad} X) = 0$ for $X \in \mathfrak{g}$.

Exercise 4.26 **(1)** For $X, Y \in \mathfrak{gl}(n, \mathbb{C})$, show that $e^{tX}e^{tY} = e^{t(X+Y)+\frac{1}{2}t^2[X,Y]+O(t^3)}$ for t near 0.
(2) Show that $e^{tX}e^{tY}e^{-tX} = e^{tY+t^2[X,Y]+O(t^3)}$ for t near 0.

Exercise 4.27 For $X, Y \in \mathfrak{gl}(n, \mathbb{C})$, show that $e^{[X,Y]} = \lim_{n\to\infty} \left(e^{\frac{X}{n}}e^{\frac{Y}{n}}e^{-\frac{X}{n}}e^{-\frac{Y}{n}}\right)^{n^2}$.

Exercise 4.28 This exercise gives a proof of Theorem 1.6. Recall the well-known fact that an n-dimensional submanifold N of an m-dimensional manifold M is regular if and only each $n \in N$ lies in an open set U of M with the property that there is a chart $\varphi : U \to \mathbb{R}^m$ of M so that $N \cap U = \varphi^{-1}(\mathbb{R}^n)$, where $\mathbb{R}^n$ is viewed as sitting in $\mathbb{R}^m$ in the usual manner ([8]). Such a chart is called cubical. Let $G \subseteq GL(n, \mathbb{C})$ be a Lie subgroup and $H \subseteq G$ be a subgroup (with no manifold assumption).
(1) Assume first that H is a regular submanifold of G and $h_i \to h$ with $h_i \in H$ and $h \in G$. Show that there is a cubical chart U of G around e and open sets $V \subseteq W \subseteq U$, so that $V^{-1}V \subseteq \overline{W} \subseteq U$. Noting that $h_i^{-1}h_j \in V^{-1}V$ for big i, j, use the definitions to show that H is closed.
(2) For the remainder, only assume H is closed. Let $\mathfrak{h} = \{X \in \mathfrak{g} \mid e^{tX} \in H, t \in \mathbb{R}\}$. Show that $e^{tX}e^{tY} = e^{t(X+Y)+O(t^2)}$, $X, Y \in \mathfrak{g}$, and use induction to see that

$$e^{t(X+Y)} = \lim_{n\to\infty} \left(e^{\frac{t}{n}X}e^{\frac{t}{n}Y}\right)^n.$$

Conclude that $\mathfrak{h}$ is a subspace and choose a complementary subspace $\mathfrak{s} \subseteq \mathfrak{g}$, so that $\mathfrak{s} \oplus \mathfrak{h} = \mathfrak{g}$.
(3) Temporarily, assume there are no neighborhoods V of 0 in $\mathfrak{g}$ with $\exp(\mathfrak{h} \cap V) = H \cap \exp V$. Using this assumption and the fact that the map $(Y, Z) \to e^Y e^Z$ is a local diffeomorphism at $(0, 0)$ from $\mathfrak{s} \oplus \mathfrak{h}$ to G, construct a nonzero sequence $Y_n \in \mathfrak{s}$, so that $Y_n \to 0$ and $e^{Y_n} \in H$. Show that you can pass to a subsequence and further assume $Y_n / \|Y_n\| \to Y$ for some nonzero $Y \in \mathfrak{s}$.
(4) For any $t \in \mathbb{R}$, show that there is $k_n \in \mathbb{Z}$ so that $k_n \|Y_n\| \to t$. Conclude that $\left(e^{Y_n}\right)^{k_n} \to e^{tY}$.
(5) Obtain a contradiction to the assumption in part (3) by showing that $Y \in \mathfrak{h}$. Conclude that there is a neighborhood V of 0 in $\mathfrak{g}$, so that exp is a diffeomorphism from V to its image in G and $\exp(\mathfrak{h} \cap V) = H \cap \exp V$.
(6) Given any $h \in H$, consider the neighborhood $U = h \exp V$ of h in G and the chart $\varphi = \exp^{-1} \circ l_h^{-1} : G \to \mathfrak{g}$ of G. Show that $\varphi^{-1}(\mathfrak{h}) = H \cap U$, so that H is a regular submanifold, as desired.