5

Abelian Lie Subgroups and Structure

Since a compact Lie group, G, can be thought of as a Lie subgroup of $U(n)$, Theorems 3.28 and 2.15, it is possible to diagonalize each $g \in G$ using conjugation in $U(n)$. In fact, the main theorem of this chapter shows it is possible to diagonalize each $g \in G$ using conjugation in G. This result will have far-reaching consequences, including various structure theorems.

5.1 Abelian Subgroups and Subalgebras

5.1.1 Maximal Tori and Cartan Subalgebras

If G is a Lie group, recall G is called *Abelian* if $g_1 g_2 = g_2 g_1$ for all $g_i \in G$. Similarly, if $\mathfrak{a}$ is a subalgebra of $\mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{a}$ is called *Abelian* if $[X, Y] = 0$ for all $X, Y \in \mathfrak{a}$.

Theorem 5.1. *Let G be a Lie subgroup of $GL(n, \mathbb{C})$.*
***(a)** For $X, Y \in \mathfrak{g}$, $[X, Y] = 0$ if and only if e^{tX} and e^{sY} commute for $s, t \in \mathbb{R}$. In this case, $e^{X+Y} = e^X e^Y$.*
***(b)** If A is a connected Lie subgroup of G with Lie algebra $\mathfrak{a}$, then A is Abelian if and only if $\mathfrak{a}$ is Abelian.*

Proof. Since part (b) follows from part (a) and Theorems 1.15 and 4.6, it suffices to prove part (a). It is a familiar fact (Exercise 4.3) that when X and Y commute, i.e., $[X, Y] = 0$, that $e^{tX+sY} = e^{tX} e^{sY}$. Since $e^{tX+sY} = e^{sY+tX}$, it follows that e^{tX} and e^{sY} commute. Conversely, if e^{tX} and e^{sY} commute, then $e^{tX} e^{sY} e^{-tX} = e^{sY}$. Applying $\frac{\partial}{\partial s}|_{s=0}$ and then $\frac{d}{dt}|_{t=0}$ yields $XY - YX = 0$, as desired. □

It is well known (Exercise 5.1) that the discrete (additive) subgroups of $\mathbb{R}^n$, up to application of an invertible linear transformation, are of the form

$$\Gamma_k = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1, \dots, x_k \in \mathbb{Z} \text{ and } x_{k+1} = \cdots x_n = 0\}.$$

In the next theorem, recall that a *torus* is a Lie group of the form $T^k = \left(S^1\right)^k \cong \mathbb{R}^k / \Gamma_k = \mathbb{R}^k / \mathbb{Z}^k$.

Theorem 5.2. ***(a)*** *The most general compact Abelian Lie group is isomorphic to $T^k \times F$, where F is a finite Abelian group. In particular, the most general compact connected Abelian Lie group is a torus.*
(b) *If G is a compact Abelian Lie group, then* exp *is a surjective map to G^0, the connected component of G.*

Proof. Let G be a compact Abelian group. By Theorem 5.1, $\exp : \mathfrak{g} \to G^0$ is a homomorphism. By Theorems 1.15 and 4.6 it follows that exp is surjective, so $G^0 \cong \mathfrak{g}/\ker(\exp)$. Since $\mathfrak{g} \cong \mathbb{R}^{\dim \mathfrak{g}}$ as a vector space and since Theorem 4.6 shows that $\ker(\exp)$ is discrete, $\ker(\exp) \cong \Gamma_k$ for some $k \leq \dim \mathfrak{g}$. But as G is compact with $G \cong \mathbb{R}^{\dim \mathfrak{g}}/\Gamma_k \cong T^k \times \mathbb{R}^{\dim \mathfrak{g}-k}$, k must be $\dim \mathfrak{g}$, so that $G^0 \cong T^{\dim \mathfrak{g}}$.

Next, G/G^0 is a finite Abelian group by compactness. It is well known that a finite Abelian group is isomorphic to a direct product of (additive) groups of the form $\mathbb{Z}/(n_i\mathbb{Z})$, $n_i \in \mathbb{N}$. For each such product, pick $g_i \in G$ whose image in G/G^0 corresponds to $1+n_i\mathbb{Z}$ in $\mathbb{Z}/(n_i\mathbb{Z})$. Then $g_i^{n_i} \in G^0$. Choose $X_i \in \mathfrak{g}$ so that $e^{n_i X_i} = g_i^{n_i}$ and let $h_i = g_i e^{-X_i}$. Then h_i is in the same connected component as is g_i, but now $h_i^{n_i} = I$. It follows easily that the map $G^0 \times \prod_i \mathbb{Z}/(n_i\mathbb{Z}) \to G$ given by mapping $(g, (m_i + n_i\mathbb{Z})) \to g \prod_i h_i^{m_i}$ is an isomorphism. □

Definition 5.3. **(a)** Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. A *maximal torus* of G is a maximal connected Abelian subgroup of G.
(b) A maximal Abelian subalgebra of $\mathfrak{g}$ is called a *Cartan subalgebra* of $\mathfrak{g}$.

By Theorem 5.2, a maximal torus T of a compact Lie group G is indeed isomorphic to a torus T^k. It should also be noted that the definition of Cartan subalgebra needs to be tweaked when working outside the category of compact Lie groups.

Theorem 5.4. *Let G be a compact Lie group and T a connected Lie subgroup of G. Then T is a maximal torus if and only if $\mathfrak{t}$ is a Cartan subalgebra. In particular, maximal tori and Cartan subalgebras exist.*

Proof. Theorems 4.14 and 5.1 show that T is a maximal torus of G if and only $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Since maximal Abelian subalgebras clearly exist, this also shows that maximal tori exist. □

5.1.2 Examples

Recall that the Lie algebras for the compact classical Lie groups were computed in §4.1.3.

5.1.2.1 *$SU(n)$* For $U(n)$ with $\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X^* = -X\}$, let

$$T = \{\operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \mid \theta_i \in \mathbb{R}\} \tag{5.5}$$
$$\mathfrak{t} = \{\operatorname{diag}(i\theta_1, \ldots, i\theta_n) \mid \theta_i \in \mathbb{R}\}$$

Clearly $\mathfrak{t}$ is the Lie algebra of the connected Lie subgroup T. Since it is easy to see $\mathfrak{t}$ is a maximal Abelian subalgebra of $\mathfrak{u}(n)$ (Exercise 5.2), it follows that T is a maximal torus and $\mathfrak{t}$ is its corresponding Cartan subalgebra.

For $SU(n)$ with $\mathfrak{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X^* = -X, \operatorname{tr} X = 0\}$, a similar construction yields a maximal torus and Cartan subalgebra. Simply use the definition for T and $\mathfrak{t}$ as in Equation 5.5 coupled with the additional requirement that $\sum_{i=1}^{n} \theta_i = 0$.

5.1.2.2 $Sp(n)$ For the first realization of $Sp(n)$ as

$$Sp(n) = \{g \in GL(n, \mathbb{H}) \mid g^* g = I\}$$

with $\mathfrak{sp}(n) = \{X \in \mathfrak{gl}(n, \mathbb{H}) \mid X^* = -X\}$, use the definition for T and $\mathfrak{t}$ as in Equation 5.5 to construct a maximal torus and Cartan subalgebra. It is straightforward to verify that $\mathfrak{t}$ is a Cartan subalgebra (Exercise 5.2).

For the second realization of $Sp(n)$ as

$$Sp(n) \cong U(2n) \cap Sp(n, \mathbb{C})$$

with $\mathfrak{sp}(n) \cong \mathfrak{u}(2n) \cap \mathfrak{sp}(n, \mathbb{C})$, let

$$\begin{aligned} T &= \{\operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}, e^{-i\theta_1}, \ldots, e^{-i\theta_n}) \mid \theta_i \in \mathbb{R}\} \\ \mathfrak{t} &= \{\operatorname{diag}(i\theta_1, \ldots, i\theta_n, -i\theta_1, \ldots, -i\theta_n) \mid \theta_i \in \mathbb{R}\} \end{aligned}$$

for $\theta_i \in \mathbb{R}$. Then T is a maximal torus and $\mathfrak{t}$ is its corresponding Cartan subalgebra (Exercise 5.2).

5.1.2.3 $SO(2n)$ For $SO(2n)$ with $\mathfrak{so}(2n) = \{X \in \mathfrak{gl}(2n, \mathbb{R}) \mid X^t = -X, \operatorname{tr} X = 0\}$, define the set of block diagonal matrices

$$T = \left\{ \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & \\ -\sin\theta_1 & \cos\theta_1 & & & \\ & & \ddots & & \\ & & & \cos\theta_n & \sin\theta_n \\ & & & -\sin\theta_n & \cos\theta_n \end{pmatrix} \mid \theta_i \in \mathbb{R} \right\}$$

$$\mathfrak{t} = \left\{ \begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \theta_n \\ & & & -\theta_n & 0 \end{pmatrix} \mid \theta_i \in \mathbb{R} \right\}.$$

Then T is a maximal torus and $\mathfrak{t}$ is its corresponding Cartan subalgebra (Exercise 5.2).

5.1.2.4 $SO(2n+1)$ For $SO(2n)$ with $\mathfrak{so}(2n) = \{X \in \mathfrak{gl}(2n+1, \mathbb{R}) \mid X^t = -X, \operatorname{tr} X = 0\}$, define the set of block diagonal matrices

$$T = \{\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & & \\ -\sin\theta_1 & \cos\theta_1 & & & & \\ & & \ddots & & & \\ & & & \cos\theta_n & \sin\theta_n & \\ & & & -\sin\theta_n & \cos\theta_n & \\ & & & & & 1 \end{pmatrix} \mid \theta_i \in \mathbb{R}\}$$

$$\mathfrak{t} = \{\begin{pmatrix} 0 & \theta_1 & & & & \\ -\theta_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & \theta_n & \\ & & & -\theta_n & 0 & \\ & & & & & 0 \end{pmatrix} \mid \theta_i \in \mathbb{R}\}.$$

Then T is a maximal torus and $\mathfrak{t}$ is its corresponding Cartan subalgebra (Exercise 5.2).

5.1.3 Conjugacy of Cartan Subalgebras

Lemma 5.6. *Let G be a compact Lie group and (π, V) a finite-dimensional representation of G.*
(a) *There exists a G-invariant inner product, $(\cdot, \cdot)$, on V and for any such G-invariant inner product on V, $d\pi X$ is skew-Hermitian, i.e., $(d\pi(X)v, w) = -(v, d\pi(X)w)$ for $X \in \mathfrak{g}$ and $v, w \in V$;*
(b) *There exists an* Ad*-invariant inner product, $(\cdot, \cdot)$, on $\mathfrak{g}$, that is, $(\mathrm{Ad}(g)Y_1, \mathrm{Ad}(g)Y_2) = (Y_1, Y_2)$ for $g \in G$ and $Y_i \in \mathfrak{g}$. For any such* Ad*-invariant inner product on $\mathfrak{g}$,* ad *is skew-symmetric, i.e., $(\mathrm{ad}(X)Y_1, Y_2) = -(Y_1, \mathrm{ad}(X)Y_2)$.*

Proof. Part (b) is simply a special case of part (a), where π is the Adjoint representation on $\mathfrak{g}$. To prove part (a), recall that Theorem 2.15 provides the existence of a G-invariant inner product on V. If $(\cdot, \cdot)$ is a G-invariant inner product on V, apply $\frac{d}{dt}|_{t=0}$ to $(\pi(e^{tX})Y_1, \pi(e^{tX})Y_2) = (Y_1, Y_2)$ to get $(d\pi(X)Y_1, Y_2) + (Y_1, d\pi(X)Y_2) = 0$. □

Lemma 5.7. *Let G be a compact Lie group and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. There exists $X \in \mathfrak{t}$, so that $\mathfrak{t} = \mathfrak{z}_{\mathfrak{g}}(X)$ where $\mathfrak{z}_{\mathfrak{g}}(X) = \{Y \in \mathfrak{g} \mid [Y, X] = 0\}$.*

Proof. By choosing a basis for $\mathfrak{t}$ and using the fact that $\mathfrak{t}$ is maximal Abelian in $\mathfrak{g}$, there exist independent $\{X_i\}_{i=1}^n$, $X_i \in \mathfrak{t}$, so that $\mathfrak{t} = \cap_i \ker(\mathrm{ad}\, X_i)$. Below we show that there exists $t \in \mathbb{R}$, so that $\ker(\mathrm{ad}(X_1 + tX_2)) = \ker(\mathrm{ad}\, X_1) \cap \ker(\mathrm{ad}\, X_2)$. Once this result is established, it is clear that an inductive argument finishes the proof.

Let $(\cdot, \cdot)$ be an invariant inner product on $\mathfrak{g}$ for which ad is skew-symmetric. Let $\mathfrak{k}_X = \ker(\mathrm{ad}\, X)$ and $\mathfrak{r}_X = (\ker(\mathrm{ad}\, X))^\perp$. It follows that $\mathfrak{r}_X$ is an $\mathrm{ad}\, X$-invariant subspace. Since $\mathfrak{g} = \mathfrak{k}_X \oplus \mathfrak{r}_X$, it is easy to see $\mathfrak{r}_X$ is the range of $\mathrm{ad}\, X$ acting on $\mathfrak{g}$.

If non-central $X, Y \in \mathfrak{t}$, then the fact that $\mathrm{ad}\, X$ and $\mathrm{ad}\, Y$ commute, Theorem 4.8, implies that $\mathrm{ad}\, Y$ preserves the subspaces $\mathfrak{k}_X$ and $\mathfrak{r}_X$. In particular,

$$\mathfrak{g} = (\mathfrak{k}_X \cap \mathfrak{k}_Y) \oplus (\mathfrak{k}_X \cap \mathfrak{r}_Y) \oplus (\mathfrak{r}_X \cap \mathfrak{k}_Y) \oplus (\mathfrak{r}_X \cap \mathfrak{r}_Y).$$

If $\mathfrak{r}_X \cap \mathfrak{r}_Y = \{0\}$, then $\mathfrak{k}_X \cap \mathfrak{k}_Y = \ker(\mathrm{ad}(X+Y))$. Otherwise, restrict $\mathrm{ad}(X+tY)$ to $\mathfrak{r}_X \cap \mathfrak{r}_Y$ and take the determinant. The resulting polynomial in t is nonzero since it is nonzero when $t = 0$. Thus there is a $t_0 \neq 0$, so $\mathrm{ad}(X+t_0Y)$ is invertible on $\mathfrak{r}_X \cap \mathfrak{r}_Y$. Hence, in either case, there exists a $t_0 \in \mathbb{R}$, so that $\ker(\mathrm{ad}(X+t_0Y)) = \mathfrak{k}_X \cap \mathfrak{k}_Y$. □

Definition 5.8. Let G be a compact Lie group and $X \in \mathfrak{g}$. If $\mathfrak{z}_\mathfrak{g}(X)$ is a Cartan subalgebra, then X is called a *regular* element of $\mathfrak{g}$.

We will see in §7.2.1 that the set of regular elements is an open dense set in $\mathfrak{g}$.

Theorem 5.9. *Let G be a compact Lie group and $\mathfrak{t}$ a Cartan subalgebra. For $X \in \mathfrak{g}$, there exists $g \in G$ so that* $\mathrm{Ad}(g)X \in \mathfrak{t}$.

Proof. Let $(\cdot,\cdot)$ be an Ad-invariant inner product on $\mathfrak{g}$. Using Lemma 5.7, write $\mathfrak{t} = \mathfrak{z}_\mathfrak{g}(Y)$ for some $Y \in \mathfrak{g}$. It is necessary to find $g_0 \in G$ so that $[\mathrm{Ad}(g_0)X, Y] = 0$. For this, it suffices to show that $([\mathrm{Ad}(g_0)X, Y], Z) = 0$ for all $Z \in \mathfrak{g}$. Using the skew-symmetry of ad, Lemma 5.6, this is equivalent to showing $(Y, [Z, \mathrm{Ad}(g_0)X]) = 0$.

Consider the continuous function f on G defined by $f(g) = (Y, \mathrm{Ad}(g)X)$. Since G is compact, choose $g_0 \in G$ so that f has a maximum at g_0. Then the function $t \to (Y, \mathrm{Ad}(e^{tZ})\,\mathrm{Ad}(g)X)$ has a maximum at $t = 0$. Applying $\frac{d}{dt}|_{t=0}$ therefore yields $0 = (Y, [Z, \mathrm{Ad}(g_0)X])$, as desired. □

The corresponding theorem is true on the group level as well, although much harder to prove (see §5.1.4).

Corollary 5.10. ***(a)*** *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Then $\mathrm{Ad}(G)$ acts transitively on the set of Cartan subalgebras of G.*
(b) *Via conjugation, G acts transitively on the set of maximal tori of G.*

Proof. For part (a), let $\mathfrak{t}_i = \mathfrak{z}_\mathfrak{g}(X_i)$, $X_i \in \mathfrak{g}$, be Cartan subalgebras. Using Theorem 5.9, there is a $g \in G$ so that $\mathrm{Ad}(g)X_1 \in \mathfrak{t}_2$. Using the fact that $\mathrm{Ad}(g)$ is a Lie algebra homomorphism, Theorem 4.8, it follows that

$$\begin{aligned}\mathrm{Ad}(g)\mathfrak{t}_1 &= \{\mathrm{Ad}(g)Y \in \mathfrak{g} \mid [Y, X_1] = 0\}\\ &= \{Y' \in \mathfrak{g} \mid [\mathrm{Ad}(g)^{-1}Y', X_1] = 0\}\\ &= \{Y' \in \mathfrak{g} \mid [Y', \mathrm{Ad}(g)X_1] = 0\} = \mathfrak{z}_\mathfrak{g}(\mathrm{Ad}(g)X_1).\end{aligned}$$

Since $\mathrm{Ad}(g)X_1 \in \mathfrak{t}_2$ and $\mathfrak{t}_2$ is Abelian, $\mathrm{Ad}(g)\mathfrak{t}_1 \supseteq \mathfrak{t}_2$. Since $\mathrm{Ad}(g)$ is a Lie algebra homomorphism, $\mathrm{Ad}(g)\mathfrak{t}_1$ is still Abelian. By maximality of Cartan subalgebras, $\mathrm{Ad}(g)\mathfrak{t}_1 = \mathfrak{t}_2$.

For part (b), let T_i be the maximal torus of G corresponding to $\mathfrak{t}_i$. Use Theorem 5.2 to write $T_i = \exp \mathfrak{t}_i$. Then Theorem 4.8 shows that

$$c_g T_1 = c_g \exp \mathfrak{t}_1 = \exp(\mathrm{Ad}(g)\mathfrak{t}_1) = \exp \mathfrak{t}_2 = T_2.$$ □

5.1.4 Maximal Torus Theorem

Lemma 5.11. *Let G be a compact connected Lie group. The kernel of the Adjoint map is the center of G, i.e.,* $\mathrm{Ad}(g) = I$ *if and only if* $g \in Z(G)$*, where* $Z(G) = \{h \in G \mid gh = hg\}$.

Proof. If $g \in Z(G)$, then c_g is the identity, so that its differential, $\mathrm{Ad}(g)$, is trivial as well. On the other hand, if $\mathrm{Ad}(g) = I$, then $c_g e^X = e^{\mathrm{Ad}(g)X} = e^X$ for $X \in \mathfrak{g}$. Thus c_g is the identity on a neighborhood of I in G. Since G is connected and c_g is a homomorphism, Theorem 1.15 shows that c_g is the identity on all of G. □

Theorem 5.12 (Maximal Torus Theorem). *Let G be a compact connected Lie group, T a maximal torus of G, and $g_0 \in G$.*
(a) *There exists $g \in G$ so that $gg_0g^{-1} \in T$.*
(b) *The exponential map is surjective, i.e.,* $G = \exp \mathfrak{g}$.

Proof. Use Theorems 5.2, 4.8, and 5.9 to observe that

$$\bigcup_{g\in G} c_g T = \bigcup_{g\in G} c_g \exp \mathfrak{t} = \bigcup_{g\in G} \exp\left(\mathrm{Ad}(g)\mathfrak{t}\right) = \exp \mathfrak{g}.$$

Thus $\bigcup_{g\in G} c_g T = G$ if and only if $\exp \mathfrak{g} = G$. In other words parts (a) and (b) are equivalent.

We will prove part (b) by induction on $\dim \mathfrak{g}$. If $\dim \mathfrak{g} = 1$, then $\mathfrak{g}$ is Abelian, and so Theorem 5.2 shows that $\exp \mathfrak{g} = G$. For $\dim \mathfrak{g} > 1$, we will use the inductive hypothesis to show that $\exp \mathfrak{g}$ is open and closed to finish the proof. Since $\bigcup_{g\in G} c_g T$ is the continuous image of the compact set $G \times T$, $\exp \mathfrak{g}$ is compact and therefore closed. Thus it remains to show that $\exp \mathfrak{g}$ is open.

Fix $X_0 \in \mathfrak{g}$ and write $g_0 = \exp X_0$. It is necessary to show that there is a neighborhood of g_0 contained in $\exp \mathfrak{g}$. By Theorem 4.6, assume $X_0 \neq 0$. Using Lemma 5.6, let $(\cdot,\cdot)$ be an Ad-invariant inner product on $\mathfrak{g}$ so that $\mathrm{Ad}(g_0)$ is unitary. Define

$$\begin{aligned}\mathfrak{a} &= \mathfrak{z}_{\mathfrak{g}}(g_0) = \{Y \in \mathfrak{g} \mid \mathrm{Ad}(g_0)Y = Y\}\\ \mathfrak{b} &= \mathfrak{a}^{\perp},\end{aligned}$$

so $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ with $\mathrm{Ad}(g_0) - I$ an invertible endomorphism of $\mathfrak{b}$. Note that $X_0 \in \mathfrak{a}$ since $\mathrm{Ad}(\exp X_0)X_0 = e^{\mathrm{ad}(X_0)}X_0 = X_0$.

Consider the smooth map $\varphi : \mathfrak{a} \oplus \mathfrak{b} \to G$ given by $\varphi(X,Y) = g_0^{-1}e^Y g_0 e^X e^{-Y}$. Under the usual tangent space identifications, the differential of φ can be computed at 0 by

$$\begin{aligned}d\varphi(X,0) &= \frac{d}{dt}\varphi(tX,0)|_{t=0} = \frac{d}{dt}e^{tX}|_{t=0} = X\\ d\varphi(0,Y) &= \frac{d}{dt}\varphi(0,tY)|_{t=0} = \frac{d}{dt}\left(g_0^{-1}e^{tY}g_0e^{-tY}\right)|_{t=0} = \left(\mathrm{Ad}(g_0) - I\right)Y.\end{aligned}$$

Thus $d\varphi$ is an isomorphism, so that $\{g_0^{-1}e^Y g_0 e^X e^{-Y} \mid X \in \mathfrak{a},\, Y \in \mathfrak{b}\}$ contains a neighborhood of I in G. Since $l_{g_0^{-1}}$ is a diffeomorphism,

$$\{e^Y g_0 e^X e^{-Y} \mid X \in \mathfrak{a}, Y \in \mathfrak{b}\}$$

contains a neighborhood of g_0 in G.

Let $A = Z_G(g_0)^0 = \{g \in G \mid gg_0 = g_0 g\}^0$, a closed and therefore compact Lie subgroup of G. Rewriting the condition $gg_0 = g_0 g$ as $c_{g_0} g = g$, the usual argument using Theorem 4.6 shows that the Lie algebra of A is $\mathfrak{a}$ (Exercise 4.22). In particular, $e^{\mathfrak{a}} \subseteq A$, so that $g_0 e^{\mathfrak{a}} \subseteq A$ since $g_0 \in A$. Thus

$$\{e^Y g_0 e^X e^{-Y} \mid X \in \mathfrak{a}, Y \in \mathfrak{b}\} \subseteq \{e^Y A e^{-Y} \mid Y \in \mathfrak{b}\},$$

and so $\bigcup_{g \in G} g^{-1} A g$ certainly contains a neighborhood of g_0 in G.

Note that $\dim \mathfrak{a} \geq 1$ as $X_0 \in \mathfrak{g}$. If $\dim \mathfrak{a} < \dim \mathfrak{g}$, the inductive hypothesis shows that $A = \exp \mathfrak{a}$, so that $\bigcup_{g \in G} g^{-1} A g = \bigcup_{g \in G} \exp(\mathrm{Ad}(g)\mathfrak{a}) \subseteq \exp \mathfrak{g}$. Thus $\exp \mathfrak{g}$ contains a neighborhood of g_0, as desired.

On the other hand, if $\dim \mathfrak{a} = \dim \mathfrak{g}$, then $\mathrm{Ad}(g_0) = I$ so that Lemma 5.11 shows $g_0 \in Z(G)$. Let $\mathfrak{t}$ be a Cartan subalgebra containing X_0. By Theorem 5.9, $\mathfrak{g} = \bigcup_{g \in G} \mathrm{Ad}(g)\mathfrak{t}$. For any $X \in \mathfrak{t}$, use the facts that $g_0 = e^{X_0} \in Z(G)$ and $[X_0, X] = 0$ to compute

$$g_0 \exp(\mathrm{Ad}(g)X) = g_0 c_g e^X = c_g \left(e^{X_0} e^X\right) = c_g e^{X_0 + X} = \exp(\mathrm{Ad}(g)(X_0 + X))$$

for $g \in G$. Since $X_0 + X \in \mathfrak{t}$, $g_0 \exp \mathfrak{g} \subseteq \exp \mathfrak{g}$. However, Theorem 4.6 shows $\exp \mathfrak{g}$ contains a neighborhood of I so that $g_0 \exp \mathfrak{g}$ contains a neighborhood of g_0. The inclusion $g_0 \exp \mathfrak{g} \subseteq \exp \mathfrak{g}$ finishes the proof. □

Corollary 5.13. *Let G be a compact connected Lie group with maximal torus T.*
(a) *Then $Z_G(T) = T$, where $Z_G(T) = \{g \in G \mid gt = tg \text{ for } t \in T\}$. In particular, T is maximal Abelian.*
(b) *The center of G is contained in T, i.e., $Z(G) \subseteq T$.*

Proof. Part (b) clearly follows from part (a). For part (a), obviously $T \subseteq Z_G(T)$. Conversely, let $g_0 \in Z_G(T)$ and consider the closed, therefore compact, connected Lie subgroup $Z_G(g_0)^0$. Using the Maximal Torus Theorem, write $g_0 = e^{X_0}$. Looking at the path $t \to e^{tX_0} \in Z_G(g_0)$, it follows that $g_0 \in Z_G(g_0)^0$. Moreover, since T is connected and contains I, clearly $T \subseteq Z_G(g_0)^0$, so that T is a maximal torus in $Z_G(g_0)^0$. By the Maximal Torus theorem applied to $Z_G(g_0)^0$, there is $h \in Z_G(g_0)^0$, so that $c_h g_0 \in T$. But by construction, $c_h g_0 = g_0$, so $g_0 \in T$, as desired. □

Note that there exist maximal Abelian subgroups that are not maximal tori (Exercise 5.6).

5.1.5 Exercises

Exercise 5.1 Let Γ be a discrete subgroup of $\mathbb{R}^n$. Pick an indivisible element $e_1 \in \Gamma$ and show that $\Gamma/\mathbb{Z}e_1$ is a discrete subgroup in $\mathbb{R}^n/\mathbb{R}e_1$. Use induction to show that Γ is isomorphic to Γ_k.

Exercise 5.2 For each compact classical Lie group in §5.1.2, show that the given subgroup T is a maximal torus and that the given subalgebra $\mathfrak{t}$ is its corresponding Cartan subalgebra.

Exercise 5.3 Show that the most general connected Abelian Lie subgroup of a general linear Lie group is isomorphic to $T^k \times \mathbb{R}^n$.

Exercise 5.4 Classify the irreducible representations of $T^k \times (\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_k\mathbb{Z})$.

Exercise 5.* *(*Kronecker's Theorem). View $T^k \cong \mathbb{R}^k/\mathbb{Z}^2$ and let $x = (x_i) \in \mathbb{R}^k$. Show that the following statements are equivalent.
(a) The set $\{1, x_1, \ldots, x_n\}$ is linearly dependent over $\mathbb{Q}$.
(b) There is a nonzero $n = (n_i) \in \mathbb{Z}^k$, so $n \cdot x \in \mathbb{Z}$.
(c) There is a nontrivial homomorphism $\pi : \mathbb{R}^k/\mathbb{Z}^k \to S^1$ with $x + \mathbb{Z}^k \in \ker \pi$.
(d) The set $\overline{\mathbb{Z}x + \mathbb{Z}^k} \neq \mathbb{R}^k/\mathbb{Z}^k$.

Exercise 5.5 Working in $\mathrm{Spin}_{2n}(\mathbb{R})$ or $\mathrm{Spin}_{2n+1}(\mathbb{R})$, let

$$T = \{(\cos t_1 + e_1e_2 \sin t_1) \cdots (\cos t_n + e_1e_2 \sin t_n) \mid t_k \in \mathbb{R}\}.$$

Show that T is a maximal torus (c.f. Exercise 1.33).

Exercise 5.6 Find a maximal Abelian subgroup of $SO(3)$ that is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and therefore not a maximal torus.

Exercise 5.7 If H is a closed connected Lie subgroup of a compact Lie group G, show that $\exp \mathfrak{h} = H$.

Exercise 5.8 Let G be a connected Lie subgroup of a general linear group. Show that the center of G, $Z(G)$, is a closed Lie subgroup of G with Lie algebra $\mathfrak{z}(\mathfrak{g})$ (c.f. Exercise 4.22).

Exercise 5.9 Let G be a compact connected Lie group. Show that the center of G, $Z(G)$, is the intersection of all maximal tori in G.

Exercise 5.10 Let G be a compact connected Lie group. For $g \in G$ and positive $n \in \mathbb{N}$, show that there exists $h \in G$ so that $h^n = g$.

Exercise 5.11 Let T be the maximal torus of $SO(3)$ given by

$$T = \left\{ \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \theta \in \mathbb{R} \right\}.$$

Find $g \in SO(3)$, so that $Z_G(g)^0 = T$ but with $Z_G(g)$ disconnected.

Exercise 5.12 Let G be a compact connected Lie group. Suppose S is a connected Abelian Lie subgroup of G.
(1) If $g \in Z_G(S)$, show that there exists a maximal torus T of G, so that $g \in T$ and $S \subseteq T$.
(2) Show that $Z_G(S)$ is the union of all maximal tori containing S and therefore is connected.
(3) For $g \in G$, show that $Z_G(g)^0$ is the union of all maximal tori containing g.

Exercise 5.13 Let G be a compact connected Lie group. Suppose that G is also a complex manifold whose group operations are holomorphic. Then the map $g \to \mathrm{Ad}(g)$, $g \in G$, is holomorphic. Show that G is Abelian and isomorphic to $\mathbb{C}^n / \Gamma$ for some discrete subgroup Γ of $\mathbb{C}^n$.

5.2 Structure

5.2.1 Exponential Map Revisited

5.2.1.1 Local Diffeomorphism Let G be a Lie subgroup of $GL(n, \mathbb{C})$. We already know from Theorem 4.6 that $\exp\colon \mathfrak{g} \to G$ is a local diffeomorphism near 0. In fact, more is true. Before beginning, use power series to define

$$\frac{I - e^{-\operatorname{ad} X}}{\operatorname{ad} X} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} (\operatorname{ad} X)^n$$

for $X \in \mathfrak{g}$.

Theorem 5.14. *(a) Let G be a Lie subgroup of $GL(n, \mathbb{C})$ and $\gamma : \mathbb{R} \to \mathfrak{g}$ a smooth curve. Then*

$$\begin{aligned} \frac{d}{dt} e^{\gamma(t)} &= e^{\gamma(t)} \left(\frac{I - e^{-\operatorname{ad} \gamma(t)}}{\operatorname{ad} \gamma(t)} \right) \left(\gamma'(t) \right) \\ &= \left[\left(\frac{e^{\operatorname{ad} \gamma(t)} - I}{\operatorname{ad} \gamma(t)} \right) \left(\gamma'(t) \right) \right] e^{\gamma(t)}. \end{aligned}$$

(b) For $X \in \mathfrak{g}$, the map $\exp\colon \mathfrak{g} \to G$ *is a local diffeomorphism near X if and only if the eigenvalues of* $\operatorname{ad} X$ *on $\mathfrak{g}$ are disjoint from $2\pi i \mathbb{Z} \backslash \{0\}$.*

Proof. In part (a), consider the special case of, say, $\gamma(t) = X + tY$ for $Y \in \mathfrak{g}$. Using the usual tangent space identifications at $t = 0$, the first part of (a) calculates the differential at X of the map $\exp : \mathfrak{g} \to G$ evaluated on Y. If $(\operatorname{ad} X)\, Y = \lambda Y$ for $\lambda \in \mathbb{C}$, then

$$\left(\frac{I - e^{-\operatorname{ad} X}}{\operatorname{ad} X} \right) (Y) = \begin{cases} \frac{1-e^{-\lambda}}{\lambda} Y & \text{if } \lambda \in 0 \\ Y & \text{if } \lambda = 0 \end{cases}$$

which is zero if and only if $\lambda \in 2\pi i\mathbb{Z}\backslash\{0\}$. Since Lemma 5.6 shows that $\operatorname{ad} X$ is normal and therefore diagonalizable, part (b) follows from the Inverse Mapping Theorem and part (a).

With sufficient patience, the proof of part (a) can be accomplished by explicit power series calculations. As is common in mathematics, we instead resort to a trick. Define $\varphi : \mathbb{R}^2 \to G$ by

$$\varphi(s,t) = e^{-s\gamma(t)} \frac{\partial}{\partial t} e^{s\gamma(t)}.$$

To prove the first part of (a), it is necessary to show $\varphi(1,t) = \left(\frac{I-e^{-\operatorname{ad}\gamma(t)}}{\operatorname{ad}\gamma(t)}\right)\left(\gamma'(t)\right)$. Begin by observing that $\varphi(0,t) = 0$, and so $\varphi(1,t) = \int_0^1 \frac{\partial}{\partial s}\varphi(s,t)\,ds$. However,

$$\begin{aligned}
\frac{\partial}{\partial s}\varphi(s,t) &= -\gamma(t)e^{-s\gamma(t)}\frac{\partial}{\partial t}e^{s\gamma(t)} + e^{-s\gamma(t)}\frac{\partial}{\partial t}\left[\gamma(t)e^{s\gamma(t)}\right] \\
&= -e^{-s\gamma(t)}\gamma(t)\frac{\partial}{\partial t}e^{s\gamma(t)} + e^{-s\gamma(t)}\gamma'(t)e^{s\gamma(t)} + e^{-s\gamma(t)}\gamma(t)\frac{\partial}{\partial t}e^{s\gamma(t)} \\
&= e^{-s\gamma(t)}\gamma'(t)e^{s\gamma(t)},
\end{aligned}$$

so that $\frac{\partial}{\partial s}\varphi(s,t) = \operatorname{Ad}(e^{-s\gamma(t)})\gamma'(t) = e^{-s\operatorname{ad}\gamma(t)}\gamma'(t)$ by Equation 4.11. Thus

$$\begin{aligned}
\varphi(1,t) &= \int_0^1 e^{-s\operatorname{ad}\gamma(t)}\gamma'(t)\,ds = \int_0^1 \sum_{n=0}^{\infty}\frac{(-s)^n}{n!}\left(\operatorname{ad}\gamma(t)\right)^n\gamma'(t)\,ds \\
&= \left(\sum_{n=0}^{\infty}\frac{(-1)^n s^{n+1}}{(n+1)!}\left(\operatorname{ad}\gamma(t)\right)^n\gamma'(t)\right)\Big|_{s=0}^{s=1} = \left(\frac{I-e^{-\operatorname{ad}\gamma(t)}}{\operatorname{ad}\gamma(t)}\right)\gamma'(t),
\end{aligned}$$

as desired. To show the second part of (a), use the relation $l_{e^{\gamma(t)}} = r_{e^{\gamma(t)}} \circ \operatorname{Ad}(e^{\gamma(t)}) = r_{e^{\gamma(t)}} \circ e^{\operatorname{ad}\gamma(t)}$, where $l_{e^{\gamma(t)}}$ and $r_{e^{\gamma(t)}}$ stand for left and right multiplication by $e^{\gamma(t)}$. □

5.2.1.2 Dynkin's Formula Let G be a Lie subgroup of $GL(n,\mathbb{C})$. For $X_i \in \mathfrak{g}$, write $[X_n, \ldots, X_3, X_2, X_1]$ for the *iterated Lie bracket*

$$\left[X_n, \ldots, \left[X_3, \left[X_2, X_1\right]\right], \ldots\right]$$

and write $[X_n^{(i_n)}, \ldots, X_1^{(i_1)}]$ for the iterated Lie bracket

$$[\overbrace{X_n, \ldots, X_n}^{i_n \text{ copies}}, \ldots, \overbrace{X_1, \ldots, X_1}^{i_1 \text{ copies}}].$$

Although now known as the *Campbell–Baker–Hausdorff Series* ([21], [5], and [49]), the following explicit formula is actually due to Dynkin ([35]). In the proof we use the well-known fact that $\ln(X)$ inverts e^X on a neighborhood of I, where $\ln(I+X) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}X^n$ converges absolutely on a neighborhood 0 (Exercise 5.15).

Theorem 5.15 (Dynkin's Formula). *Let G be a Lie subgroup of $GL(n,\mathbb{C})$. For $X, Y \in \mathfrak{g}$ in a sufficiently small neighborhood of* 0*,*

$$e^X e^Y = e^Z,$$

where Z is given by the formula

$$Z = \sum \frac{(-1)^{n+1}}{n} \frac{1}{(i_1 + j_1) + \cdots + (i_n + j_n)} \frac{[X^{(i_1)}, Y^{(j_1)}, \ldots, X^{(i_1)}, Y^{(j_n)}]}{i_1! j_1! \cdots i_n! j_n!},$$

where the sum is taken over all $2n$-tuples $(i_1, \ldots, i_n, j_1, \ldots, j_n) \in \mathbb{N}^{2n}$ satisfying $i_k + j_k \geq 1$ for positive $n \in \mathbb{N}$.

Proof. The approach of this proof follows [34]. Using Theorem 4.6, choose a neighborhood U_0 of 0 in $\mathfrak{g}$ on which exp is a local diffeomorphism and where ln is well defined on $\exp U$. Let $U \subseteq U_0$ be an open ball about of 0 in $\mathfrak{g}$, so that $(\exp U)^2 (\exp U)^{-2} \subseteq \exp U_0$ (by continuity of the group structure as in Exercise 1.4). For $X, Y \in U$, define $\gamma(t) = e^{tX} e^{tY}$ mapping a neighborhood of $[0, 1]$ to $\exp U$. Therefore there is a unique smooth curve $Z(t) \in U_0$, so that $e^{Z(t)} = e^{tX} e^{tY}$. Apply $\frac{d}{dt}$ to this equation and use Theorem 5.14 to see that

$$\left[\left(\frac{e^{\operatorname{ad} Z(t)} - I}{\operatorname{ad} Z(t)}\right) \left(Z'(t)\right)\right] e^{Z(t)} = X e^{Z(t)} + e^{Z(t)} Y.$$

Since $Z(t) \in U_0$, exp is a local diffeomorphism near $Z(t)$. Thus the proof of Theorem 5.14 shows that $\left(\frac{I - e^{-\operatorname{ad} Z(t)}}{\operatorname{ad} Z(t)}\right)$ is an invertible map on $\mathfrak{g}$. As $e^{Z(t)} = e^{tX} e^{tY}$, $\operatorname{Ad}(e^{Z(t)}) = \operatorname{Ad}\left(e^{tX}\right) \operatorname{Ad}\left(e^{tY}\right)$, so that $e^{\operatorname{ad} Z(t)} = e^{t \operatorname{ad} X} e^{t \operatorname{ad} Y}$ by Equation 4.11. Thus

$$\begin{aligned} Z'(t) &= \left(\frac{\operatorname{ad} Z(t)}{e^{\operatorname{ad} Z(t)} - I}\right) \left(X + \operatorname{Ad}(e^{Z(t)}) Y\right) = \left(\frac{\operatorname{ad} Z(t)}{e^{\operatorname{ad} Z(t)} - I}\right) \left(X + e^{\operatorname{ad} Z(t)} Y\right) \\ &= \left(\frac{\operatorname{ad} Z(t)}{e^{\operatorname{ad} Z(t)} - I}\right) \left(X + e^{t \operatorname{ad} X} e^{t \operatorname{ad} Y} Y\right) = \left(\frac{\operatorname{ad} Z(t)}{e^{\operatorname{ad} Z(t)} - I}\right) \left(X + e^{t \operatorname{ad} X} Y\right). \end{aligned}$$

Using the relation $A = \ln(I + (e^A - I)) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(e^A - I\right)^n$ for $A = \operatorname{ad} Z(t)$ and $e^A = e^{t \operatorname{ad} X} e^{t \operatorname{ad} Y}$, we get

$$\frac{\operatorname{ad} Z(t)}{e^{\operatorname{ad} Z(t)} - I} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(e^{t \operatorname{ad} X} e^{t \operatorname{ad} Y} - I\right)^{n-1}.$$

Hence

$$Z'(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(e^{t \operatorname{ad} X} e^{t \operatorname{ad} Y} - I\right)^{n-1} \left(X + e^{t \operatorname{ad} X} Y\right)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left[\sum_{i,j=0,\ (i,j)\neq(0,0)}^{\infty} \frac{t^{i+j}}{i!\, j!} (\operatorname{ad} X)^i (\operatorname{ad} Y)^j\right]^{n-1} \left(X + \left(\sum_{i=0}^{\infty} \frac{t^i}{i!} (\operatorname{ad} X)^i\right) Y\right)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left[\sum \frac{t^{i_1+j_1+\cdots i_{n-1}+j_{n-1}}}{i_1!j_1!\cdots i_{n-1}!j_{n-1}!} [(X)^{i_1}, (Y)^{j_1}, \ldots, (X)^{i_{k-1}}, (Y)^{j_{k-1}}, X] \right.$$
$$\left. + \sum \frac{t^{i_1+j_1+\cdots i_{n-1}+j_{n-1}+i_n}}{i_1!j_1!\cdots i_{n-1}!j_{n-1}!i_n!} [(X)^{i_1}, (Y)^{j_1}, \ldots, (X)^{i_{k-1}}, (Y)^{j_{k-1}}, (X)^{i_n}, Y] \right]$$

where the second and third sum are taken over all $i_k, j_k \in \mathbb{N}$ with $i_k + j_k \geq 1$. Since $Z(0) = 0$, $Z(1) = \int_0^1 \frac{d}{dt} Z(t)\, dt$. Integrating the above displayed equation finishes the proof. □

The explicit formula for Z in Dynkin's Formula is actually not important. In practice it is much too difficult to use. However, what *is* important is the fact that such a formula exists using only Lie brackets.

Corollary 5.16. *Let N be a connected Lie subgroup of $GL(n, \mathbb{C})$ whose Lie algebra $\mathfrak{n}$ lies in the set of* strictly upper triangular *matrices, i.e., if $X \in \mathfrak{n}$, then $X_{i,j} = 0$ when $i \geq j$. Then the map* $\exp\colon \mathfrak{n} \to N$ *is surjective, i.e.,* $N = \exp \mathfrak{n}$.

Proof. It is a simple exercise to see that $[X_n, \ldots, X_3, X_2, X_1] = 0$ for any strictly upper triangular $X, X_i \in \mathfrak{gl}(n, \mathbb{C})$ and that e^X is polynomial in X (Exercise 5.18). In particular, for $X, Y \in \mathfrak{n}$ near 0, Dynkin's Formula gives a polynomial expression for $Z \in \mathfrak{n}$ solving $e^X e^Y = e^Z$. Since both sides of this expression are polynomials in X and Y that agree on a neighborhood, they agree everywhere. Because the formula for Z involves only the algebra structure of $\mathfrak{n}$, Z remains in $\mathfrak{n}$ for $X, Y \in \mathfrak{n}$. In other words, $(\exp \mathfrak{n})^2 \subseteq \exp \mathfrak{n}$. Since $\exp \mathfrak{n}$ generates N by Theorem 1.15, this shows that $\exp \mathfrak{n} = N$. □

5.2.2 Lie Algebra Structure

If G_i are Lie subgroups of a linear group, then, as in the proof of Theorem 4.16, recall that the *direct sum* of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, may be viewed as the Lie algebra of $G_1 \times G_2$ with $[X_1 + X_2, Y_1 + Y_2] = [X_1, X_2] + [Y_1, Y_2]$ for $X_i, Y_i \in \mathfrak{g}_i$.

Definition 5.17. (a) Let $\mathfrak{g}$ be the Lie algebra of a Lie subgroup of a linear group. Then $\mathfrak{g}$ is called *simple* if $\mathfrak{g}$ has no proper ideals and if $\dim \mathfrak{g} > 1$, i.e., if the only ideals of $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$ and $\mathfrak{g}$ is non-Abelian.
(b) The Lie algebra $\mathfrak{g}$ is called *semisimple* if $\mathfrak{g}$ is a direct sum of simple Lie algebras.
(c) The Lie algebra $\mathfrak{g}$ is called *reductive* if $\mathfrak{g}$ is a direct sum of a semisimple Lie algebra and an Abelian Lie algebra.
(d) Let $\mathfrak{g}'$ be the ideal of $\mathfrak{g}$ spanned by $[\mathfrak{g}, \mathfrak{g}]$.

Theorem 5.18. *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is reductive. If $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$, i.e., $\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] = 0\}$, then*

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}(\mathfrak{g}),$$

$\mathfrak{g}'$ is semisimple, and $\mathfrak{z}(\mathfrak{g})$ is Abelian. Moreover, there are simple ideals $\mathfrak{s}_i$ of $\mathfrak{g}'$, so that

$$\mathfrak{g}' = \bigoplus_{i=1}^{k} \mathfrak{s}_i$$

with $[\mathfrak{s}_i, \mathfrak{s}_j] = 0$ *for* $i \neq j$ *and* $\operatorname{span}[\mathfrak{s}_i, \mathfrak{s}_i] = \mathfrak{s}_i$.

Proof. Using Lemma 5.6, let $(\cdot, \cdot)$ be an Ad-invariant inner product on $\mathfrak{g}$, so that $\operatorname{ad} X$, $X \in \mathfrak{g}$, is skew-Hermitian. If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{a}^\perp$ is also an ideal. It follows that $\mathfrak{g}$ can be written as a direct sum of minimal ideals

$$\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k \oplus \mathfrak{z}_1 \oplus \cdots \oplus \mathfrak{z}_n, \tag{5.19}$$

where $\dim \mathfrak{s}_i > 1$ and $\dim \mathfrak{z}_j = 1$. Since $\mathfrak{s}_i$ is an ideal, $[\mathfrak{s}_i, \mathfrak{s}_j] \subseteq \mathfrak{s}_i \cap \mathfrak{s}_j$. Thus $[\mathfrak{s}_i, \mathfrak{s}_j] = 0$ for $i \neq j$ and $[\mathfrak{s}_i, \mathfrak{s}_i] \subseteq \mathfrak{s}_i$. Similarly, $[\mathfrak{s}_i, \mathfrak{z}_j] = 0$ and $[\mathfrak{z}_i, \mathfrak{z}_j] = 0$ for $i \neq j$. Moreover, $[\mathfrak{z}_i, \mathfrak{z}_i] = 0$ since $\dim \mathfrak{z}_i = 1$ and $[\cdot, \cdot]$ is skew-symmetric.

In particular, $\mathfrak{z}_1 \oplus \cdots \oplus \mathfrak{z}_n \subseteq \mathfrak{z}(\mathfrak{g})$. On the other hand, if $Z \in \mathfrak{z}(\mathfrak{g})$ is decomposed according to Equation 5.19 as $Z = \sum_i S_i + \sum_j Z_j$, then $0 = [Z, \mathfrak{s}_i] = [S_i, \mathfrak{s}_i]$. This suffices to show that $S_i \in \mathfrak{z}(\mathfrak{g})$ which, by construction of $\mathfrak{s}_i$ as a minimal ideal with $\dim \mathfrak{s}_i > 1$, implies that $S_i = 0$. Thus $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}_1 \oplus \cdots \oplus \mathfrak{z}_n$. The remainder of the proof follows by showing that $\operatorname{span}[\mathfrak{s}_i, \mathfrak{s}_i] = \mathfrak{s}_i$. However, this too follows from the construction of $\mathfrak{s}_i$ as a minimal ideal. Since $\mathfrak{s}_i$ is not central, $\dim(\operatorname{span}[\mathfrak{s}_i, \mathfrak{s}_i]) \geq 1$. As a result, $\dim(\operatorname{span}[\mathfrak{s}_i, \mathfrak{s}_i])$ cannot be less than $\dim \mathfrak{s}_i$ either, or else $\operatorname{span}[\mathfrak{s}_i, \mathfrak{s}_i]$ would be a proper ideal. □

It is an important theorem from the study of Lie algebras (see [56], [61], or [70]) that the simple Lie algebras are classified. It is rather remarkable that there are, relatively speaking, so few of them. In §6.1.2 we will discuss the complexification of our Lie algebras. In that setting, there are four infinite families of simple complex Lie algebras. They arise from the compact classical Lie groups $SU(n)$, $SO(2n+1)$, $Sp(n)$, and $SO(2n)$. Beside these families, there are only five other simple complex Lie algebras. They are called exceptional and go by the names G_2, F_4, E_6, E_7, and E_8. They have dimensions are 14, 52, 78, 133, and 248, respectively.

5.2.3 Commutator Theorem

Definition 5.20. Let G be a Lie subgroup of a linear group. The *commutator subgroup*, G', is the normal subgroup of G generated by

$$\{g_1 g_2 g_1^{-1} g_2^{-1} \mid g_i \in G\}.$$

In a more general setting, G' need not be closed, however this nuisance does not arise for compact Lie groups.

Theorem 5.21. *Let G be a compact connected Lie group. Then G' is a connected closed normal Lie subgroup of G with Lie algebra $\mathfrak{g}'$.*

Proof. As usual, Theorem 3.28, assume G is a closed Lie subgroup of $U(n)$. With respect to the standard representation on $\mathbb{C}^n$, decompose $\mathbb{C}^n$ into its irreducible summands under the action of G, $\mathbb{C}^n \cong \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_k}$ with $n_1 + \cdots + n_k = n$ and $n_i \geq 1$. Thus G can be viewed as a closed Lie subgroup of $U(n_1) \times \cdots \times U(n_k)$, so that the induced projection $\pi_i : G \to U(n_i)$ yields an irreducible representation of G.

Let $\varphi : G \to S^1 \times \cdots \times S^1$ (k copies) be the homomorphism induced by taking the determinant of each $\pi_i(g)$. Define H to be the closed Lie subgroup of G given by $H = \ker \varphi$. We will show that $\mathfrak{h} = \mathfrak{g}'$ and that $H^0 = G'$, which will finish the proof.

Recall that it follows easily from Theorem 4.6 that $\mathfrak{h} = \ker d\varphi$ (Exercise 4.24) and that the Lie algebra of $Z(G)$ is $\mathfrak{z}(\mathfrak{g})$ (c.f. Exercises 4.22 or 5.8). Now if $Z \in \mathfrak{z}(\mathfrak{g})$, then $e^{tZ} \in Z(G)$, so that Schur's Lemma implies $\pi_i e^{tZ} = c_i(t) I$ for some scalar $c_i(t)$ with $c_i(0) = 1$. Evaluating at $\frac{d}{dt}|_0$, this means that

$$Z = \operatorname{diag}(\overbrace{c_1'(0), \dots, c_1'(0)}^{n_1 \text{ copies}}, \dots, \overbrace{c_k'(0), \dots, c_k'(0)}^{n_k \text{ copies}}).$$

Hence $d\varphi(Z) = (n_1 c_1'(0), \dots, n_k c_k'(0))$ and $Z \in \ker \varphi$ if and only if $Z = 0$. On the other hand, since $\mathfrak{g}'$ is spanned by $[\mathfrak{g}, \mathfrak{g}]$, clearly $\operatorname{tr}(d\pi_i X) = 0$ for $X \in \mathfrak{g}'$ (Exercise 5.20). Thus $\det \pi_i e^{tX} = 1$ (Exercise 4.3), so that $\mathfrak{g}' \subseteq \ker d\varphi$. Combined with the decomposition from Theorem 5.18, it follows that $\mathfrak{h} = \mathfrak{g}'$.

Turning to G', let $U = \{g_1 g_2 g_1^{-1} g_2^{-1} \mid g_i \in G\}$. Since U is the continuous image of $G \times G$ under the obvious map, U is connected. As $I \in U$, $I \in U^j$ and since $G' = \cup_j U^j$, G' is therefore connected.

Next, by the multiplicative nature of determinants and the definition of the commutator, it follows that $\pi_i G' \subseteq SU(n_i)$ so that $G' \subseteq H$. It only remains to see $H^0 \subseteq G'$ since G' is connected. For this, it suffices to show that G' contains a neighborhood of I in H by Theorem 1.15.

To this end, for $X, Y \in \mathfrak{h}$, define $c_{X,Y}(t) \in H \cap G'$, $t \in \mathbb{R}$, by

$$c_{X,Y}(t) = \begin{cases} e^{\sqrt{t}X} e^{\sqrt{t}Y} e^{-\sqrt{t}X} e^{-\sqrt{t}Y} & t \geq 0 \\ e^{\sqrt{|t|}X} e^{-\sqrt{|t|}Y} e^{-\sqrt{|t|}X} e^{\sqrt{|t|}Y} & t < 0. \end{cases}$$

Using either Dynkin's Formula (Exercise 5.21, c.f. Exercises 4.26 and 5.16) or elementary power series calculations, it easily follows that $c_{X,Y}(t) = e^{t[X,Y] + O(|t|^{\frac{3}{2}})}$ for t near 0, so that $c_{X,Y}$ is continuously differentiable with $c_{X,Y}'(0) = [X, Y]$.

Let $\{[X_i, Y_i]\}_{i=1}^p$ be a basis for $\mathfrak{g}'$ and consider the map $c : \mathbb{R}^p \to H$ given by $c(t_1, \dots, t_p) = \prod_i c_{X_i, Y_i}(t)$. As $c_{X,Y}'(0) = [X, Y]$, the differential of c at 0 is an isomorphism to $\mathfrak{h}$ (c.f. Exercise 4.12). Thus the image of c contains a neighborhood of I in H and, since $c(t)$ is also in G', the proof is finished. □

5.2.4 Compact Lie Group Structure

Theorem 5.22. ***(a)*** *Let G be a compact connected Lie group. Then $G = G' Z(G)^0$, $Z(G') = G' \cap Z(G)$ is a finite Abelian group, $Z(G)^0$ is a torus, and*

$$G \cong \left[G' \times Z(G)^0\right]/F,$$

where the finite Abelian group $F = G' \cap Z(G)^0$ *is embedded in* $G' \times Z(G)^0$ *as* $\{(f, f^{-1}) \mid f \in F\}$.

(b) *Decompose* $\mathfrak{g}' = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k$ *into simple ideals as in Theorem 5.18 and let* $S_i = \exp \mathfrak{s}_i$. *Then* S_i *is a connected closed normal Lie subgroup of* G' *with Lie algebra* $\mathfrak{s}_i$. *The only proper closed normal Lie subgroups of* S_i *are discrete, finite, and central in* G. *Moreover, the map* $(s_1, \ldots, s_k) \to s_i \cdots s_k$ *from* $S_1 \times \cdots \times S_k$ *to* G' *is a surjective homomorphism with finite central kernel,* F', *so that*

$$G' \cong [S_1 \times \cdots \times S_k]/F'.$$

Proof. For part (a), first note that G' is closed and therefore compact. It follows from the Maximal Torus Theorem that $G' = \exp \mathfrak{g}'$. Using the decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}(\mathfrak{g})$ and the fact that the Lie algebra of $Z(G)$ is $\mathfrak{z}(\mathfrak{g})$, Theorems 5.1 and 5.2 show $\exp \mathfrak{g} = G'Z(G)^0$. Thus $G = G'Z(G)^0$. This relation also shows that $Z(G') = G' \cap Z(G)$.

Using the machinery from the proof Theorem 5.21, any $Z \in Z(G)$ must be of the form $Z = \operatorname{diag}(c_1, \ldots, c_1, \ldots, c_k, \ldots, c_k)$. If also $Z \in G'$, then $c_1^{n_1} = \ldots, c_k^{n_k} = 1$, so that c_i is an n_i^{th}-root of unity. In particular, $G' \cap Z(G)$ is a finite Abelian group. Finally, consider the surjective homomorphism mapping $(g, z) \to gz$ from $G' \times Z(G)^0 \to G$. Clearly the kernel is $\{(f, f^{-1}) \mid f \in F\}$, as desired.

For part (b), the fact that $\mathfrak{s}_i$ and $\mathfrak{s}_j$, $i \neq j$, commute and $G' = \exp \mathfrak{g}'$ show $G' = S_1 \cdots S_k$ with S_i and S_j, $i \neq j$, commuting. It is necessary to verify that S_i is a closed Lie subgroup. To this end, write $\operatorname{Ad}(g)|_{\mathfrak{s}_j}$ for $\operatorname{Ad}(g)$ restricted to $\mathfrak{s}_j$ and let $K_i = \{g \in G' \mid \operatorname{Ad}(g)|_{\mathfrak{s}_j} = I,\ j \neq i\}^0$. Obviously K_i is a connected closed Lie subgroup of G'. We will show that $K_i = S_i$.

Now $X \in \mathfrak{k}_i$ if and only if $e^{tX} \in K_i$, $t \in \mathbb{R}$, if and only if $\operatorname{Ad}(e^{tX})|_{\mathfrak{s}_j} = e^{t\,\operatorname{ad}(X)}|_{\mathfrak{s}_j} = I$ for $j \neq i$. Using $\frac{d}{dt}|_{t=0}$, $X \in \mathfrak{k}_i$ if and only if $[X, \mathfrak{s}_j] = 0$ for $j \neq i$. Since $\mathfrak{s}_j$ is an ideal with $\operatorname{span}[\mathfrak{s}_j, \mathfrak{s}_j] = \mathfrak{s}_j$ and $[\mathfrak{s}_i, \mathfrak{s}_j] = 0$, decomposing X according to $\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k$ shows $\mathfrak{k}_i = \mathfrak{s}_i$. Thus $K_i = \exp \mathfrak{k}_i = S_i$ and in particular, S_i is a closed connected normal Lie subgroup with Lie algebra $\mathfrak{s}_i$.

If N is a normal Lie subgroup of S_i, then $c_s N = N$ for $s \in S_i$. Since $\operatorname{Ad}(s)$ is the differential of c_s, $\operatorname{Ad}(s)\mathfrak{n} = \mathfrak{n}$. Since ad is the differential of Ad, $\operatorname{ad}(X)\mathfrak{n} \subseteq \mathfrak{n}$, $X \in \mathfrak{s}_i$, so that $\mathfrak{n}$ is an ideal (c.f. Exercise 4.23). By construction, this forces $\mathfrak{n}$ to be $\mathfrak{s}_i$ or $\{0\}$, so that $N = S_i$ or $N^0 = I$. Assuming further that N is proper and closed, therefore compact, N must be discrete and finite. Lemma 1.21 shows that N is central. Finally, the differential of the map $(s_1, \ldots, s_k) \to s_i \cdots s_k$ is obviously the identity map, so that F' is discrete and normal. As above, this shows that F' is finite and central as well. □

The effect of Theorem 5.22 is to reduce the study of connected compact groups to the study of connected compact groups with simple Lie algebras.

5.2.5 Exercises

Exercise 5.14 Let $X \in \mathfrak{gl}(n, \mathbb{C})$ be diagonalizable with eigenvalues $\{\lambda_i\}_{i=1}^n$. Show that $\operatorname{ad}(X)$ has eigenvalues $\{\lambda_i - \lambda_j\}_{i,j=1}^n$.

Exercise 5.15 Show that $\ln(I+X)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}X^n$ is well defined for X in a neighborhood of I in $GL(n,\mathbb{C})$. On that neighborhood, show that $\ln X$ is the inverse function to e^X.

Exercise 5.16 Let G be a Lie subgroup of $GL(n,\mathbb{C})$. For $X,Y\in\mathfrak{g}$ in a sufficiently small neighborhood of 0, show that

$$e^Xe^Y=e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]+\frac{1}{12}[Y,[Y,X]]+\frac{1}{24}[Y,[X,[Y,X]]]+\cdots}.$$

Exercise 5.17 Let G be a Lie subgroup of $GL(n,\mathbb{C})$. For $X,Y\in\mathfrak{g}$ in a sufficiently small neighborhood of 0, write $e^Xe^Y=e^Z$. Modify the proof of Dynkin's Formula to show Z is also given by the formula

$$Z=\sum\frac{(-1)^n}{n+1}\frac{1}{i_1+\cdots+i_n+1}\frac{[X^{(i_1)},Y^{(j_1)},\ldots,X^{(i_1)},Y^{(j_n)},X]}{i_1!j_1!\cdots i_n!j_n!}$$

by starting with $e^{Z(t)}=e^{tX}e^Y$.

Exercise 5.18 Let $X,X_i\in\mathfrak{gl}(n,\mathbb{C})$ be strictly upper triangular. Show that $X_n\cdots X_2X_1=0$, so that $\left[X_n,\ldots\left[X_3,\left[X_{2,}X_1\right]\right]\ldots\right]=0$ and e^X is a polynomial in X.

Exercise 5.19 Let N_i be connected Lie subgroups of $GL(n,\mathbb{C})$ whose Lie algebras $\mathfrak{n}_i$ lie in the set of strictly upper triangular matrices. Suppose $\psi:\mathfrak{n}_1\to\mathfrak{n}_2$ is a linear map. Show ψ descends to a well-defined homomorphism of groups $\varphi:N_1\to N_2$ by $\varphi(e^X)=e^{\psi X}$ if and only if ψ is a Lie algebra homomorphism.

Exercise 5.20 For $X,Y\in\mathfrak{gl}(n,\mathbb{C})$, show that $\operatorname{tr}XY=\operatorname{tr}YX$.

Exercise 5.21 In the proof of Theorem 5.21, verify that $c_{X,Y}(t)=e^{t[X,Y]+O(|t|^{\frac{3}{2}})}$.

Exercise 5.22 **(1)** Take advantage of diagonalization to show directly that $U(n)'=SU(n)$.
(2) Show that $GL(n,\mathbb{F})'=SL(n,\mathbb{F})$.

Exercise 5.23 For a Lie subgroup $G\subseteq U(n)$, show that the differential of the determinant $\det:G\to S^1$ is the trace.

Exercise 5.24 **(1)** Let G be a Lie subgroup of $GL(n,\mathbb{C})$. Show that G' is the smallest normal subgroup subgroup of G whose quotient group in G is commutative.
(2) Show that $\mathfrak{g}'$ is the smallest ideal of $\mathfrak{g}$ whose quotient algebra in $\mathfrak{g}$ is commutative.

Exercise 5.25 Let G be a compact connected Lie group and write $G=S_1\cdots S_kZ(G)^0$ as in Theorem 5.22. Show that any closed normal Lie subgroup of G is a product of some of the S_i with a central subgroup.