6

Roots and Associated Structures

By examining the joint eigenvalues of a Cartan subalgebra under the ad-action, a great deal of information about a Lie group and its Lie algebra may be encoded. For instance, the fundamental group can be read off from this data (§6.3.3). Moreover, this encoding is a key step in the classification of irreducible representations (§7).

6.1 Root Theory

6.1.1 Representations of Lie Algebras

Definition 6.1. **(a)** Let $\mathfrak{g}$ be the Lie algebra of a Lie subgroup of $GL(n, \mathbb{C})$. A *representation* of $\mathfrak{g}$ is a pair (ψ, V), where V is a finite-dimensional complex vector space and ψ is a linear map $\psi : \mathfrak{g} \to \operatorname{End}(V)$, satisfying $\psi([X, Y]) = \psi(X) \circ \psi(Y) - \psi(Y) \circ \psi(X)$ for $X, Y \in \mathfrak{g}$.
(b) The representation (ψ, V) is said to be *irreducible* if there are no proper $\psi(\mathfrak{g})$-invariant subspaces, i.e., the only $\psi(\mathfrak{g})$-invariant subspaces are $\{0\}$ and V. Otherwise (ψ, V) is called *reducible*.

As with group representations, a Lie algebra representation (ψ, V) may simply be written as ψ or as V when no ambiguity can arise. Also similar to the group case, $\psi(X)v$, $v \in V$, may be denoted by $X \cdot v$ or by Xv.

It should be noted that if V is m-dimensional, a choice of basis allows us to view a representation of $\mathfrak{g}$ as a homomorphism $\psi : \mathfrak{g} \to \mathfrak{gl}(m, \mathbb{C})$, i.e., ψ is linear and satisfies $\psi[X, Y] = [\psi X, \psi Y]$. We will often make this identification without comment.

Theorem 6.2. ***(a)*** *Let G be a Lie subgroup of $GL(n, \mathbb{C})$ and (π, V) a finite-dimensional representation of G. Then $(d\pi, V)$ is a representation of $\mathfrak{g}$ satisfying $e^{d\pi X} = \pi(e^X)$, where the differential of π is given by $d\pi(X) = \frac{d}{dt}\pi(e^{tX})|_{t=0}$ for $X \in \mathfrak{g}$. If G is connected, π is completely determined by $d\pi$.*
(b) *For connected G, a subspace $W \subseteq V$ is $\pi(G)$-invariant if and only if it is $d\pi(\mathfrak{g})$-invariant. In particular, V is irreducible under G if and only if it is irreducible under $\mathfrak{g}$.*

(c) *For connected compact G, V is irreducible if and only if the only endomorphisms of V commuting with all the operators $d\pi(\mathfrak{g})$ are scalar multiples of the identity map.*

Proof. Part (a) follows immediately from Theorem 4.8 by looking at the homomorphism $\pi : G \to GL(V)$ and choosing a basis for V. Part (b) follows from the relation $e^{d\pi X} = \pi(e^X)$, the definition of $d\pi$, and the fact that $\exp \mathfrak{g}$ generates G. For part (c), let $T \in \operatorname{End}(V)$ and embed G in $GL(n, \mathbb{C})$. Observe that $[T, d\pi X] = 0$ if and only if $e^{t\,\mathrm{ad}(d\pi X)}T = T$, $t \in \mathbb{R}$, if and only if $\operatorname{Ad}(e^{td\pi X})T = T$ if and only if T commutes with $e^{td\pi X} = \pi(e^{tX})$. Using the fact that G is connected, part (c) follows from part (b) and Schur's Lemma. □

As an example, let G be a Lie subgroup of $GL(n, \mathbb{C})$ and let $(\pi, \mathbb{C})$ be the trivial representation of G. Then $d\pi = 0$. This representation of $\mathfrak{g}$ is called the *trivial representation.*

As a second example, let G be a Lie subgroup of $GL(n, \mathbb{C})$ and let $(\pi, \mathbb{C}^n)$ be the standard representation. Then $d\pi(X)v = Xv$ for $v \in \mathbb{C}^n$. This representation is called the *standard representation.* In the cases of G equal to $GL(n, \mathbb{F})$, $SL(n, \mathbb{F})$, $U(n)$, $SU(n)$, or $SO(n)$, the standard representation is known to be irreducible on the Lie group level (§2.2.2), so that each is irreducible on the Lie algebra level.

As a last example, consider the representation $V_n(\mathbb{C}^2)$ of $SU(2)$ from §2.1.2.2 given by

$$\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \cdot z_1^k z_2^{n-k} = (\overline{a}z_1 + \overline{b}z_2)^k(-bz_1 + az_2)^{n-k}.$$

From §4.1.3, $\mathfrak{su}(2) = \{X = \begin{pmatrix} ix & -\overline{w} \\ w & -ix \end{pmatrix} \mid x \in \mathbb{R}, w \in \mathbb{C}\}$. Using either power series calculations or Corollary 4.9, $\exp tX = (\cos \lambda t)\, I + \left(\frac{1}{\lambda} \sin \lambda t\right) X$ where $\lambda = \sqrt{\det X}$ (Exercise 6.2). It follows that the Lie algebra acts by

$$\begin{aligned} X \cdot (z_1^k z_2^{n-k}) &= \frac{d}{dt}\left(\begin{pmatrix} \cos \lambda t + \frac{ix}{\lambda} \sin \lambda t & -\frac{\overline{w}}{\lambda} \sin \lambda t \\ \frac{w}{\lambda} \sin \lambda t & \cos \lambda t - \frac{ix}{\lambda} \sin \lambda t \end{pmatrix} \cdot z_1^k z_2^{n-k}\right)\Big|_{t=0} \\ &= k\,(-ixz_1 + \overline{w}z_2)\, z_1^{k-1} z_2^{n-k} + (n-k)\,(-wz_1 + ixz_2)\, z_1^k z_2^{n-k-1} \\ &= k\overline{w}\, z_1^{k-1} z_2^{n-k+1} + i\,(n-2k)\, x\, z_1^k z_2^{n-k} + (k-n)w\, z_1^{k+1} z_2^{n-k-1}. \end{aligned} \tag{6.3}$$

It is easy to use Equation 6.3 and Theorem 6.2 to show that $V_n(\mathbb{C}^2)$ is irreducible. In fact, this is the idea underpinning the argument given in §2.1.2.2.

As in the case of representations of Lie groups, new Lie algebra representations can be created using linear algebra. It is straightforward to verify (Exercise 6.1) that differentials of the Lie group representations listed in Definition 2.10 yield the following Lie algebra representations.

Definition 6.4. Let V and W be representations of a Lie algebra $\mathfrak{g}$ of a Lie subgroup of $GL(n, \mathbb{C})$.

(1) $\mathfrak{g}$ acts on $V \oplus W$ by $X(v, w) = (Xv, Xw)$.

(2) $\mathfrak{g}$ acts on $V \otimes W$ by $X \sum v_i \otimes w_j = \sum X v_i \otimes w_j + \sum v_i \otimes X w_j$.
(3) $\mathfrak{g}$ acts on $\mathrm{Hom}(V, W)$ by $(XT)(v) = XT(v) - T(Xv)$.
(4) $\mathfrak{g}$ acts on $\bigotimes^k V$ by $X \sum v_{i_1} \otimes \cdots v_{i_k} = \sum \left(X v_{i_1}\right) \otimes \cdots v_{i_k} + \cdots \sum v_{i_1} \otimes \cdots \left(X v_{i_k}\right)$.
(5) $\mathfrak{g}$ acts on $\bigwedge^k V$ by $X \sum v_{i_1} \wedge \cdots v_{i_k} = \sum \left(X v_{i_1}\right) \wedge \cdots v_{i_k} + \cdots \sum v_{i_1} \wedge \cdots \left(X v_{i_k}\right)$.
(6) $\mathfrak{g}$ acts on $S^k(V)$ by $X \sum v_{i_1} \cdots v_{i_k} = \sum \left(X v_{i_1}\right) \cdots v_{i_k} + \cdots \sum v_{i_1} \cdots \left(X v_{i_k}\right)$.
(7) $\mathfrak{g}$ acts on V^* by $(XT)(v) = -T(Xv)$.
(8) $\mathfrak{g}$ acts on $\overline{V}$ by the same action as it does on V.

6.1.2 Complexification of Lie Algebras

Definition 6.5. (a) Let $\mathfrak{g}$ be the Lie algebra of a Lie subgroup of $GL(n, \mathbb{C})$. The *complexification* of $\mathfrak{g}$, $\mathfrak{g}_{\mathbb{C}}$, is defined as $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. The Lie bracket on $\mathfrak{g}$ is extended to $\mathfrak{g}_{\mathbb{C}}$ by $\mathbb{C}$-linearity.
(b) If (ψ, V) is a representation of $\mathfrak{g}$, extend the domain of ψ to $\mathfrak{g}_{\mathbb{C}}$ by $\mathbb{C}$-linearity. Then (ψ, V) is said to be *irreducible* under $\mathfrak{g}_{\mathbb{C}}$ if there are no proper $\psi(\mathfrak{g}_{\mathbb{C}})$-invariant subspaces.

Writing a matrix in terms of its skew-Hermitian and Hermitian parts, observe that $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) \oplus i\mathfrak{u}(n)$. It follows that if $\mathfrak{g}$ is the Lie algebra of a compact Lie group G realized with $G \subseteq U(n)$, $\mathfrak{g}_{\mathbb{C}}$ may be identified with $\mathfrak{g} \oplus i\mathfrak{g}$ equipped with the standard Lie bracket inherited from $\mathfrak{gl}(n, \mathbb{C})$ (Exercise 6.3). We will often make this identification without comment. In particular, $\mathfrak{u}(n)_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$. Similarly, $\mathfrak{su}(n)_{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(n)_{\mathbb{C}}$ is realized by

$$\mathfrak{so}(n, \mathbb{C}) = \{X \in \mathfrak{sl}(n, \mathbb{C}) \mid X^t = -X\},$$

and, realizing $\mathfrak{sp}(n)$ as $\mathfrak{u}(2n) \cap \mathfrak{sp}(n, \mathbb{C})$ as in §4.1.3, $\mathfrak{sp}(n)_{\mathbb{C}}$ is realized by $\mathfrak{sp}(n, \mathbb{C})$ (Exercise 6.3).

Lemma 6.6. *Let $\mathfrak{g}$ be the Lie algebra of a Lie subgroup of $GL(n, \mathbb{C})$ and let (ψ, V) be a representation of $\mathfrak{g}$. Then V is irreducible under $\mathfrak{g}$ if and only if it is irreducible under $\mathfrak{g}_{\mathbb{C}}$.*

Proof. Simply observe that since a subspace $W \subseteq V$ is a complex subspace, W is $\psi(\mathfrak{g})$-invariant if and only if it is $\psi(\mathfrak{g}_{\mathbb{C}})$-invariant. □

For example, $\mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$ is equipped with the *standard basis*

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(c.f. Exercise 4.21). Since $E = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{i}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, Equation 6.3 shows that the resulting action of E on $V_n(\mathbb{C}^2)$ is given by

$$
\begin{aligned}
E \cdot (z_1^k z_2^{n-k}) &= \frac{1}{2}\left[-k\, z_1^{k-1} z_2^{n-k+1} - (k-n)\, z_1^{k+1} z_2^{n-k-1}\right] \\
&\quad - \frac{i}{2}\left[-ik\, z_1^{k-1} z_2^{n-k+1} + i(k-n)\, z_1^{k+1} z_2^{n-k-1}\right] \\
&= -k\, z_1^{k-1} z_2^{n-k+1}.
\end{aligned}
$$

Similarly (Exercise 6.4), the action of H and F on $V_n(\mathbb{C}^2)$ is given by

$$
\begin{aligned}
H \cdot (z_1^k z_2^{n-k}) &= (n-2k)\; z_1^k z_2^{n-k} \\
F \cdot (z_1^k z_2^{n-k}) &= (k-n)\, z_1^{k+1} z_2^{n-k-1}.
\end{aligned}
\tag{6.7}
$$

Irreducibility of $V_n(\mathbb{C}^2)$ is immediately apparent from these formulas (Exercise 6.7).

6.1.3 Weights

Let G be a compact Lie group and (π, V) a finite-dimensional representation of G. Fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and write $\mathfrak{t}_{\mathbb{C}}$ for its complexification. By Theorem 5.6, there exists an inner product, $(\cdot,\cdot)$, on V that is G-invariant and for which $d\pi$ is skew-Hermitian on $\mathfrak{g}$ and is Hermitian on $i\mathfrak{g}$. Thus $\mathfrak{t}_{\mathbb{C}}$ acts on V as a family of commuting normal operators and so V is simultaneously diagonalizable under the action of $\mathfrak{t}_{\mathbb{C}}$. In particular, the following definition is well defined.

Definition 6.8. Let G be a compact Lie group, (π, V) a finite-dimensional representation of G, and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. There is a finite set $\Delta(V) = \Delta(V, \mathfrak{t}_{\mathbb{C}}) \subseteq \mathfrak{t}_{\mathbb{C}}^*$, called the *weights* of V, so that

$$
V = \bigoplus_{\alpha \in \Delta(V)} V_\alpha,
$$

where

$$
V_\alpha = \{v \in V \mid d\pi(H)v = \alpha(H)v,\;\; H \in \mathfrak{t}_{\mathbb{C}}\}
$$

is nonzero. The above displayed equation is called the *weight space decomposition* of V with respect to $\mathfrak{t}_{\mathbb{C}}$.

As an example, take $G = SU(2)$, $V = V_n(\mathbb{C}^2)$, and $\mathfrak{t}$ to be the diagonal matrices in $\mathfrak{su}(2)$. Define $\alpha_m \in \mathfrak{t}_{\mathbb{C}}^*$ by requiring $\alpha_m(H) = m$. Then Equation 6.7 shows that the weight space decomposition for $V_n(\mathbb{C}^2)$ is $V_n(\mathbb{C}^2) = \bigoplus_{k=0}^n V_n(\mathbb{C}^2)_{\alpha_{n-2k}}$, where $V_n(\mathbb{C}^2)_{\alpha_{n-2k}} = \mathbb{C} z_1^k z_2^{n-k}$.

Theorem 6.9. ***(a)*** *Let G be a compact Lie group, (π, V) a finite-dimensional representation of G, T a maximal torus of G, and $V = \bigoplus_{\alpha \in \Delta(V,\mathfrak{t}_{\mathbb{C}})} V_\alpha$ the weight space decomposition. For each weight $\alpha \in \Delta(V)$, α is purely imaginary on $\mathfrak{t}$ and is real valued on $i\mathfrak{t}$.*
(b) *For $t \in T$, choose $H \in \mathfrak{t}$ so that $e^H = t$. Then $tv_\alpha = e^{\alpha(H)} v_\alpha$ for $v_\alpha \in V_\alpha$.*

Proof. Part (a) follows from the facts that $d\pi$ is skew-Hermitian on $\mathfrak{t}$ and is Hermitian on $i\mathfrak{t}$. Part (b) follows from the fact that $\exp \mathfrak{t} = T$ and the relation $e^{d\pi H} = \pi(e^H)$. $\square$

By $\mathbb{C}$-linearity, $\alpha \in \Delta(V)$ is completely determined by its restriction to either $\mathfrak{t}$ or $i\mathfrak{t}$. Thus we permit ourselves to interchangeably view α as an element of any of the dual spaces $\mathfrak{t}_{\mathbb{C}}^*$, $(i\mathfrak{t})^*$ (real valued), or $\mathfrak{t}^*$ (purely imaginary valued). In alternate notation (not used in this text), $i\mathfrak{t}$ is sometimes written $\mathfrak{t}_{\mathbb{C}}(\mathbb{R})$.

6.1.4 Roots

Let G be a compact Lie group. For $g \in G$, extend the domain of $\mathrm{Ad}(g)$ from $\mathfrak{g}$ to $\mathfrak{g}_{\mathbb{C}}$ by $\mathbb{C}$-linearity. Then $(\mathrm{Ad}, \mathfrak{g}_{\mathbb{C}})$ is a representation of G with differential given by ad (extended by $\mathbb{C}$-linearity). It has a weight space decomposition

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})} \mathfrak{g}_\alpha$$

that is important enough to warrant its own name. Notice the zero weight space is $\mathfrak{g}_0 = \{Z \in \mathfrak{g}_{\mathbb{C}} \mid [H, Z] = 0,\ H \in \mathfrak{t}_{\mathbb{C}}\}$. Thus

$$\mathfrak{g}_0 = \mathfrak{t}_{\mathbb{C}}$$

since $\mathfrak{t}$ is a maximal Abelian subspace of $\mathfrak{g}$. In the definition below, it turns out to be advantageous to separate this zero weight space from the remaining nonzero weight spaces.

Definition 6.10. Let G be a compact Lie group and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. There is a finite set of *nonzero* elements $\Delta(\mathfrak{g}_{\mathbb{C}}) = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \subseteq \mathfrak{t}_{\mathbb{C}}^*$, called the *roots* of $\mathfrak{g}_{\mathbb{C}}$, so that

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{Z \in \mathfrak{g}_{\mathbb{C}} \mid [H, Z] = \alpha(H)Z,\ H \in \mathfrak{t}_{\mathbb{C}}\}$ is nonzero. The above displayed equation is called the *root space decomposition* of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$.

Theorem 6.11. ***(a)*** *Let G be a compact Lie group, (π, V) a finite-dimensional representation of G, and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. For $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$ and $\beta \in \Delta(V)$, $d\pi(\mathfrak{g}_\alpha)V_\beta \subseteq V_{\alpha+\beta}$.*
(b) *In particular for $\alpha, \beta \in \Delta(\mathfrak{g}_{\mathbb{C}}) \cup \{0\}$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.*
(c) *Let $(\cdot, \cdot)$ be an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}_{\mathbb{C}}$. For $\alpha, \beta \in \Delta(\mathfrak{g}_{\mathbb{C}}) \cup \{0\}$, $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ when $\alpha + \beta \neq 0$.*
(d) *If $\mathfrak{g}$ has trivial center (i.e., if $\mathfrak{g}$ is semisimple), then $\Delta(\mathfrak{g}_{\mathbb{C}})$ spans $\mathfrak{t}_{\mathbb{C}}^*$.*

Proof. For part (a), let $H \in \mathfrak{t}_{\mathbb{C}}$, $X_\alpha \in \mathfrak{g}_\alpha$, and $v_\beta \in V_\beta$ and calculate

$$\begin{aligned} d\pi(H)d\pi(X_\alpha)v_\beta &= (d\pi(X_\alpha)d\pi(H) + [d\pi(H), d\pi(X_\alpha)])\, v_\beta \\ &= (d\pi(X_\alpha)d\pi(H) + d\pi\,[H, X_\alpha])\, v_\beta \\ &= (d\pi(X_\alpha)d\pi(H) + \alpha(H)d\pi\,(X_\alpha))\, v_\beta \\ &= (\beta(H) + \alpha(H))\, d\pi(X_\alpha)v_\beta, \end{aligned}$$

so that $d\pi(X_\alpha)v_\beta \in V_{\alpha+\beta}$ as desired. Part (b) clearly follows from part (a).

For part (c), recall that Lemma 5.6 shows that ad is skew-Hermitian. Thus

$$\alpha(H)(X_\alpha, X_\beta) = ([H, X_\alpha], X_\beta) = -(X_\alpha, [H, X_\beta]) = -\beta(H)(X_\alpha, X_\beta).$$

For part (d), suppose $H \in \mathfrak{t}_{\mathbb{C}}$ satisfies $\alpha(H) = 0$ for all $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$. It suffices to show that $H = 0$. However, the condition $\alpha(H) = 0$ for all $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$ is equivalent to saying that H is central in $\mathfrak{g}_{\mathbb{C}}$. Since it is easy to see that $\mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) = \mathfrak{z}(\mathfrak{g})_{\mathbb{C}}$ (Exercise 6.5), it follows from semisimplicity and Theorem 5.18 that $H = 0$. □

In §6.2.3 we will further see that $\dim \mathfrak{g}_\alpha = 1$ for $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$ and that the only multiples of α in $\Delta(\mathfrak{g}_{\mathbb{C}})$ are $\pm\alpha$.

6.1.5 Compact Classical Lie Group Examples

The root space decomposition for the complexification of the Lie algebra of each compact classical Lie group is given below. The details are straightforward to verify (Exercise 6.10).

6.1.5.1 $\mathfrak{su}(n)$ For $G = U(n)$ with $\mathfrak{t} = \{\mathrm{diag}(i\theta_1, \ldots, i\theta_n) \mid \theta_i \in \mathbb{R}\}$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{t}_{\mathbb{C}} = \{\mathrm{diag}(z_1, \ldots, z_n) \mid z_i \in \mathbb{C}\}$. For $G = SU(n)$ with $\mathfrak{t} = \{\mathrm{diag}(i\theta_1, \ldots, i\theta_n) \mid \theta_i \in \mathbb{R}, \sum_i \theta_i = 0\}$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{t}_{\mathbb{C}} = \{\mathrm{diag}(z_1, \ldots, z_n) \mid z_i \in \mathbb{C}, \sum_i z_i = 0\}$. In either case, it is straightforward to check that the set of roots is given by

$$\Delta(\mathfrak{g}_{\mathbb{C}}) = \{\pm(\epsilon_i - \epsilon_j) \mid 1 \le i < j \le n\},$$

where $\epsilon_i(\mathrm{diag}(z_1, \ldots, z_n)) = z_i$. In the theory of Lie algebras, this root system is called A_{n-1}. The corresponding root space is

$$\mathfrak{g}_{\epsilon_i - \epsilon_j} = \mathbb{C}E_{i,j},$$

where $\{E_{i,j}\}$ is the standard basis for $n \times n$ matrices.

6.1.5.2 $\mathfrak{sp}(n)$ For $G = Sp(n)$ realized as $Sp(n) \cong U(2n) \cap Sp(n, \mathbb{C})$ with $\mathfrak{t} = \{\mathrm{diag}(i\theta_1, \ldots, i\theta_n, -i\theta_1, \ldots, -i\theta_n) \mid \theta_i \in \mathbb{R}\}$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}(n, \mathbb{C})$ and $\mathfrak{t}_{\mathbb{C}} = \{\mathrm{diag}(z_1, \ldots, z_n, -z_1, \ldots, -z_n) \mid z_i \in \mathbb{C}\}$. Then

$$\Delta(\mathfrak{g}_{\mathbb{C}}) = \{\pm(\epsilon_i - \epsilon_j) \mid 1 \le i < j \le n\} \cup \{\pm\left(\epsilon_i + \epsilon_j\right) \mid 1 \le i \le j \le n\},$$

where $\epsilon_i(\mathrm{diag}(z_1, \ldots, z_n, -z_1, \ldots, -z_n)) = z_i$. In the theory of Lie algebras, this root system is called C_n. The corresponding root spaces are

$$\mathfrak{g}_{\epsilon_i - \epsilon_j} = \mathbb{C}\left(E_{i,j} - E_{j+n,i+n}\right)$$

$$\mathfrak{g}_{\epsilon_i + \epsilon_j} = \mathbb{C}\left(E_{i,j+n} + E_{j,i+n}\right),\ \mathfrak{g}_{-\epsilon_i - \epsilon_j} = \mathbb{C}\left(E_{i+n,j} + E_{j+n,i}\right)$$

$$\mathfrak{g}_{2\epsilon_i} = \mathbb{C}E_{i,i+n},\ \mathfrak{g}_{-2\epsilon_i} = \mathbb{C}E_{i+n,i}.$$

6.1.5.3 $\mathfrak{so}(E_n)$ For $SO(n)$, it turns out that the root space decomposition is a bit messy (see Exercise 6.14 for details). The results are much cleaner if we diagonalize by making a change of variables. In other words, we will examine an isomorphic copy of $SO(n)$ instead of $SO(n)$ itself. Define

$$T_{2m} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_m & I_m \\ iI_m & -iI_m \end{pmatrix}, \; E_{2m} = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix},$$
$$T_{2m+1} = \begin{pmatrix} T_{2m} & 0 \\ 0 & 1 \end{pmatrix}, \; E_{2m+1} = \begin{pmatrix} E_{2m} & 0 \\ 0 & 1 \end{pmatrix},$$

$$SO(E_n) = \{g \in SL(n, \mathbb{C}) \mid \overline{g} = E_n g E_n, \, g^t E_n g = E_n\},$$

$$\mathfrak{so}(E_n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid \overline{X} = E_n X E_n, \, X^t E_n + E_n X = 0\},$$

$$\mathfrak{so}(E_n, \mathbb{C}) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X^t E_n + E_n X = 0\}.$$

Notice that $E_n = T_n^t T_n$ and $\overline{T}_n = T_n^{-1,t}$. The following lemma is straightforward and left as an exercise (Exercise 6.12).

Lemma 6.12. ***(a)*** *$SO(E_n)$ is a compact Lie subgroup of $SU(n)$ with Lie algebra $\mathfrak{so}(E_n)$ and with complexified Lie algebra $\mathfrak{so}(E_n, \mathbb{C})$.*
(b) *The map $g \to T_n^{-1} g T_n$ induces an isomorphism of Lie groups $SO(n) \cong SO(E_n)$.*
(c) *The map $X \to T_n^{-1} X T_n$ induces an isomorphism of Lie algebras $\mathfrak{so}(n) \cong \mathfrak{so}(E_n)$ and $\mathfrak{so}(n, \mathbb{C}) \cong \mathfrak{so}(E_n, \mathbb{C})$.*
(d) *For $n = 2m$, a maximal torus is given by*

$$T = \{\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_m}, e^{-i\theta_1}, \ldots, e^{-i\theta_m}) \mid \theta_i \in \mathbb{R}\}$$

with corresponding Cartan subalgebra

$$\mathfrak{t} = \{\mathrm{diag}(i\theta_1, \ldots, i\theta_m, -i\theta_1, \ldots, -i\theta_m) \mid \theta_i \in \mathbb{R}\}$$

and complexification

$$\mathfrak{t}_{\mathbb{C}} = \{\mathrm{diag}(z_1, \ldots, z_m, -z_1, \ldots, -z_m) \mid z_i \in \mathbb{C}\}.$$

(e) *For $n = 2m + 1$, a maximal torus is given by*

$$T = \{\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_m}, e^{-i\theta_1}, \ldots, e^{-i\theta_m}, 1) \mid \theta_i \in \mathbb{R}\}$$

with corresponding Cartan subalgebra

$$\mathfrak{t} = \{\mathrm{diag}(i\theta_1, \ldots, i\theta_m, -i\theta_1, \ldots, -i\theta_m, 0) \mid \theta_i \in \mathbb{R}\}$$

and complexification

$$\mathfrak{t}_{\mathbb{C}} = \{\mathrm{diag}(z_1, \ldots, z_m, -z_1, \ldots, -z_m, 0) \mid z_i \in \mathbb{C}\}.$$

6.1.5.4 $\mathfrak{so}(2n)$ Working with $G = SO(E_{2n})$ and the Cartan subalgebra from Lemma 6.12, the set of roots is

$$\Delta(\mathfrak{g}_{\mathbb{C}}) = \{\pm\left(\epsilon_i \pm \epsilon_j\right) \mid 1 \le i < j \le n\},$$

where $\epsilon_i(\mathrm{diag}(z_1, \ldots, z_n, -z_1, \ldots, -z_n)) = z_i$. In the theory of Lie algebras, this root system is called D_n. The corresponding root spaces are

$$\mathfrak{g}_{\epsilon_i-\epsilon_j} = \mathbb{C}\left(E_{i,j} - E_{j+n,i+n}\right),\ \mathfrak{g}_{-\epsilon_i+\epsilon_j} = \mathbb{C}\left(E_{j,i} - E_{i+n,j+n}\right)$$

$$\mathfrak{g}_{\epsilon_i+\epsilon_j} = \mathbb{C}\left(E_{i,j+n} - E_{j,i+n}\right),\ \mathfrak{g}_{-\epsilon_i-\epsilon_j} = \mathbb{C}\left(E_{i+n,j} - E_{j+n,i}\right).$$

6.1.5.5 $\mathfrak{so}(2n+1)$ Working with $G = SO(E_{2n+1})$ and with the Cartan subalgebra from Lemma 6.12, the set of roots is

$$\Delta(\mathfrak{g}_{\mathbb{C}}) = \{\pm\left(\epsilon_i \pm \epsilon_j\right) \mid 1 \le i < j \le n\} \cup \{\pm\epsilon_i \mid 1 \le i \le n\},$$

where $\epsilon_i(\mathrm{diag}(z_1, \ldots, z_n, -z_1, \ldots, -z_n, 0)) = z_i$. In the theory of Lie algebras, this root system is called B_n. The corresponding root spaces are

$$\mathfrak{g}_{\epsilon_i-\epsilon_j} = \mathbb{C}\left(E_{i,j} - E_{j+n,i+n}\right),\ \mathfrak{g}_{-\epsilon_i+\epsilon_j} = \mathbb{C}\left(E_{j,i} - E_{i+n,j+n}\right)$$

$$\mathfrak{g}_{\epsilon_i+\epsilon_j} = \mathbb{C}\left(E_{i,j+n} - E_{j,i+n}\right),\ \mathfrak{g}_{-\epsilon_i-\epsilon_j} = \mathbb{C}\left(E_{i+n,j} - E_{j+n,i}\right).$$

$$\mathfrak{g}_{\epsilon_i} = \mathbb{C}\left(E_{i,2n+1} - E_{2n+1,i+n}\right),\ \mathfrak{g}_{-\epsilon_i} = \mathbb{C}\left(E_{i+n,2n+1} - E_{2n+1,i}\right).$$

6.1.6 Exercises

Exercise 6.1 Verify that the differentials of the actions given in Definition 2.10 give rise to the actions given in Definition 6.4.

Exercise 6.2 For $X = \begin{pmatrix} ix & z \\ -\bar{z} & -ix \end{pmatrix}$, $x \in \mathbb{R}$ and $z \in \mathbb{C}$, let $\lambda = \sqrt{x^2 + |z|^2}$. Show that $\exp X = (\cos\lambda)\, I + \frac{\sin\lambda}{\lambda} X$.

Exercise 6.3 **(1)** Show that $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) \oplus i\mathfrak{u}(n)$.
(2) Suppose $\mathfrak{g}$ is the Lie algebra of a compact Lie group G with G a Lie subgroup of $U(n)$. Show that there is an isomorphism of algebras $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g} \oplus i\mathfrak{g}$ induced by mapping $X \otimes (a + ib)$ to $aX + ibX$ for $X \in \mathfrak{g}$ and $a, b \in \mathbb{R}$.
(3) Show that $\mathfrak{su}(n)_{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$ and that $\mathfrak{so}(n)_{\mathbb{C}} = \mathfrak{so}(n, \mathbb{C})$.
(4) Show that $\mathfrak{sp}(n)_{\mathbb{C}} \cong \mathfrak{sp}(n, \mathbb{C})$ and that

$$\mathfrak{sp}(n, \mathbb{C}) = \{\begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix} \mid X, Y, Z \in \mathfrak{gl}(n, \mathbb{C}),\ Y^t = Y,\ Z^t = Z\}.$$

(5) Show that $\mathfrak{so}(E_{2n})_{\mathbb{C}} = \mathfrak{so}(E_{2n}, \mathbb{C})$ and that

$$\mathfrak{so}(E_{2n},\mathbb{C}) = \{\begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix} \mid X, Y, Z \in \mathfrak{gl}(n,\mathbb{C}),\, Y^t = -Y,\, Z^t = -Z\}.$$

(6) Show that$\mathfrak{so}(E_{2n+1})_{\mathbb{C}} = \mathfrak{so}(E_{2n+1},\mathbb{C})$ and that

$$\mathfrak{so}(E_{2n+1},\mathbb{C}) = \{\begin{pmatrix} X & Y & u \\ Z & -X^t & v \\ -v^t & -u^t & 0 \end{pmatrix} \mid X, Y, Z \in \mathfrak{gl}(n,\mathbb{C}),\, Y^t = -Y,\, Z^t = -Z,\, u, v \in \mathbb{C}^n\}.$$

Exercise 6.4 Verify Equation 6.7.

Exercise 6.5 Let G be a compact Lie group. Show that $\mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) = \mathfrak{z}(\mathfrak{g})_{\mathbb{C}}$.

Exercise 6.6 **(1)** Let G be a Lie subgroup of $GL(n,\mathbb{C})$ and assume $\mathfrak{g}$ is semisimple. Show that any one-dimensional representation of $\mathfrak{g}$ is trivial, i.e., $\mathfrak{g}$ acts by 0.
(2) Show that any one-dimensional representation of G is trivial.

Exercise 6.7 Use Equation 6.7 to verify that $V_n(\mathbb{C}^2)$ is an irreducible representation of $SU(2)$.

Exercise 6.8 This exercise gives an algebraic proof of the classification of irreducible representations of $SU(2)$ (c.f. Theorem 3.32).
(1) Given any irreducible representation V of $SU(2)$, show that there is a nonzero $v_0 \in V$, so that $Hv_0 = \lambda v_0$, $\lambda \in \mathbb{C}$, and so that $Ev_0 = 0$.
(2) Let $v_i = F^i v_0$. Show that $Hv_i = (\lambda - 2i)v_i$ and $Ev_i = i(\lambda - i + 1)v_{i-1}$.
(3) Let m be the smallest natural number satisfying $v_{m+1} = 0$. Show that $\{v_i\}_{i=0}^m$ is a basis for V.
(4) Show that the trace of the H action on V is zero.
(5) Show that $\lambda = m$ and use this to show that $V \cong V_m(\mathbb{C}^2)$.

Exercise 6.9 **(1)** Find the weight space decomposition for the standard representation of $SU(n)$ on $\mathbb{C}^n$.
(2) Find the weight space decomposition for the standard representation of $SO(n)$ on $\mathbb{C}^n$.

Exercise 6.10 Verify that the roots and root spaces listed in §6.1.5 are correct (c.f., Exercise 6.3).

Exercise 6.11 Let G be a compact Lie group and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. Use root theory to show directly that there exists $X \in \mathfrak{t}$, so that $\mathfrak{t} = \mathfrak{z}_{\mathfrak{g}}(X)$ (c.f. Lemma 5.7).

Exercise 6.12 Prove Lemma 6.12.

Exercise 6.13 **(1)** Let $\mathfrak{g}$ be the Lie algebra of a Lie subgroup of a linear group. Then $\mathfrak{g}_{\mathbb{C}}$ is called *simple* if $\mathfrak{g}_{\mathbb{C}}$ has no (complex) proper ideals and if $\dim_{\mathbb{C}} \mathfrak{g} > 1$, i.e., if the only ideals of $\mathfrak{g}_{\mathbb{C}}$ are $\{0\}$ and $\mathfrak{g}_{\mathbb{C}}$ and if $\mathfrak{g}_{\mathbb{C}}$ is non-Abelian. Show that $\mathfrak{g}$ is simple if and only if $\mathfrak{g}_{\mathbb{C}}$ is simple.

(2) Use the root decomposition to show that $\mathfrak{sl}(n,\mathbb{C})$ is simple, $n \geq 2$.
(3) Show that $\mathfrak{sp}(n,\mathbb{C})$ is simple, $n \geq 1$.
(4) Show that $\mathfrak{so}(2n,\mathbb{C})$ is simple for $n \geq 3$, but that $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$.
(5) Show that $\mathfrak{so}(2n+1,\mathbb{C})$ is simple, $n \geq 1$.

Exercise 6.14 **(1)** For $G = SO(2n)$ and

$$\mathfrak{t} = \{\text{blockdiag}\left(\begin{pmatrix} 0 & \theta_1 \\ -\theta_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \theta_n \\ -\theta_n & 0 \end{pmatrix}\right) \mid \theta_i \in \mathbb{R}\}$$

as in §5.1.2, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2n,\mathbb{C})$ and

$$\mathfrak{t}_{\mathbb{C}} = \{\text{blockdiag}\left(\begin{pmatrix} 0 & z_1 \\ -z_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & z_n \\ -z_n & 0 \end{pmatrix}\right) \mid z_i \in \mathbb{C}\}.$$

Show that

$$\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\pm\left(\epsilon_i - \epsilon_j\right),\ \ \pm\left(\epsilon_i + \epsilon_j\right) \mid 1 \leq i < j \leq n\},$$

where $\epsilon_j(\text{blockdiag}\left(\begin{pmatrix} 0 & z_1 \\ -z_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & z_n \\ -z_n & 0 \end{pmatrix}\right)) = -iz_j$. Partition each $2n \times 2n$ matrix into n^2 blocks of size 2×2. For $\alpha = \pm\epsilon_i \pm \epsilon_j$, show that the root space is $\mathfrak{g}_\alpha = \mathbb{C}E_\alpha$, where E_α is 0 on all 2×2 blocks except for the ij^{th} block and the ji^{th} block. Show that E_α is given by the matrix X_α on the ij^{th} block and by $-X_\alpha^t$ on the ji^{th} block, where

$$X_{\epsilon_i-\epsilon_j} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix},\ X_{-\epsilon_i+\epsilon_j} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$
$$X_{\epsilon_i+\epsilon_j} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},\ X_{-\epsilon_i-\epsilon_j} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

(2) For $G = SO(2n+1)$ and

$$\mathfrak{t} = \{\text{blockdiag}\left(\begin{pmatrix} 0 & \theta_1 \\ -\theta_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \theta_n \\ -\theta_n & 0 \end{pmatrix}, 0\right) \mid \theta_i \in \mathbb{R}\}$$

as in §5.1.2, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2n+1,\mathbb{C})$ and

$$\mathfrak{t}_{\mathbb{C}} = \{\text{blockdiag}\left(\begin{pmatrix} 0 & z_1 \\ -z_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & z_n \\ -z_n & 0 \end{pmatrix}, 0\right) \mid z_i \in \mathbb{C}\}.$$

Show that

$$\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\pm\left(\epsilon_i \pm \epsilon_j\right) \mid 1 \leq i < j \leq n\} \cup \{\pm\epsilon_i \mid 1 \leq i \leq n\},$$

where

$$\epsilon_j(\text{blockdiag}\left(\begin{pmatrix} 0 & z_1 \\ -z_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & z_n \\ -z_n & 0 \end{pmatrix}, 0\right)) = -iz_j.$$

For $\alpha = \pm\left(\epsilon_i \pm \epsilon_j\right)$, show that the root space is obtained by embedding the corresponding root space from $\mathfrak{so}(2n,\mathbb{C})$ into $\mathfrak{so}(2n+1,\mathbb{C})$ via the map $X \to \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$. For $\alpha = \pm\epsilon_j$, show that the root space is $\mathfrak{g}_\alpha = \mathbb{C}E_\alpha$, where E_α is 0 except on the last column and last row. Writing $v \in \mathbb{C}^{2n+1}$ for the last column, show the last row of E_α is given by $-v^t$, where v is given in terms of the standard basis vectors by $v = e_{2j-1} \mp ie_{2j}$.

6.2 The Standard $\mathfrak{sl}(2,\mathbb{C})$ Triple

6.2.1 Cartan Involution

Definition 6.13. Let G be a compact Lie group. The *Cartan involution*, θ, of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{g}$ is the Lie algebra involution of $\mathfrak{g}_\mathbb{C}$ given by $\theta(X \otimes z) = X \otimes \overline{z}$ for $X \in \mathfrak{g}$ and $z \in \mathbb{C}$. In other words, if $Z \in \mathfrak{g}_\mathbb{C}$ is uniquely written as $Z = X + iY$ for $X, Y \in \mathfrak{g} \otimes 1$, then $\theta Z = X - iY$.

It must be verified that θ is a Lie algebra involution, but this follows from a simple calculation (Exercise 6.15). Under the natural embedding of $\mathfrak{g}$ in $\mathfrak{g}_\mathbb{C}$, notice that the $+1$ eigenspace of θ on $\mathfrak{g}_\mathbb{C}$ is $\mathfrak{g}$ and that the -1 eigenspace is $i\mathfrak{g}$. Notice also that when $\mathfrak{g} \subseteq \mathfrak{u}(n)$, then $\theta Z = -Z^*$ for $Z \in \mathfrak{g}_\mathbb{C}$ since $X^* = -X$ for $X \in \mathfrak{u}(n)$. In particular,

$$\theta Z = -Z^*$$

when $\mathfrak{g}$ is $\mathfrak{u}(n)$, $\mathfrak{su}(n)$, $\mathfrak{sp}(n)$, $\mathfrak{so}(n)$, or $\mathfrak{so}(E_n)$.

Lemma 6.14. *Let G be a compact Lie group and $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$.*
(a) *If $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$, then $-\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$ and $\mathfrak{g}_{-\alpha} = \theta\mathfrak{g}_\alpha$.*
(b) *$\theta\mathfrak{t}_\mathbb{C} = \mathfrak{t}_\mathbb{C}$.*

Proof. Let $\alpha \in \Delta(\mathfrak{g}_\mathbb{C}) \cup \{0\}$. Recalling that θ is an involution, it suffices to show $\theta\mathfrak{g}_\alpha \subseteq \mathfrak{g}_{-\alpha}$. Write $Z \in \mathfrak{g}_\alpha$ uniquely as $Z = X + iY$ for $X, Y \in \mathfrak{g} \otimes 1$. Then for $H \in \mathfrak{t}$,

$$\alpha(H)(X + iY) = [H, X + iY] = [H, X] + i[H, Y].$$

Since $\alpha(H) \in i\mathbb{R}$ by Theorem 6.9 and since $[H, X], [H, Y] \in \mathfrak{g} \otimes 1$,

$$\alpha(H)X = i[H, Y] \text{ and } \alpha(H)Y = -i[H, X].$$

Thus

$$[H, \theta Z] = [H, X] - i[H, Y] = -\alpha(H)(X - iY) = -\alpha(H)\left(\theta Z\right),$$

so that $\theta Z \in \mathfrak{g}_{-\alpha}$, as desired. □

In particular, notice that $\mathfrak{g}$ is spanned by elements of the form $Z + \theta Z$ for $Z \in \mathfrak{g}_\alpha$, $\alpha \in \Delta(\mathfrak{g}_\mathbb{C}) \cup \{0\}$.

6.2.2 Killing Form

Definition 6.15. Let $\mathfrak{g}$ be the Lie algebra of a Lie subgroup of $GL(n, \mathbb{C})$. For $X, Y \in \mathfrak{g}_{\mathbb{C}}$, the symmetric complex bilinear form $B(X, Y) = \operatorname{tr}(\operatorname{ad} X \circ \operatorname{ad} Y)$ on $\mathfrak{g}_{\mathbb{C}}$ is called the *Killing form.*

Theorem 6.16. *Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group G.*
(a) *For $X, Y \in \mathfrak{g}$, $B(X, Y) = \operatorname{tr}(\operatorname{ad} X \circ \operatorname{ad} Y)$ on $\mathfrak{g}$.*
(b) *B is* Ad*-invariant, i.e., $B(X, Y) = B(\operatorname{Ad}(g)X, \operatorname{Ad}(g)Y)$ for $g \in G$ and $X, Y \in \mathfrak{g}_{\mathbb{C}}$.*
(c) *B is skew* ad*-invariant, i.e., $B(\operatorname{ad}(Z)X, Y) = -B(X, \operatorname{ad}(Z)Y)$ for $Z, X, Y \in \mathfrak{g}_{\mathbb{C}}$.*
(d) *B restricted to $\mathfrak{g}' \times \mathfrak{g}'$ is negative definite.*
(e) *B restricted to $\mathfrak{g}_\alpha \times \mathfrak{g}_\beta$ is zero when $\alpha + \beta \neq 0$ for $\alpha, \beta \in \Delta(\mathfrak{g}_{\mathbb{C}}) \cup \{0\}$.*
(f) *B is nonsingular on $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$. If $\mathfrak{g}$ is semisimple with a Cartan subalgebra $\mathfrak{t}$, then B is also nonsingular on $\mathfrak{t}_{\mathbb{C}} \times \mathfrak{t}_{\mathbb{C}}$.*
(g) *The* radical *of B,* $\operatorname{rad} B = \{X \in \mathfrak{g}_{\mathbb{C}} \mid B(X, \mathfrak{g}_{\mathbb{C}}) = 0\}$, *is the center of $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{z}(\mathfrak{g}_{\mathbb{C}})$.*
(h) *If $\mathfrak{g}$ is semisimple, the form $(X, Y) = -B(X, \theta Y)$, $X, Y \in \mathfrak{g}_{\mathbb{C}}$, is an* Ad*-invariant inner product on $\mathfrak{g}_{\mathbb{C}}$.*
(i) *Let $\mathfrak{g}$ be simple and choose a linear realization of G, so that $\mathfrak{g} \subseteq \mathfrak{u}(n)$. Then there exists a positive $c \in \mathbb{R}$, so that $B(X, Y) = c \operatorname{tr}(XY)$ for $X, Y \in \mathfrak{g}_{\mathbb{C}}$.*

Proof. Part (a) is elementary. For part (b), recall that $\operatorname{Ad} g$ preserves the Lie bracket by Theorem 4.8. Thus $\operatorname{ad}(\operatorname{Ad}(g)X) = \operatorname{Ad}(g) \operatorname{ad}(X) \operatorname{Ad}(g^{-1})$ and part (b) follows. As usual, part (c) follows from part (b) by examining the case of $g = \exp tZ$ and applying $\frac{d}{dt}|_{t=0}$ when $Z \in \mathfrak{g}$. For $Z \in \mathfrak{g}_{\mathbb{C}}$, use the fact that B is complex bilinear.

For part (d), let $X \in \mathfrak{g}$. Using Theorem 5.9, choose a Cartan subalgebra $\mathfrak{t}$ containing X. Then the root space decomposition shows $B(X, X) = \sum_{\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})} \alpha^2(X)$. Since G is compact, $\alpha(X) \in i\mathbb{R}$ by Theorem 6.9. Thus B is negative semidefinite on $\mathfrak{g}$. Moreover, $B(X, X) = 0$ if and only if $\alpha(X) = 0$ for all $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$, i.e., if and only if $X \in \mathfrak{z}(\mathfrak{g})$. Thus the decomposition $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}'$ from Theorem 5.18 finishes part (d).

For part (e), let $X_\alpha \in \mathfrak{g}_\alpha$ and $H \in \mathfrak{t}$. Use part (c) to see that

$$0 = B(\operatorname{ad}(H)X_\alpha, X_\beta) + B(X_\alpha, \operatorname{ad}(H)X_\beta) = [(\alpha + \beta)(H)]\,(X_\alpha, X_\beta).$$

In particular (e) follows.

For part (f), recall that $\mathfrak{g}_{-\alpha} = \theta\mathfrak{g}_\alpha$. Thus if $X_\alpha = U_\alpha + iV_\alpha$ with $U_\alpha, V_\alpha \in \mathfrak{g}$, then $U_\alpha - iV_\alpha \in \mathfrak{g}_{-\alpha}$ and

$$B(U_\alpha + iV_\alpha, U_\alpha - iV_\alpha) = B(U_\alpha, U_\alpha) + B(V_\alpha, V_\alpha). \tag{6.17}$$

In light of part (d), the above expression is zero if and only if $X_\alpha \in \mathbb{C}\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}_{\mathbb{C}})$ (Exercise 6.5). Since $\mathfrak{g}_\alpha \subseteq (\mathfrak{g}')_{\mathbb{C}}$ for $\alpha \neq 0$, part (f) is complete.

For part (g), first observe that $\mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) \subseteq \operatorname{rad} B$ since $\operatorname{ad} Z = 0$ for $Z \in \mathfrak{z}(\mathfrak{g}_{\mathbb{C}})$. On the other hand, since $\mathfrak{g}_{\mathbb{C}} = \mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) \oplus (\mathfrak{g}')_{\mathbb{C}}$, the root space decomposition and part (f) finishes part (g).

Except for verifying positive definiteness, the assertion in part (h) follows from the definitions. To check positive definiteness, use the root space decomposition, the relation $\mathfrak{g}_{-\alpha} = \theta\mathfrak{g}_\alpha$, parts (d), (e) and (f), and Equation 6.17.

For part (i), first note that the trace form mapping $X, Y \in \mathfrak{g}_\mathbb{C}$ to $\operatorname{tr}(XY)$ is Ad-invariant since $\operatorname{Ad}(g)X = gXg^{-1}$. For $X \in \mathfrak{u}(n)$, X is diagonalizable with eigenvalues in $i\mathbb{R}$. In particular, the trace form is negative definite on $\mathfrak{g}$. Arguing as in Equation 6.17, this shows the trace form is nondegenerate on $\mathfrak{g}_\mathbb{C}$. In particular, both $-B(X, \theta Y)$ and $-\operatorname{tr}(X\theta Y)$ are Ad-invariant inner products on $\mathfrak{g}_\mathbb{C}$. However, since $\mathfrak{g}$ is simple, $\mathfrak{g}_\mathbb{C}$ is an irreducible representation of $\mathfrak{g}$ under ad (Exercise 6.17) and therefore an irreducible representation of G under Ad by Lemma 6.6 and Theorem 6.2. Corollary 2.20 finishes the argument. □

6.2.3 The Standard $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{su}(2)$ Triples

Let G be a compact Lie algebra and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. When $\mathfrak{g}$ is semisimple, recall that B is negative definite on $\mathfrak{t}$ by Theorem 6.16. It follows that B restricts to a real inner product on the real vector space $i\mathfrak{t}$. Continuing to write $(i\mathfrak{t})^*$ for the set of $\mathbb{R}$-linear functionals on $i\mathfrak{t}$, B induces an isomorphism between $i\mathfrak{t}$ and $(i\mathfrak{t})^*$ as follows.

Definition 6.18. Let G be a compact Lie group with semisimple Lie algebra, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$, and $\alpha \in (i\mathfrak{t})^*$. Let $u_\alpha \in i\mathfrak{t}$ be uniquely determined by the equation

$$\alpha(H) = B(H, u_\alpha)$$

for all $H \in i\mathfrak{t}$ and, when $\alpha \neq 0$, let

$$h_\alpha = \frac{2u_\alpha}{B(u_\alpha, u_\alpha)}.$$

In case $\mathfrak{g}$ is not semisimple, define $u_\alpha \in i\mathfrak{t}' \subseteq i\mathfrak{t} \subseteq \mathfrak{t}$ by first restricting B to $i\mathfrak{t}'$. For $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$, recall that α is determined by its restriction to $i\mathfrak{t}$. On $i\mathfrak{t}$, α is a real-valued linear functional by Theorem 6.9. Viewing α as an element of $(i\mathfrak{t})^*$, define u_α and h_α via Definition 6.18. Note that the equation $\alpha(H) = B(H, u_\alpha)$ now holds for all $H \in \mathfrak{t}_\mathbb{C}$ by $\mathbb{C}$-linear extension. An alternate notation for h_α is $\alpha^\vee$, and so we write

$$\Delta(\mathfrak{g}_\mathbb{C})^\vee = \{h_\alpha \mid \alpha \in \Delta(\mathfrak{g}_\mathbb{C})\}.$$

When $\mathfrak{g} \subseteq \mathfrak{u}(n)$ is simple, Theorem 6.16 shows that there exists a positive $c \in \mathbb{R}$, so that $B(X, Y) = c\operatorname{tr}(XY)$ for $X, Y \in \mathfrak{g}_\mathbb{C}$. Thus if $\alpha \in (i\mathfrak{t})^*$ and $u'_\alpha, h'_\alpha \in i\mathfrak{t}$ are determined by the equations $\alpha(H) = \operatorname{tr}(Hu'_\alpha)$ and $h'_\alpha = \frac{2u'_\alpha}{\operatorname{tr}(u'_\alpha u'_\alpha)}$, it follows that $u'_\alpha = cu_\alpha$ but that $h'_\alpha = h_\alpha$. In particular, h_α can be computed with respect to the trace form instead of the Killing form.

For the classical compact groups, this calculation is straightforward (see §6.1.5 and Exercise 6.21). Notice also that $h_{-\alpha} = -h_\alpha$.

For $SU(n)$ with $\mathfrak{t} = \{\text{diag}(i\theta_1, \ldots, i\theta_n) \mid \theta_i \in \mathbb{R}, \sum_i \theta_i = 0\}$, that is the A_{n-1} root system,

$$h_{\epsilon_i - \epsilon_j} = E_i - E_j,$$

where $E_i = \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 in the i^{th} position

For $Sp(n)$ realized as $Sp(n) \cong U(2n) \cap Sp(n, \mathbb{C})$ with

$$\mathfrak{t} = \{\text{diag}(i\theta_1, \ldots, i\theta_n, -i\theta_1, \ldots, -i\theta_n) \mid \theta_i \in \mathbb{R}\},$$

that is the C_n root system,

$$\begin{aligned} h_{\epsilon_i - \epsilon_j} &= \left(E_i - E_j\right) - \left(E_{i+n} - E_{j+n}\right) \\ h_{\epsilon_i + \epsilon_j} &= \left(E_i + E_j\right) - \left(E_{i+n} + E_{j+n}\right) \\ h_{2\epsilon_i} &= E_i - E_{i+n}. \end{aligned}$$

For $SO(E_{2n})$ with $\mathfrak{t} = \{\text{diag}(i\theta_1, \ldots, i\theta_n, -i\theta_1, \ldots, -i\theta_n) \mid \theta_i \in \mathbb{R}\}$, that is the D_n root system,

$$\begin{aligned} h_{\epsilon_i - \epsilon_j} &= \left(E_i - E_j\right) - \left(E_{i+n} - E_{j+n}\right) \\ h_{\epsilon_i + \epsilon_j} &= \left(E_i + E_j\right) - \left(E_{i+n} + E_{j+n}\right). \end{aligned}$$

For $SO(E_{2n+1})$ with $\mathfrak{t} = \{\text{diag}(i\theta_1, \ldots, i\theta_n, -i\theta_1, \ldots, -i\theta_n, 0) \mid \theta_i \in \mathbb{R}\}$, that is the B_n root system,

$$\begin{aligned} h_{\epsilon_i - \epsilon_j} &= \left(E_i - E_j\right) - \left(E_{i+n} - E_{j+n}\right) \\ h_{\epsilon_i + \epsilon_j} &= \left(E_i + E_j\right) - \left(E_{i+n} + E_{j+n}\right) \\ h_{\epsilon_i} &= 2E_i - 2E_{i+n}. \end{aligned}$$

Lemma 6.19. *Let G be a compact Lie group, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$, and $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$.*
(a) *Then $\alpha(h_\alpha) = 2$.*
(b) *For $E \in \mathfrak{g}_\alpha$ and $F \in \mathfrak{g}_{-\alpha}$,*

$$[E, F] = B(E, F)u_\alpha = \frac{1}{2}B(E, F)B(u_\alpha, u_\alpha)h_\alpha.$$

(c) *Given a nonzero $E \in \mathfrak{g}_\alpha$, E may be rescaled by an element of $\mathbb{R}$, so that $[E, F] = h_\alpha$, where $F = -\theta E$.*

Proof. For part (a) simply use the definitions

$$\alpha(h_\alpha) = \frac{2\alpha(u_\alpha)}{B(u_\alpha, u_\alpha)} = \frac{2B(u_\alpha, u_\alpha)}{B(u_\alpha, u_\alpha)} = 2.$$

For part (b), first note that $[E, F] \subseteq \mathfrak{t}_{\mathbb{C}}$ by Theorem 6.11. Given any $H \in \mathfrak{t}_{\mathbb{C}}$, calculate

$$B([E,F],H) = B(E,[F,H]) = \alpha(H)B(E,F) = B(u_\alpha,H)B(E,F)$$
$$= B(B(E,F)u_\alpha,H).$$

Since B is nonsingular on $\mathfrak{t}_\mathbb{C}$ by Theorem 6.16, part (b) is finished. For part (c), replace E by cE, where

$$c^2 = \frac{2}{-B(E,\theta E)B(u_\alpha,u_\alpha)},$$

and use Theorem 6.16 to check that $-B(E,\theta E) > 0$ and $B(u_\alpha,u_\alpha) > 0$. □

For the next theorem, recall that B is nonsingular on $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$ by Theorem 6.16 and that $-B(X,\theta Y)$ is an Ad-invariant inner product on $\mathfrak{g}'_\mathbb{C}$ for $X, Y \in \mathfrak{g}_\mathbb{C}$.

Theorem 6.20. *Let G be a compact Lie group, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$, and $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$. Fix a nonzero $E_\alpha \in \mathfrak{g}_\alpha$ and let $F_\alpha = -\theta E_\alpha$. Using Lemma 6.19, rescale E_α (and therefore F_α), so that $[E_\alpha, F_\alpha] = H_\alpha$ where $H_\alpha = h_\alpha$.*
***(a)** Then $\mathfrak{sl}(2,\mathbb{C}) \cong \operatorname{span}_\mathbb{C}\{E_\alpha, H_\alpha, F_\alpha\}$ with $\{E_\alpha, H_\alpha, F_\alpha\}$ corresponding to the standard basis*

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of $\mathfrak{sl}(2,\mathbb{C})$.
***(b)** Let $\mathcal{I}_\alpha = iH_\alpha$, $\mathcal{J}_\alpha = -E_\alpha + F_\alpha$, and $\mathcal{K}_\alpha = -i(E_\alpha + F_\alpha)$. Then $\mathcal{I}_\alpha, \mathcal{J}_\alpha, \mathcal{K}_\alpha \in \mathfrak{g}$ and $\mathfrak{su}(2) \cong \operatorname{span}_\mathbb{R}\{\mathcal{I}_\alpha, \mathcal{J}_\alpha, \mathcal{K}_\alpha\}$ with $\{\mathcal{I}_\alpha, \mathcal{J}_\alpha, \mathcal{K}_\alpha\}$ corresponding to the basis*

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\quad \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

of $\mathfrak{su}(2)$ (c.f. Exercise 4.2 for the isomorphism $\operatorname{Im}(\mathbb{H}) \cong \mathfrak{su}(2)$).
***(c)** There exists a Lie algebra homomorphism $\varphi_\alpha : SU(2) \to G$, so that $d\varphi : \mathfrak{su}(2) \to \mathfrak{g}$ implements the isomorphism in part (b) and whose complexification $d\varphi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}_\mathbb{C}$ implements the isomorphism in part (a).*
***(d)** The image of φ_α in G is a Lie subgroup of G isomorphic to either $SU(2)$ or $SO(3)$ depending on whether the kernel of φ_α is $\{I\}$ or $\{\pm I\}$.*

Proof. For part (a), Lemma 6.19 and the definitions show that $[H_\alpha, E_\alpha] = 2E_\alpha$, $[H_\alpha, F_\alpha] = -2F_\alpha$, and $[E_\alpha, F_\alpha] = H_\alpha$. Since these are the bracket relations for the standard basis of $\mathfrak{sl}(2,\mathbb{C})$, part (a) is finished (c.f. Exercise 4.21). For part (b), observe that θ fixes $\mathcal{I}_\alpha$, $\mathcal{J}_\alpha$, and $\mathcal{K}_\alpha$ by construction, so that $\mathcal{I}_\alpha, \mathcal{J}_\alpha, \mathcal{K}_\alpha \in \mathfrak{g}$. The bracket relations for $\mathfrak{sl}(2,\mathbb{C})$ then quickly show that $[\mathcal{I}_\alpha, \mathcal{J}_\alpha] = 2\mathcal{K}_\alpha$, $[\mathcal{J}_\alpha, \mathcal{K}_\alpha] = 2\mathcal{I}_\alpha$, and $[\mathcal{K}_\alpha, \mathcal{I}_\alpha] = 2\mathcal{J}_\alpha$, so that $\mathfrak{su}(2) \cong \operatorname{span}_\mathbb{R}\{\mathcal{I}_\alpha, \mathcal{J}_\alpha, \mathcal{K}_\alpha\}$ (Exercise 4.2). For part (c), recall that $SU(2)$ is simply connected since, topologically, it is isomorphic to S^3. Thus, Theorem 4.16 provides the existence of φ_α. For part (d), observe that $d\varphi_\alpha$ is an isomorphism by definition. Thus, the kernel of φ_α is discrete and normal and therefore central by Lemma 1.21. Since the center of $SU(2)$ is $\pm I$ and since $SO(3) \cong SU(2)/\{\pm I\}$ by Lemma 1.23, the proof is complete. □

Definition 6.21. Let G be a compact Lie group, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$, and $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$. Continuing the notation from Theorem 6.20, the set $\{E_\alpha, H_\alpha, F_\alpha\}$ is called a *standard* $\mathfrak{sl}(2,\mathbb{C})$*-triple* associated to α and the set $\{\mathcal{I}_\alpha, \mathcal{J}_\alpha, \mathcal{K}_\alpha\}$ is called a *standard* $\mathfrak{su}(2)$*-triple* associated to α.

Corollary 6.22. *Let G be a compact Lie group, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$, and $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$.*
(a) *The only multiple of α in $\Delta(\mathfrak{g}_{\mathbb{C}})$ is $\pm\alpha$.*
(b) $\dim \mathfrak{g}_\alpha = 1$.
(c) *If $\beta \in \Delta(\mathfrak{g}_{\mathbb{C}})$, then $a(h_\beta) \in \pm\{0,1,2,3\}$.*
(d) *If (π, V) is a representation of G and $\lambda \in \Delta(V)$, then $\lambda(h_\alpha) \in \mathbb{Z}$.*

Proof. Let $\{E_\alpha, H_\alpha, F_\alpha\}$ be a standard $\mathfrak{sl}(2,\mathbb{C})$-triple associated to α and $\{\mathcal{I}_\alpha, \mathcal{J}_\alpha, \mathcal{K}_\alpha\}$ the standard $\mathfrak{su}(2)$-triple associated to α with $\varphi_\alpha : SU(2) \to G$ the corresponding embedding. Since $e^{2\pi i H} = I$, applying $\mathrm{Ad}\circ\varphi_\alpha$ shows that

$$I = \mathrm{Ad}(\varphi_\alpha e^{2\pi i H}) = \mathrm{Ad}(e^{2\pi d\varphi_\alpha i H}) = \mathrm{Ad}(e^{2\pi i H_\alpha}) = e^{2\pi i \,\mathrm{ad}\, H_\alpha}$$

on $\mathfrak{g}_{\mathbb{C}}$. Using the root decomposition, it follows that $\beta(H_\alpha) = \frac{2B(u_\beta,u_\alpha)}{\|u_\alpha\|^2} \in \mathbb{Z}$ where $\|\cdot\|$ is the norm corresponding to the Killing form. Now if $k\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$, then $u_{k\alpha} = ku_\alpha$, so that $\frac{2}{k} = \frac{2B(u_\alpha,ku_\alpha)}{\|ku_\alpha\|^2} = \alpha(H_{k\alpha}) \in \mathbb{Z}$ and $2k = \frac{2B(ku_\alpha,u_\alpha)}{\|u_\alpha\|^2} = (k\alpha)(H_\alpha) \in \mathbb{Z}$. Thus $k \in \pm\{\frac{1}{2}, 1, 2\}$.

For part (a), it therefore suffices to show that $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$ implies $\pm 2\alpha \notin \Delta(\mathfrak{g}_{\mathbb{C}})$. For this, let $\mathfrak{l}_\alpha = \mathrm{span}_{\mathbb{R}}\{\mathcal{I}_\alpha, \mathcal{J}_\alpha, \mathcal{K}_\alpha\} \cong \mathfrak{su}(2)$, so that $(\mathfrak{l}_\alpha)_{\mathbb{C}} = \mathrm{span}_{\mathbb{C}}\{E_\alpha, H_\alpha, F_\alpha\} \cong \mathfrak{sl}(2,\mathbb{C})$. Also let $V = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathbb{C}H_\alpha \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{g}_{\pm 2\alpha}$ is possibly zero in this case. By Lemma 6.19 and Theorem 6.11, V is invariant under $(\mathfrak{l}_\alpha)_{\mathbb{C}}$ with respect to the ad-action. In particular, V is a representation of $\mathfrak{l}_\alpha$. Of course, $(\mathfrak{l}_\alpha)_{\mathbb{C}} \subseteq V$ is an $\mathfrak{l}_\alpha$-invariant subspace. Thus V decomposes under the $\mathfrak{l}$-action as $V = (\mathfrak{l}_\alpha)_{\mathbb{C}} \oplus V'$ for some submodule V' of V. To finish parts (a) and (b), it suffices to show $V' = \{0\}$.

From the discussion in §6.1.3, we know that H_α acts on the $(n+1)$-dimensional irreducible representation of $\mathfrak{su}(2)$ with eigenvalues $\{n, n-2, \ldots, -n+2, -2n\}$. In particular, if V' were nonzero, V' would certainly contain an eigenvector of H_α corresponding to an eigenvalue of either 0 or 1. Now the eigenvalues of H_α on V are contained in $\pm\{0,2,4\}$ by construction. Since the 0-eigenspace has multiplicity one in V and is already contained in $(\mathfrak{l}_\alpha)_{\mathbb{C}}$, V' must be $\{0\}$.

For part (c), let $\beta \in \Delta(\mathfrak{g}_{\mathbb{C}})$ and write $B(u_\beta, u_\alpha) = \|u_\beta\| \|u_\alpha\| \cos\theta$, where θ is the angle between u_β and u_β. Thus $4\cos^2\theta = \frac{2B(u_\alpha,u_\beta)}{\|u_\beta\|^2}\frac{2B(u_\beta,u_\alpha)}{\|u_\alpha\|^2} \in \mathbb{Z}$. As $\cos^2\theta \le 1$, $4\cos^2\theta = \alpha(H_\beta)\beta(H_\alpha) \in \{0,1,2,3,4\}$. To finish part (c), it only remains to rule out the possibility that $\{\alpha(H_\beta), \beta(H_\alpha)\} = \pm\{1,4\}$. Clearly $\alpha(H_\beta)\beta(H_\alpha) = 4$ only when $\theta = 0$, i.e., when α and β are multiples of each other. By part (a), this occurs only when $\beta = \pm\alpha$ in which case $\alpha(H_\beta) = \beta(H_\alpha) = \pm 2$. In particular, $\{\alpha(H_\beta), \beta(H_\alpha)\} \ne \pm\{1,4\}$. Thus $\alpha(H_\beta), \beta(H_\alpha) \in \pm\{0,1,2,3\}$.

For part (d), simply apply $\pi \circ \varphi_\alpha$ to $e^{2\pi i H} = I$ to get $e^{2\pi i (d\pi) H_\alpha} = I$ on V. As in the first paragraph, the weight decomposition shows that $\lambda(H_\alpha) \in \mathbb{Z}$. $\square$

It turns out that the above condition $\alpha(h_\beta) \in \pm\{0,1,2,3\}$ is strict. In other words, there exist compact Lie groups for which each of these values are achieved.

6.2.4 Exercises

Exercise 6.15 Show that θ is a Lie algebra involution of $\mathfrak{g}_\mathbb{C}$, i.e., that θ is $\mathbb{R}$-linear, $\theta^2 = I$, and that $\theta[Z_1, Z_2] = [\theta Z_1, \theta Z_2]$ for $Z_i \in \mathfrak{g}_\mathbb{C}$.

Exercise 6.16 Let G be a connected compact Lie group and $g \in G$. Use the Maximal Torus Theorem, Lemma 6.14, and Theorem 6.9 to show that $\det \operatorname{Ad} g = 1$ on $\mathfrak{g}_\mathbb{C}$ and therefore on $\mathfrak{g}$ as well.

Exercise 6.17 Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Show that $\mathfrak{g}$ is simple if and only if $\mathfrak{g}_\mathbb{C}$ is an irreducible representation of $\mathfrak{g}$ under ad.

Exercise 6.18 **(1)** Let G be a compact Lie group with simple Lie algebra $\mathfrak{g}$. If $(\cdot, \cdot)$ is an Ad-invariant symmetric bilinear form on $\mathfrak{g}_\mathbb{C}$, show that there is a constant $c \in \mathbb{C}$ so that $(\cdot, \cdot) = cB(\cdot, \cdot)$.
(2) If $(\cdot, \cdot)$ is nonzero and $B(\cdot, \cdot)$ is replaced by $(\cdot, \cdot)$ in Definition 6.18, show that h_α is unchanged.

Exercise 6.19 Let G be a compact Lie group with a simple (c.f. Exercise 6.13) Lie algebra $\mathfrak{g} \subseteq \mathfrak{u}(n)$. Theorem 6.16 shows that there is a positive $c \in \mathbb{R}$, so that $B(X, Y) = c \operatorname{tr}(XY)$ for $X, Y \in \mathfrak{g}_\mathbb{C}$. In the special cases below, show that c is given as stated.
(1) $c = 2n$ for $G = SU(n), n \geq 2$
(2) $c = 2(n+1)$ for $G = Sp(n), n \geq 1$
(3) $c = 2(n-1)$ for $G = SO(2n), n \geq 3$
(4) $c = 2n - 1$ for $G = SO(2n+1), n \geq 1$.

Exercise 6.20 Let G be a compact Lie group with semisimple Lie algebra $\mathfrak{g}$, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$, and $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$. If $\beta \in \Delta(\mathfrak{g}_\mathbb{C})$ with $\beta \neq \pm a$ and $B(u_\alpha, u_\alpha) \leq B(u_\beta, u_\beta)$, show that $a(h_\beta) \in \pm\{0, 1\}$.

Exercise 6.21 For each compact classical Lie group, this section lists h_α for each root α. Verify these calculations.

Exercise 6.22 Let G be a compact Lie group with semisimple Lie algebra $\mathfrak{g}$, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$, and $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$. Let V be a finite-dimensional representation of G and $\lambda \in \Delta(V)$. The *α-string through* λ is the set of all weights of the form $\lambda + n\alpha$, $n \in \mathbb{Z}$.
(1) Make use of a standard $\mathfrak{sl}(2)$-triple $\{E_\alpha, H_\alpha, F_\alpha\}$ and consider the space $\bigoplus_n V_{\lambda+n\alpha}$ to show the α-string through β is of the form $\{\lambda + n\alpha \mid -p \leq n \leq q\}$, where $p, q \in \mathbb{Z}^{\geq 0}$ with $p - q = \lambda(h_\alpha)$.
(2) If $\lambda(h_\alpha) < 0$, show show $\lambda + \alpha \in \Delta(V)$. If $\lambda(h_\alpha) > 0$, show that $\lambda - \alpha \in \Delta(V)$.
(3) Show that $d\pi(E_\alpha)^{p+q} V_{\lambda - p\alpha} \neq 0$.
(4) If $\alpha, \beta, \alpha + \beta \in \Delta(\mathfrak{g}_\mathbb{C})$, show that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

Exercise 6.23 Show that $SL(2, \mathbb{C})$ has no nontrivial finite-dimensional unitary representations. To this end, argue by contradiction. Assume (π, V) is such a representation and compare the form $B(X, Y)$ on $\mathfrak{sl}(2, \mathbb{C})$ to the form $(X, Y)' = \operatorname{tr}(d\pi(X) \circ d\pi(Y))$.

6.3 Lattices

6.3.1 Definitions

Let G be a compact Lie group, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$, and $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$. As noted in §6.1.3, α may be viewed as an element of $(i\mathfrak{t})^*$. Use Definition 6.18 to transport the Killing form from $i\mathfrak{t}$ to $(i\mathfrak{t})^*$ by setting

$$B(\lambda_1, \lambda_2) = B(u_{\lambda_1}, u_{\lambda_2})$$

for $\lambda_1, \lambda_2 \in (i\mathfrak{t})^*$. In particular, for $\lambda \in (i\mathfrak{t})^*$,

$$\lambda(h_\alpha) = \frac{2B(\lambda, \alpha)}{B(\alpha, \alpha)}.$$

For the sake of symmetry, also note that

$$\alpha(H) = \frac{2B(H, h_\alpha)}{B(h_\alpha, h_\alpha)}$$

for $H \in i\mathfrak{t}$.

Definition 6.23. Let G be a compact Lie group and T a maximal torus of G with corresponding Cartan subalgebra $\mathfrak{t}$.
(a) The *root lattice*, $R = R(\mathfrak{t})$, is the lattice in $(i\mathfrak{t})^*$ given by

$$R = \operatorname{span}_{\mathbb{Z}}\{\alpha \mid \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})\}.$$

(b) The *weight lattice* (alternately called the set of *algebraically integral weights*), $P = P(\mathfrak{t})$, is the lattice in $(i\mathfrak{t})^*$ given by

$$P = \{\lambda \in (i\mathfrak{t})^* \mid \lambda(h_\alpha) \in \mathbb{Z} \text{ for } \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})\},$$

where $\lambda \in (i\mathfrak{t})^*$ is extended to an element of $(\mathfrak{t}_{\mathbb{C}})^*$ by $\mathbb{C}$-linearity.
(c) The set of *analytically integral weights*, $A = A(T)$, is the lattice in $(i\mathfrak{t})^*$ given by

$$A = \{\lambda \in (i\mathfrak{t})^* \mid \lambda(H) \in 2\pi i\mathbb{Z} \text{ whenever } \exp H = I \text{ for } H \in \mathfrak{t}\}.$$

To the lattices R, P, and A, there are also a number of associated dual lattices.

Definition 6.24. Let G be a compact Lie group and T a maximal torus of G with corresponding Cartan subalgebra $\mathfrak{t}$.
(a) The *dual root lattice*, $R^\vee = R^\vee(\mathfrak{t})$, is the lattice in $i\mathfrak{t}$ given by

$$R^\vee = \operatorname{span}_{\mathbb{Z}}\{h_\alpha \mid \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})\}.$$

(b) The *dual weight lattice*, $P^\vee = P^\vee(\mathfrak{t})$, is the lattice in $i\mathfrak{t}$ given by

$$P^\vee = \{H \in i\mathfrak{t} \mid \alpha(H) \in \mathbb{Z} \text{ for } \alpha \in \Delta(\mathfrak{g}_\mathbb{C})\}.$$

(c) Let $\ker \mathcal{E} = \ker \mathcal{E}(T)$ be the lattice in $i\mathfrak{t}$ given by

$$\ker \mathcal{E} = \{H \in i\mathfrak{t} \mid \exp(2\pi i H) = I\}.$$

(d) In general, if Λ_1 is a lattice in $(i\mathfrak{t})^*$ that spans $(i\mathfrak{t})^*$ and if Λ_2 is a lattice in $i\mathfrak{t}$ that spans $i\mathfrak{t}$, define the *dual lattices*, Λ_1^* and Λ_2^* in $i\mathfrak{t}$ and $(i\mathfrak{t})^*$, respectively, by

$$\Lambda_1^* = \{H \in i\mathfrak{t} \mid \lambda(H) \in \mathbb{Z} \text{ for } \lambda \in \Lambda_1\}$$
$$\Lambda_2^* = \{\lambda \in (i\mathfrak{t})^* \mid \lambda(H) \in \mathbb{Z} \text{ for } H \in \Lambda_2\}.$$

It is well known that Λ_1^* and Λ_2^* are lattices and that they satisfy $\Lambda_i^{**} = \Lambda_i$ (Exercise 6.24). Notice $\ker \mathcal{E}$ is a lattice by the proof of Theorem 5.2.

6.3.2 Relations

Lemma 6.25. *Let G be a compact connected Lie group with Cartan subalgebra $\mathfrak{t}$. For $H \in \mathfrak{t}$, $\exp H \in Z(G)$ if and only if $\alpha(H) \in 2\pi i\mathbb{Z}$ for all $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$.*

Proof. Let $g = \exp H$ and recall from Lemma 5.11 that $g \in Z(G)$ if and only if $\mathrm{Ad}(g)X = X$ for all $X \in \mathfrak{g}$. Now for $\alpha \in \Delta(\mathfrak{g}_\mathbb{C}) \cup \{0\}$ and $X \in \mathfrak{g}_\alpha$, $\mathrm{Ad}(g)X = e^{\mathrm{ad}\, H}X = e^{\alpha(H)}X$. The root decomposition finishes the proof. □

Definition 6.26. Let G be a compact Lie group and T a maximal torus. Write $\chi(T)$ for the *character group* on T, i.e., $\chi(T)$ is the set of all Lie homomorphisms $\xi : T \to \mathbb{C}\backslash\{0\}$.

Theorem 6.27. *Let G be a compact Lie group with a maximal torus T.*
***(a)** $R \subseteq A \subseteq P$.*
***(b)** Given $\lambda \in (i\mathfrak{t})^*$, $\lambda \in A$ if and only if there exists $\xi_\lambda \in \chi(T)$ satisfying*

$$\xi_\lambda(\exp H) = e^{\lambda(H)} \tag{6.28}$$

for $H \in \mathfrak{t}$, where $\lambda \in (i\mathfrak{t})^*$ is extended to an element of $(\mathfrak{t}_\mathbb{C})^*$ by $\mathbb{C}$-linearity. The map $\lambda \to \xi_\lambda$ establishes a bijection

$$A \longleftrightarrow \chi(T).$$

***(c)** For semisimple $\mathfrak{g}$, $|P/R|$ is finite.*

Proof. Let $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$ and suppose $H \in \mathfrak{t}$ with $\exp H = e$. Lemma 6.25 shows that $\alpha(H) \in 2\pi i\mathbb{Z}$, so that $R \subseteq A$. Next choose a standard $\mathfrak{sl}(2,\mathbb{C})$-triple $\{E_\alpha, h_\alpha, F_\alpha\}$ associated to α. As in the proof of Corollary 6.22, $\exp 2\pi i h_\alpha = I$. Thus if $\lambda \in A$, $\lambda(2\pi i h_\alpha) \in 2\pi i\mathbb{Z}$, so that $A \subseteq P$ which finishes part (a).

For part (b), start with $\lambda \in A$. Using the fact that $\exp \mathfrak{t} = T$ and using Lemma 6.25, Equation 6.28 uniquely defines a well-defined function ξ_λ on T. It is a homomorphism by Theorem 5.1. Conversely, if there is a $\xi_\lambda \in \chi(T)$ satisfying Equation 6.28, then clearly $\lambda(H) \in 2\pi i\mathbb{Z}$ whenever $\exp H = I$, so that $\lambda \in A$. Finally, to see that there is a bijection from A to $\chi(T)$, it remains to see that the map $\lambda \to \xi_\lambda$ is surjective. However, this requirement follows immediately by taking the differential of an element of $\chi(T)$ and extending via $\mathbb{C}$-linearity. Theorem 6.9 shows the differential can be viewed as an element of $(i\mathfrak{t})^*$.

Next, Theorem 6.11 shows that R spans $(i\mathfrak{t})^*$ for semisimple $\mathfrak{g}$. Part (c) therefore follows immediately from elementary lattice theory (e.g., see [3]). In fact, it is straightforward to show $|P/R|$ is equal to the determinant of the so-called *Cartan matrix* (Exercise 6.42). □

Theorem 6.29. *Let G be a compact Lie group with a semisimple Lie algebra $\mathfrak{g}$ and let T be a maximal torus of G with corresponding Cartan subalgebra $\mathfrak{t}$.*
(a) $R^* = P^\vee$.
(b) $P^* = R^\vee$.
(c) $A^* = \ker \mathcal{E}$.
(d) $P^* \subseteq A^* \subseteq R^*$, *i.e.*, $R^\vee \subseteq \ker \mathcal{E} \subseteq P^\vee$.

Proof. The equalities $R^* = P^\vee$, $\left(R^\vee\right)^* = P$, and $(\ker \mathcal{E})^* = A$ follow immediately from the definitions. This proves parts (a), (b), and (c) (Exercise 6.24). Part (d) follows from Theorem 6.27 (Exercise 6.24). □

6.3.3 Center and Fundamental Group

The proof of part (b) of the following theorem will be given in §7.3.6. However, for the sake of comparison, part (b) is stated now.

Theorem 6.30. *Let G be a connected compact Lie group with a semisimple Lie algebra and maximal torus T.*
(a) $Z(G) \cong P^\vee / \ker \mathcal{E} \cong A/R$.
(b) $\pi_1(G) \cong \ker \mathcal{E}/R^\vee \cong P/A$.

Proof (part (a) only). By Theorem 5.1, Corollary 5.13, and Lemma 6.25, the exponential map induces an isomorphism

$$\begin{aligned} Z(G) &\cong \{H \in \mathfrak{t} \mid \alpha(H) \in 2\pi i\mathbb{Z} \text{ for } \alpha \in \Delta(\mathfrak{g}_\mathbb{C})\} \,/\, \{H \in \mathfrak{t} \mid \exp H = I\} \\ &= (2\pi i\, P^\vee) \,/\, (2\pi i\, \ker \mathcal{E}), \end{aligned}$$

so that $Z(G) \cong P^\vee / \ker \mathcal{E}$. Basic lattice theory shows $R^*/A^* \cong A/R$ (Exercise 6.24) which finishes the proof. □

While the proof of part (b) of Theorem 6.30 is postponed until §7.3.6, in this section we at least prove the simply connected covering of a compact semisimple Lie group is still compact.

Let G be a compact connected Lie group and let $\widetilde{G}$ be the simply connected covering of G. A priori, it is not known that $\widetilde{G}$ is a *linear group* and thus our development of the theory of Lie algebras and, in particular, the exponential map is not directly applicable to $\widetilde{G}$. Indeed for more general groups, $\widetilde{G}$ may not be linear. As usual though, compact groups are nicely behaved. Instead of redoing our theory in the context of arbitrary Lie groups, we instead use the lifting property of covering spaces. Write $\exp_G : \mathfrak{g} \to G$ for the standard exponential map and let

$$\exp_{\widetilde{G}} : \mathfrak{g} \to \widetilde{G}$$

be the unique smooth lift of $\exp_G$ satisfying $\exp_{\widetilde{G}}(0) = \widetilde{e}$ and $\exp_G = \pi \circ \exp_{\widetilde{G}}$.

Lemma 6.31. *Let G be a compact connected Lie group, T a maximal torus of G, $\widetilde{G}$ the simply connected covering of G, $\pi : \widetilde{G} \to G$ the associated covering homomorphism, and $\widetilde{T} = \left[\pi^{-1}(T)\right]^0$.*
(a) *Restricted to $\mathfrak{t}$, $\exp_{\widetilde{G}}$ induces an isomorphism of Lie groups $\widetilde{T} \cong \mathfrak{t}/\left(\mathfrak{t} \cap \ker \exp_{\widetilde{G}}\right)$.*
(b) *If $\mathfrak{g}$ is semisimple, then $\widetilde{T}$ is compact.*

Proof. Elementary covering theory shows that $\widetilde{T}$ is a covering of T. From this it follows that $\widetilde{T}$ is Abelian on a neighborhood of $\widetilde{e}$ and, since $\widetilde{T}$ is connected, $\widetilde{T}$ is Abelian everywhere. Since $\pi \exp_{\widetilde{G}} \mathfrak{t} = \exp_G \mathfrak{t} = T$ and since $\exp_{\widetilde{G}} \mathfrak{t}$ is connected, $\exp_{\widetilde{G}} \mathfrak{t} \subseteq \widetilde{T}$. In particular, $\exp_{\widetilde{G}} : \mathfrak{t} \to \widetilde{T}$ is the unique lift of $\exp_G : \mathfrak{t} \to T$ satisfying $\exp_{\widetilde{G}}(0) = \widetilde{e}$. In turn, uniqueness of the lifting easily shows $\exp_{\widetilde{G}}(t_0 + t) = \exp_{\widetilde{G}}(t_0) \exp_{\widetilde{G}}(t)$. To finish part (a), it suffices to show $\exp_{\widetilde{G}} \mathfrak{t}$ contains a neighborhood $\widetilde{e}$ in $\widetilde{T}$. For this, it suffices to show the differential of $\exp_{\widetilde{G}}$ at 0 is invertible. But since π is a local diffeomorphism and since $\exp_G$ is a local diffeomorphism near 0, we are done.

For part (b), it suffices to show that $\widetilde{T}$ is a finite cover of T when $\mathfrak{g}$ is semisimple. For this, first observe that $\ker \exp_{\widetilde{G}} \subseteq \ker \exp_G = 2\pi i \ker \mathcal{E}$ since $\exp_G = \pi \circ \exp_{\widetilde{G}}$. As $T \cong \mathfrak{t}/(2\pi i \ker \mathcal{E})$, it follows that the $\ker \pi$ restricted to $\widetilde{T}$ is isomorphic to $(2\pi i \ker \mathcal{E}) / \left(\mathfrak{t} \cap \ker \exp_{\widetilde{G}}\right)$. By Theorems 6.27 and 6.29, it therefore suffices to show that $2\pi i R^\vee \subseteq \mathfrak{t} \cap \ker \exp_{\widetilde{G}}$.

Given $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$, let $\{\mathcal{I}_\alpha, \mathcal{J}_\alpha, \mathcal{K}_\alpha\}$ be a standard $\mathfrak{su}(2)$-triple in $\mathfrak{g}$ associated to α. Write $\varphi_\alpha : SU(2) \to G$ for the corresponding homomorphism. Since $SU(2)$ is simply connected, write $\widetilde{\varphi}_\alpha : SU(2) \to \widetilde{G}$ for the unique lift of φ_α mapping I to $\widetilde{e}$. Using the uniqueness of lifting from $\mathfrak{su}(2)$ to $\widetilde{G}$, if follows easily that $\widetilde{\varphi}_\alpha \circ \exp_{SU(2)} = \exp_{\widetilde{G}} \circ d\varphi_\alpha$. Therefore by construction,

$$\widetilde{e} = \widetilde{\varphi}_\alpha(I) = \widetilde{\varphi}_\alpha(\exp_{SU(2)} 2\pi i H) = \exp_{\widetilde{G}} (2\pi i \, d\varphi_\alpha H) = \exp_G (2\pi i \, h_\alpha),$$

which finishes the proof. $\square$

Lemma 6.32. *Let G be a compact connected Lie group, T a maximal torus of G, $\widetilde{G}$ the simply connected covering of of G, $\pi : \widetilde{G} \to G$ the associated covering homomorphism, and $\widetilde{T} = \left[\pi^{-1}(T)\right]^0$.*
(a) $\widetilde{G} = \bigcup_{\widetilde{g} \in \widetilde{G}} \left(c_{\widetilde{g}} \widetilde{T}\right)$.
(b) $\widetilde{G} = \exp_{\widetilde{G}}(\mathfrak{g})$.

Proof. The proof of this lemma is a straightforward generalization of the proof of the Maximal Torus theorem, Theorem 5.12 (Exercise 6.26). □

Corollary 6.33. *Let G be a compact connected Lie group with semisimple Lie algebra $\mathfrak{g}$, T a maximal torus of G, $\widetilde{G}$ the simply connected covering of of G, $\pi : \widetilde{G} \to G$ the associated covering homomorphism, and $\widetilde{T} = \left[\pi^{-1}(T)\right]^0$.*
(a) *$\widetilde{G}$ is compact.*
(b) *$\mathfrak{g}$ may be identified with the Lie algebra of $\widetilde{G}$, so that $\exp_{\widetilde{G}}$ is the corresponding exponential map.*
(c) *$\widetilde{T} = \pi^{-1}(T)$ and $\widetilde{T}$ is a maximal torus of $\widetilde{G}$.*
(d) $\ker \pi \subseteq Z(\widetilde{G}) \subseteq \widetilde{T}$.

Proof. For part (a), observe that $\widetilde{G} = \bigcup_{\widetilde{g}\in\widetilde{G}} \left(c_{\widetilde{g}}\widetilde{T}\right)$ by Lemma 6.32. Thus $\widetilde{G}$ is the continuous image of the compact set $\widetilde{G}/Z(\widetilde{G}) \times \widetilde{T} \cong G/Z(G) \times \widetilde{T}$ (Exercise 6.26).

For part (b), recall from Corollary 4.9 that there is a one-to-one correspondence between one-parameter subgroups of $\widetilde{G}$ and the Lie algebra of $\widetilde{G}$. By the uniqueness of lifting, $\exp_{\widetilde{G}}(tX)\exp_{\widetilde{G}}(sX) = \exp_{\widetilde{G}}((t+s)X)$ for $X \in \mathfrak{g}$ and $t, s \in \mathbb{R}$, so that $t \to \exp_{\widetilde{G}}(tX)$ is a one-parameter subgroup of $\widetilde{G}$. On the other hand, if $\gamma : \mathbb{R} \to \widetilde{G}$ is a one-parameter subgroup, then so is $\pi \circ \gamma : \mathbb{R} \to G$. Thus there is a unique $X \in \mathfrak{g}$, so that $\pi(\gamma(t)) = \exp_G(tX)$. As usual, the uniqueness property of lifting from $\mathbb{R}$ to $\widetilde{G}$ shows that $\gamma(t) = \exp_{\widetilde{G}}(tX)$, which finishes part (b).

For parts (c) and (d), we already know from Lemma 6.31 that $\widetilde{T} = \exp_{\widetilde{G}}(\mathfrak{t})$. Since $\mathfrak{t}$ is a Cartan subalgebra, Theorem 5.4 shows that $\widetilde{T}$ is a maximal torus of $\widetilde{G}$. By Lemma 1.21 and Corollary 5.13, $\ker \pi \subseteq Z(\widetilde{G}) \subseteq \widetilde{T}$ so that $\pi^{-1}(T) = \widetilde{T}(\ker \pi) = \widetilde{T}$ is, in fact, connected. □

6.3.4 Exercises

Exercise 6.24 Suppose Λ_i is a lattice in $(i\mathfrak{t})^*$ that spans $(i\mathfrak{t})^*$.
(1) Show that Λ_i^* is a lattice in $i\mathfrak{t}$.
(2) Show that $\Lambda_i^{**} = \Lambda_i$.
(3) If $\Lambda_1 \subseteq \Lambda_2$, show that $\Lambda_2^* \subseteq \Lambda_1^*$.
(4) If $\Lambda_1 \subseteq \Lambda_2$, show that $\Lambda_2/\Lambda_1 \cong \Lambda_1^*/\Lambda_2^*$.

Exercise 6.25 **(1)** Use the standard root system notation from §6.1.5. In the following table, write (θ_i) for the element $\operatorname{diag}(\theta_1, \ldots, \theta_n)$ in the case of $G = SU(n)$, for the element $\operatorname{diag}(\theta_1, \ldots, \theta_n, -\theta_1, \ldots, -\theta_n)$ in the cases of $G = Sp(n)$ or $SO(E_{2n})$, and for the element $\operatorname{diag}(\theta_1, \ldots, \theta_n, -\theta_1, \ldots, -\theta_n, 0)$ in the case of $G = SO(E_{2n+1})$. Verify that the following table is correct.

G	$R^\vee$	$\ker \mathcal{E}$	$P^\vee$	$P^\vee/R^\vee$
$SU(n)$	$\{(\theta_i) \mid \theta_i \in \mathbb{Z},$ $\sum_{i=1}^n \theta_i = 0\}$	$R^\vee$	$\{(\theta_i + \frac{\theta_0}{n}) \mid \theta_i \in \mathbb{Z},$ $\sum_{i=0\theta i}^n = 0\}$	$\mathbb{Z}_n$
$Sp(n)$	$\{(\theta_i) \mid \theta_i \in \mathbb{Z}\}$	$R^\vee$	$\{(\theta_i + \frac{\theta_0}{2}) \mid \theta_i \in \mathbb{Z}\}$	$\mathbb{Z}_2$
$SO(E_{2n})$	$\{(\theta_i) \mid \theta_i \in \mathbb{Z},$ $\sum_{i=1}^n \theta_i \in 2\mathbb{Z}\}$	$\{(\theta_i) \mid \theta_i \in \mathbb{Z}\}$	$\{(\theta_i + \frac{\theta_0}{2}) \mid \theta_i \in \mathbb{Z}\}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$ n even $\mathbb{Z}_4$ n odd
$SO(E_{2n+1})$	$\{(\theta_i) \mid \theta_i \in \mathbb{Z},$ $\sum_{i=1}^n \theta_i \in 2\mathbb{Z}\}$	$P^\vee$	$\{(\theta_i) \mid \theta_i \in \mathbb{Z}\}$	$\mathbb{Z}_2$.

(2) In the following table, write (λ_i) for the element $\sum_i \lambda_i \epsilon_i$. Verify that the following table is correct.

G	R	A	P	P/R
$SU(n)$	$\{(\lambda_i) \mid \lambda_i \in \mathbb{Z},$ $\sum_{i=1}^n \lambda_i = 0\}$	P	$\{(\lambda_i + \frac{\lambda_0}{n}) \mid \lambda_i \in \mathbb{Z},$ $\sum_{i=0}^n \lambda_i = 0\}$	$\mathbb{Z}_n$
$Sp(n)$	$\{(\lambda_i) \mid \lambda_i \in \mathbb{Z},$ $\sum_{i=1}^n \lambda_i \in 2\mathbb{Z}\}$	P	$\{(\lambda_i) \mid \lambda_i \in \mathbb{Z}\}$	$\mathbb{Z}_2$
$SO(E_{2n})$	$\{(\lambda_i) \mid \lambda_i \in \mathbb{Z},$ $\sum_{i=1}^n \lambda_i \in 2\mathbb{Z}\}$	$\{(\lambda_i) \mid \lambda_i \in \mathbb{Z}\}$	$\{(\lambda_i + \frac{\lambda_0}{2}) \mid \lambda_i \in \mathbb{Z}\}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$ n even $\mathbb{Z}_4$ n odd
$SO(E_{2n+1})$	$\{(\lambda_i) \mid \lambda_i \in \mathbb{Z}\}$	R	$\{(\lambda_i + \frac{\lambda_0}{2}) \mid \lambda_i \in \mathbb{Z}\}$	$\mathbb{Z}_2$.

Exercise 6.26 Let G be a compact connected Lie group, T a maximal torus of G, $\widetilde{G}$ the simply connected covering of of G, $\pi : \widetilde{G} \to G$ the associated covering homomorphism, and $\widetilde{T} = \left[\pi^{-1}(T)\right]^0$. This exercise generalizes the proof of the Maximal Torus theorem, Theorem 5.12, to show that $\widetilde{G} = \bigcup_{\widetilde{g} \in \widetilde{G}} \left(c_{\widetilde{g}} \widetilde{T}\right)$ and $\widetilde{G} = \exp_{\widetilde{G}}(\mathfrak{g})$.
(1) Make use of Lemma 5.11 and the fact that $\ker \pi$ is discrete to show that $\ker(\operatorname{Ad} \circ \pi) = Z(\widetilde{G})$.
(2) Suppose $\widetilde{\varphi} : \mathfrak{g} \to \widetilde{G}$ is lift of a map $\varphi : \mathfrak{g} \to G$. Use the fact that π is a local diffeomorphism to show that $\widetilde{\varphi}$ is a local diffeomorphism if and only if φ is a local diffeomorphism.
(3) Use the uniqueness property of lifting to show that $\exp_{\widetilde{G}} \circ \operatorname{Ad}(\pi g) = c_g \circ \exp_{\widetilde{G}}$ for $g \in \widetilde{G}$.
(4) Show that $\bigcup_{g \in \widetilde{G}} c_g \widetilde{T} = \exp_{\widetilde{G}}(\mathfrak{g})$.
(5) If $\dim \mathfrak{g} = 1$, show that $G \cong S^1$ and $\mathfrak{g} \cong \widetilde{G} \cong \mathbb{R}$ with $\exp_{\widetilde{G}}$ being the identity map. Conclude that $\widetilde{G} = \exp_{\widetilde{G}}(\mathfrak{g})$.
(6) Assume $\dim \mathfrak{g} > 1$ and use induction on $\dim \mathfrak{g}$ to show that $\widetilde{G} = \exp_{\widetilde{G}}(\mathfrak{g})$ as outlined in the remaining steps. First, in the case where $\dim \mathfrak{g}' < \dim \mathfrak{g}$, show that $G \cong \left[G' \times T^k\right]/F$, where F is a finite Abelian group. Conclude that $\widetilde{G} \cong \widetilde{G}_{ss} \times \mathbb{R}^k$. Use the fact that the exponential map from $\mathbb{R}^k$ to T^k is surjective and the inductive hypothesis to show $\widetilde{G} = \exp_{\widetilde{G}}(\mathfrak{g})$.
(7) For the remainder, assume $\mathfrak{g}$ is semisimple, so that $\widetilde{T}$ is compact. Use Lemma 1.21 to show that $\ker \pi \subseteq Z(\widetilde{G})$. Conclude that $\widetilde{G}/Z(\widetilde{G}) \cong G/Z(G)$ and use this to show that $\exp_{\widetilde{G}}(\mathfrak{g})$ is compact and therefore closed.
(8) It remains to show that $\exp_{\widetilde{G}}(\mathfrak{g})$ is open. Fix $X_0 \in \mathfrak{g}$ and write $g_0 = \exp_{\widetilde{G}}(X_0)$. Use Theorem 4.6 to show that it suffices to consider $X_0 \neq 0$.

(9) As in the proof of Theorem 5.12, let $\mathfrak{a} = \mathfrak{z}_{\mathfrak{g}}(\pi g_0)$ and $\mathfrak{b} = \mathfrak{a}^{\perp}$. Consider the map $\widetilde{\varphi} : \mathfrak{a} \oplus \mathfrak{b} \to \widetilde{G}$ given by $\widetilde{\varphi}(X, Y) = g_0^{-1} \exp_{\widetilde{G}}(Y)\, g_0 \exp_{\widetilde{G}}(X) \exp_{\widetilde{G}}(-Y)$. Show that $\widetilde{\varphi}$ is a local diffeomorphism near 0. Conclude that $\{\exp_{\widetilde{G}}(Y) g_0 \exp_{\widetilde{G}}(X) \exp_{\widetilde{G}}(-Y) \mid X \in \mathfrak{a}, Y \in \mathfrak{b}\}$ contains a neighborhood of g_0 in $\widetilde{G}$.
(10) Let $\widetilde{A} = \left(\pi^{-1} A\right)^0$, a covering of the compact Lie subgroup $A = Z_G(\pi g_0)^0$ of G. Show that $\exp_{\widetilde{G}}(\mathfrak{a}) \subseteq \widetilde{A}$. Conclude that $\bigcup_{g \in \widetilde{G}} g^{-1} \widetilde{A} g$ contains a neighborhood of g_0 in $\widetilde{G}$.
(11) If $\dim \mathfrak{a} < \dim \mathfrak{g}$, use the inductive hypothese to show that $\widetilde{A} = \exp_{\widetilde{G}}(\mathfrak{a})$. Conclude that $\bigcup_{g \in \widetilde{G}} g^{-1} \widetilde{A} g = \bigcup_{g \in \widetilde{G}} \exp_{\widetilde{G}} (\mathrm{Ad}(\pi g)\mathfrak{a})$, so that $\exp_{\widetilde{G}}(\mathfrak{g})$ contains a neighborhood of g_0.
(12) Finally, if $\dim \mathfrak{a} = \dim \mathfrak{g}$, show that $g_0 \in Z(\widetilde{G})$. Let $\mathfrak{t}'$ be a Cartan subalgebra containing X_0 so that $\mathfrak{g} = \bigcup_{g \in \widetilde{G}} \mathrm{Ad}(\pi g)\mathfrak{t}'$. Show that $g_0 \exp_{\widetilde{G}}(\mathfrak{g}) \subseteq \exp_{\widetilde{G}}(\mathfrak{g})$. Conclude that $\exp_{\widetilde{G}}(\mathfrak{g})$ contains a neighborhood of g_0.

6.4 Weyl Group

6.4.1 Group Picture

Definition 6.34. Let G be a compact connected Lie group with maximal torus T. Let $N = N(T)$ be the normalizer in G of T, $N = \{g \in G \mid gTg^{-1} = T\}$. The *Weyl group* of G, $W = W(G) = W(G, T)$, is defined by $W = N/T$.

If T' is another maximal torus of G, Corollary 5.10 shows that there is a $g \in G$, so $c_g T = T'$. In turn, this shows that $c_g N(T) = N(T')$, so that $W(G, T) \cong W(G, T')$. Thus, up to isomorphism, the Weyl group is independent of the choice of maximal torus.

Given $w \in N$, $H \in \mathfrak{t}$, and $\lambda \in \mathfrak{t}^*$, define an action of N on $\mathfrak{t}$ and $\mathfrak{t}^*$ by

$$\begin{aligned} w(H) &= \mathrm{Ad}(w)H \\ [w(\lambda)]\,(H) &= \lambda(w^{-1}(H)) = \lambda(\mathrm{Ad}(w^{-1})H). \end{aligned} \tag{6.35}$$

As usual, extend this to an action of N on $\mathfrak{t}_{\mathbb{C}}$, $i\mathfrak{t}$, $\mathfrak{t}_{\mathbb{C}}^*$, and $(i\mathfrak{t})^*$ by $\mathbb{C}$-linearity. As $\mathrm{Ad}(T)$ acts trivially on $\mathfrak{t}$, the action of N descends to an action of $W = N/T$.

Theorem 6.36. *Let G be a compact connected Lie group with a maximal torus T.*
(a) *The action of W on $i\mathfrak{t}$ and on $(i\mathfrak{t})^*$ is faithful, i.e., a Weyl group element acts trivially if and only it is the identity element.*
(b) *For $w \in N$ and $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}) \cup \{0\}$, $\mathrm{Ad}(w)\mathfrak{g}_\alpha = \mathfrak{g}_{w\alpha}$.*
(c) *The action of W on $(i\mathfrak{t})^*$ preserves and acts faithfully on $\Delta(\mathfrak{g}_{\mathbb{C}})$.*
(d) *The action of W on $i\mathfrak{t}$ preserves and acts faithfully on $\{h_\alpha \mid \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})\}$. Moreover, $wh_\alpha = h_{w\alpha}$.*
(e) *W is a finite group.*
(f) *Given $t_i \in T$, there exists $g \in G$ so $c_g t_1 = t_2$ if and only if there exists $w \in N$, so $c_w t_1 = t_2$.*

Proof. For part (a), suppose $w \in N$ acts trivially on $\mathfrak{t}$ via Ad. Since $\exp \mathfrak{t} = T$ and since $c_w \circ \exp = \exp \circ \operatorname{Ad}(w)$, this implies that $w \in Z_G(T)$. However, Corollary 5.13 shows that $Z_G(T) = T$ so that $w \in T$, as desired.

For part (b), let $w \in N$, $H \in \mathfrak{t}_{\mathbb{C}}$, and $X_\alpha \in \mathfrak{g}_\alpha$ and calculate

$$[H, \operatorname{Ad}(w)X_\alpha] = [\operatorname{Ad}(w^{-1})H, X_\alpha] = \alpha(\operatorname{Ad}(w^{-1})H)X_\alpha = [(w\alpha)(H)]\, X_\alpha,$$

which shows that $\operatorname{Ad}(w)\mathfrak{g}_\alpha \subseteq \mathfrak{g}_{w\alpha}$. Since $\dim \mathfrak{g}_\alpha = 1$ and since $\operatorname{Ad}(w)$ is invertible, $\operatorname{Ad}(w)\mathfrak{g}_\alpha = \mathfrak{g}_{w\alpha}$ and, in particular, $w\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$. Noting that W acts trivially on $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{t}$, we may reduce to the case where $\mathfrak{g}$ is semisimple. As $\Delta(\mathfrak{g}_{\mathbb{C}})$ spans $(i\mathfrak{t})^*$, parts (b) and (c) are therefore finished.

For part (d), calculate

$$B(u_{w\alpha}, H) = B(w\alpha)(H) = \alpha(w^{-1}H) = B(u_\alpha, w^{-1}H) = B(wu_\alpha, H),$$

so that $u_{w\alpha} = wu_\alpha$. Since the action of w preserves the Killing form, it follows that $wh_\alpha = h_{w\alpha}$, which finishes part (d). As $\Delta(\mathfrak{g}_{\mathbb{C}})$ is finite and the action is faithful, part (e) is also done.

For part (f), suppose $c_g t_1 = t_2$ for $g \in G$. Consider the connected compact Lie subgroup $Z_G(t_2)^0 = \{h \in G \mid c_{t_2} h = h\}^0$ of G with Lie algebra $\mathfrak{z}_{\mathfrak{g}}(t_2) = \{X \in \mathfrak{g} \mid \operatorname{Ad}(t_2)X = X\}$ (Exercise 4.22). Clearly $\mathfrak{t} \subseteq \mathfrak{z}_{\mathfrak{g}}(t_2)$ and $\mathfrak{t}$ is still a Cartan subalgebra of $\mathfrak{z}_{\mathfrak{g}}(t_2)$. Therefore $T \subseteq Z_G(t_2)$ and T is a maximal torus of $Z_G(t_2)$. On the other hand, $\operatorname{Ad}(t_2)\operatorname{Ad}(g)H = \operatorname{Ad}(g)\operatorname{Ad}(t_1)H = \operatorname{Ad}(g)H$ for $H \in \mathfrak{t}$. Thus $\operatorname{Ad}(g)\mathfrak{t}$ is also a Cartan subalgebra in $\mathfrak{z}_{\mathfrak{g}}(t_2)$, and so $c_g T$ is a maximal torus in $Z_G(t_2)^0$. By Corollary 5.10, there is a $z \in Z_G(t_2)$, so that $c_z\left(c_g T\right) = T$, i.e., $zg \in N(T)$. Since $c_{zg} t_1 = c_z t_2 = t_2$, the proof is finished. □

6.4.2 Classical Examples

Here we calculate the Weyl group for each of the compact classical Lie groups. The details are straightforward matrix calculations and are mostly left as an exercise (Exercise 6.27).

6.4.2.1 $U(n)$ **and** $SU(n)$ For $U(n)$ let $T_{U(n)} = \{\operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \mid \theta_i \in \mathbb{R}\}$ be a maximal torus. Write $\mathcal{S}_n$ for the set of $n \times n$ *permutation matrices*. Recall that an element of $GL(n, \mathbb{C})$ is a permutation matrix if the entries of each row and column consists of a single one and $(n-1)$ zeros. Thus $\mathcal{S}_n \cong S_n$ where S_n is the *permutation group* on n letters. Since the set of eigenvalues is invariant under conjugation, any $w \in N$ must permute, up to scalar, the standard basis of $\mathbb{R}^n$. In particular, this shows that

$$N(T_{U(n)}) = \mathcal{S}_n T_{U(n)}$$
$$W \cong S_n$$
$$|W| = n!.$$

Write (θ_i) for the element $\operatorname{diag}(\theta_1, \ldots, \theta_n) \in \mathfrak{t}$ and (λ_i) for the element $\sum_i \lambda_i \epsilon_i \in (i\mathfrak{t})^*$. It follows that W acts on $i\mathfrak{t}_{U(n)} = \{(\theta_i) \mid \theta_i \in \mathbb{R}\}$ and on $(i\mathfrak{t}_{U(n)})^* = \{(\lambda_i) \mid \lambda_i \in \mathbb{R}\}$ by all permutations of the coordinates.

For $SU(n)$, let $T_{SU(n)} = T_{U(n)} \cap SU(n) = \{\operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \mid \theta_i \in \mathbb{R}, \sum_i \theta_i = 0\}$ be a maximal torus. Note that $U(n) \cong \left(SU(n) \times S^1\right)/\mathbb{Z}_n$ with S^1 central, so that $W(SU(n)) \cong W(U(n))$. In particular for the A_{n-1} root system,

$$\begin{aligned} N(T_{SU(n)}) &= \left(\mathcal{S}_n T_{U(n)}\right) \cap SU(n) \\ W &\cong S_n \\ |W| &= n!. \end{aligned}$$

As before, W acts on $i\mathfrak{t}_{SU(n)} = \{(\theta_i) \mid \theta_i \in \mathbb{R}, \sum_i \theta_i = 0\}$ and $(i\mathfrak{t}_{SU(n)})^* = \{(\lambda_i) \mid \lambda_i \in \mathbb{R}, \sum_i \lambda_i = 0\}$ by all permutations of the coordinates.

6.4.2.2 *$Sp(n)$* For $Sp(n)$ realized as $Sp(n) \cong U(2n) \cap Sp(n, \mathbb{C})$, let

$$T = \{\operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}, e^{-i\theta_1}, \ldots, e^{-i\theta_n}) \mid \theta_i \in \mathbb{R}\}.$$

For $1 \le i \le n$, write $s_{1,i}$ for the matrix realizing the linear transformation that maps e_i, the i^{th} standard basis vector of $\mathbb{R}^{2n}$, to $-e_{i+n}$, maps e_{i+n} to e_n, and fixes the remaining standard basis vectors. In particular, $s_{1,i}$ is just the natural embedding of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ into $Sp(n)$ in the $i \times (n+i)^{\text{th}}$ submatrix. By considering eigenvalues, it is straightforward to check that for the C_n root system,

$$\begin{aligned} N(T) &= \left\{ \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \mid s \in \mathcal{S}_n \right\} \left\{ \prod_i s_{1,i}^{k_i} \mid 1 \le i \le n, k_i \in \{0, 1\} \right\} T \\ W &\cong S_n \ltimes (\mathbb{Z}_2)^n \\ |W| &= 2^n n!. \end{aligned}$$

Write (θ_i) for the element $\operatorname{diag}(\theta_1, \ldots, \theta_n, -\theta_1, \ldots, -\theta_n) \in i\mathfrak{t}$ and (λ_i) for the element $\sum_i \lambda_i \epsilon_i \in (i\mathfrak{t})^*$. Then W acts on $i\mathfrak{t} = \{(\theta_i) \mid \theta_i \in \mathbb{R}\}$ and on $(i\mathfrak{t})^* = \{(\lambda_i) \mid \lambda_i \in \mathbb{R}\}$ by all permutations and all sign changes of the coordinates.

6.4.2.3 *$SO(E_{2n})$* For $G = SO(E_{2n})$, let

$$T = \{\operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}, e^{-i\theta_1}, \ldots, e^{-i\theta_n}) \mid \theta_i \in \mathbb{R}\}$$

be a maximal torus. For $1 \le i \le n$, write $s_{2,i}$ for the matrix realizing the linear transformation that maps e_i, the i^{th} standard basis vector of $\mathbb{R}^{2n}$, to e_{i+n}, maps e_{i+n} to e_n, and fixes the remaining standard basis vectors. In particular, $s_{2,i}$ is just the natural embedding of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ into $O(E_{2n})$ in the $i \times (n+i)^{\text{th}}$ submatrix. Then for the D_n root system,

$$\begin{aligned} N(T) &= \{ \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \mid s \in \mathcal{S}_n \} \{ \prod_i s_{2,i}^{k_i} \mid 1 \le i \le n, k_i \in \{0, 1\}, \sum_i k_i \in 2\mathbb{Z} \} T \\ W &\cong S_n \ltimes (\mathbb{Z}_2)^{n-1} \\ |W| &= 2^{n-1} n!. \end{aligned}$$

Write (θ_i) for the element $\operatorname{diag}(\theta_1, \ldots, \theta_n, -\theta_1, \ldots, -\theta_n) \in i\mathfrak{t}$ and (λ_i) for the element $\sum_i \lambda_i \epsilon_i \in (i\mathfrak{t})^*$. Then W acts on $i\mathfrak{t} = \{(\theta_i) \mid \theta_i \in \mathbb{R}\}$ and on $(i\mathfrak{t})^* = \{(\lambda_i) \mid \lambda_i \in \mathbb{R}\}$ by all permutations and all even sign changes of the coordinates.

6.4.2.4 $SO(E_{2n+1})$ For $G = SO(E_{2n+1})$, let

$$T = \{\operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}, e^{-i\theta_1}, \ldots, e^{-i\theta_n}, 1) \mid \theta_i \in \mathbb{R}\}$$

be a maximal torus. For $1 \leq i \leq n$, write $s_{3,i}$ for the matrix realizing the linear transformation that maps e_i, the i^{th} standard basis vector of $\mathbb{R}^{2n+1}$, to e_{i+n}, maps e_{i+n} to e_n, maps e_{2n+1} to $-e_{2n+1}$, and fixes the remaining standard basis vectors. In particular, $s_{3,i}$ is just the natural embedding of $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ into $SO(E_{2n+1})$ in the $i \times (n+i) \times (2n+1)^{\text{th}}$ submatrix. Then for the B_n root system,

$$\begin{aligned} N(T) &= \left\{ \begin{pmatrix} s & & \\ & s & \\ & & 1 \end{pmatrix} \mid s \in \mathcal{S}_n \right\} \left\{ \prod_i s_{3,i}^{k_i} \mid 1 \leq i \leq n, k_i \in \{0,1\} \right\} T \\ W &\cong S_n \ltimes (\mathbb{Z}_2)^n \\ |W| &= 2^n n!. \end{aligned}$$

Write (θ_i) for the element $\operatorname{diag}(\theta_1, \ldots, \theta_n, -\theta_1, \ldots, -\theta_n, 0) \in i\mathfrak{t}$ and (λ_i) for the element $\sum_i \lambda_i \epsilon_i \in (i\mathfrak{t})^*$. Then W acts on $i\mathfrak{t} = \{(\theta_i) \mid \theta_i \in \mathbb{R}\}$ and on $(i\mathfrak{t})^* = \{(\lambda_i) \mid \lambda_i \in \mathbb{R}\}$ by all permutations and all sign changes of the coordinates.

6.4.3 Simple Roots and Weyl Chambers

Definition 6.37. Let G be compact Lie group with a Cartan subalgebra $\mathfrak{t}$. Write $\mathfrak{t}' = \mathfrak{g}' \cap \mathfrak{t}$.
(a) A *system of simple roots*, $\Pi = \Pi(\mathfrak{g}_{\mathbb{C}})$, is a subset of $\Delta(\mathfrak{g}_{\mathbb{C}})$ that is a basis of $\left(i\mathfrak{t}'\right)^*$ and satisfies the property that any $\beta \in \Delta(\mathfrak{g}_{\mathbb{C}})$ may be written as

$$\beta = \sum_{\alpha \in \Pi} k_\alpha \alpha$$

with either $\{k_\alpha \mid \alpha \in \Pi\} \subseteq \mathbb{Z}_{\geq 0}$ or $\{k_\alpha \mid \alpha \in \Pi\} \subseteq \mathbb{Z}_{\leq 0}$, where $\mathbb{Z}_{\geq 0} = \{k \in \mathbb{Z} \mid k \geq 0\}$ and $\mathbb{Z}_{\leq 0} = \{k \in \mathbb{Z} \mid k \leq 0\}$. The elements of Π are called *simple roots*.
(b) Given a system of simple roots Π, the set of *positive roots* with respect to Π is

$$\Delta^+(\mathfrak{g}_{\mathbb{C}}) = \{\beta \in \Delta(\mathfrak{g}_{\mathbb{C}}) \mid \beta = \sum_{\alpha \in \Pi} k_\alpha \alpha \text{ with } k_\alpha \in \mathbb{Z}_{\geq 0}\}$$

and the set of *negative roots* with respect to Π is

$$\Delta^-(\mathfrak{g}_{\mathbb{C}}) = \{\beta \in \Delta(\mathfrak{g}_{\mathbb{C}}) \mid \beta = \sum_{\alpha \in \Pi} k_\alpha \alpha \text{ with } k_\alpha \in \mathbb{Z}_{\leq 0}\},$$

so that $\Delta(\mathfrak{g}_{\mathbb{C}}) = \Delta^+(\mathfrak{g}_{\mathbb{C}}) \coprod \Delta^-(\mathfrak{g}_{\mathbb{C}})$ and $\Delta^-(\mathfrak{g}_{\mathbb{C}}) = -\Delta^+(\mathfrak{g}_{\mathbb{C}})$.

As matters stand at the moment, we are not guaranteed that simple systems exist. In Lemma 6.42 below, this shortcoming will be rectified using the following definition.

Definition 6.38. Let G be compact Lie group with a Cartan subalgebra $\mathfrak{t}$.
(a) The connected components of $(i\mathfrak{t}')^*\backslash\left(\cup_{\alpha\in\Delta(\mathfrak{g}_\mathbb{C})}\alpha^\perp\right)$ are called the (open) *Weyl chambers* of $(i\mathfrak{t})^*$. The connected components of $i\mathfrak{t}'\backslash\left(\cup_{\alpha\in\Delta(\mathfrak{g}_\mathbb{C})}h_\alpha^\perp\right)$ are called the (open) *Weyl chambers* of $i\mathfrak{t}$.
(b) If C is a Weyl chamber of $(i\mathfrak{t})^*$, $\alpha\in\Delta(\mathfrak{g}_\mathbb{C})$ is called C*-positive* if $B(C,\alpha)>0$ and C*-negative* if $B(C,\alpha)<0$. If α is C-positive, it is called *decomposable* with respect to C if there exist C-positive $\beta,\gamma\in\Delta(\mathfrak{g}_\mathbb{C})$, so that $\alpha=\beta+\gamma$. Otherwise α is called *indecomposable* with respect to C.
(c) If $C^\vee$ is a Weyl chamber of $i\mathfrak{t}$, $\alpha\in\Delta(\mathfrak{g}_\mathbb{C})$ is called $C^\vee$*-positive* if $\alpha(C^\vee)>0$ and C*-negative* if $\alpha(C^\vee)<0$. If α is $C^\vee$-positive, it is called *decomposable* with respect to $C^\vee$ if there exist $C^\vee$-positive $\beta,\gamma\in\Delta(\mathfrak{g}_\mathbb{C})$, so that $\alpha=\beta+\gamma$. Otherwise α is called *indecomposable* with respect to $C^\vee$.
(d) If C is a Weyl chamber of $(i\mathfrak{t})^*$, let

$$\Pi(C)=\{\alpha\in\Delta(\mathfrak{g}_\mathbb{C})\mid\alpha\text{ is }C\text{-positive and indecomposable}\}.$$

If $C^\vee$ is a Weyl chamber of $i\mathfrak{t}$, let

$$\Pi(C^\vee)=\{\alpha\in\Delta(\mathfrak{g}_\mathbb{C})\mid\alpha\text{ is }C^\vee\text{-positive and indecomposable}\}.$$

(e) If Π is a system of simple roots, the *associated Weyl chamber* of $(i\mathfrak{t})^*$ is

$$C(\Pi)=\{\lambda\in(i\mathfrak{t})^*\mid B(\lambda,\alpha)>0\text{ for }\alpha\in\Pi\}$$

and the *associated Weyl chamber* of $i\mathfrak{t}$ is

$$C^\vee(\Pi)=\{H\in i\mathfrak{t}\mid\alpha(H)>0\text{ for }\alpha\in\Pi\}.$$

Each Weyl chamber is a polyhedral convex cone and its closure is called the *closed Weyl chamber*. For the sake of symmetry, note that the condition $\alpha(H)>0$ above is equivalent to the condition $B(H,h_\alpha)>0$. In Lemma 6.42 we will see that the mapping $C\to\Pi(C)$ establishes a one-to-one correspondence between Weyl chambers and simple systems. For the time being, we list the standard simple systems and corresponding Weyl chamber of $(i\mathfrak{t})^*$ for the classical compact groups. The details are straightforward and left to Exercise 6.30 (see §6.1.5 for the roots and notation).

In addition to a simple system and its corresponding Weyl chamber, two other pieces of data are given below. For the first, write the given simple system as $\Pi=\{\alpha_1,\ldots,\alpha_l\}$. Define the *fundamental weights* to be the basis $\{\pi_1,\ldots,\pi_l\}$ of $(i\mathfrak{t})^*$ determined by $2\frac{B(\pi_i,\alpha_i)}{B(\alpha_i,\alpha_i)}=\delta_{i,j}$ and define $\rho=\rho(\Pi)\in(i\mathfrak{t})^*$ as

$$\rho=\sum_i\pi_i \tag{6.39}$$

Notice $\rho(h_{\alpha_i})=2\frac{B(\rho,\alpha_i)}{B(\alpha_i,\alpha_i)}=1$, so that $\rho\in P$ (c.f. Exercise 6.34).

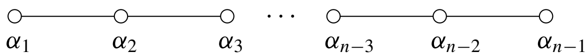
The second piece of data given below is called the *Dynkin diagram* of the simple system Π. The Dynkin diagram is a graph with one vertex for each simple root, α_i, and turns out to be independent of the choice of simple system. Whenever $B(\alpha_i, \alpha_j) \neq 0$, $i \neq j$, the vertices corresponding to α_i and α_j are joined by an edge of multiplicity $m_{ij} = \alpha_i(h_{\alpha_j})\alpha_j(h_{\alpha_i})$. In this case, from the proof of Corollary 6.22 (*c.f.* Exercise 6.20), it turns out that $m_{ij} = m_{ji} = \frac{\|\alpha_i\|^2}{\|\alpha_j\|^2} \in \{1, 2, 3\}$ when $\|\alpha_i\|^2 \geq \|\alpha_j\|^2$. Furthermore, when two vertices corresponding to roots of unequal length are connected by an edge, the edge is oriented by an arrow pointing towards the vertex corresponding to the shorter root.

6.4.3.1 *$SU(n)$* For $SU(n)$ with $\mathfrak{t} = \{\mathrm{diag}(i\theta_1, \ldots, i\theta_n) \mid \theta_i \in \mathbb{R}, \sum_i \theta_i = 0\}$, i.e., the A_{n-1} root system,

$$\begin{aligned}
\Pi &= \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n-1\} \\
C &= \{\mathrm{diag}(\theta_1, \ldots, \theta_n) \mid \theta_i > \theta_{i+1}, \theta_i \in \mathbb{R}\} \\
\rho &= \frac{1}{2}\left((n-1)\epsilon_1 + (n-3)\epsilon_2 + \cdots + (-n+1)\epsilon_n\right)
\end{aligned}$$

and the corresponding Dynkin diagram is

A_n



6.4.3.2 *$Sp(n)$* For $Sp(n)$ realized as $Sp(n) \cong U(2n) \cap Sp(n, \mathbb{C})$ with

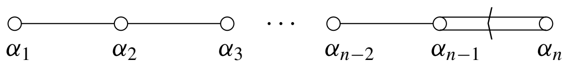
$$\mathfrak{t} = \{\mathrm{diag}(i\theta_1, \ldots, i\theta_n, -i\theta_1, \ldots, -i\theta_n) \mid \theta_i \in \mathbb{R}\},$$

i.e., the C_n root system,

$$\begin{aligned}
\Pi &= \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\alpha_n = 2\epsilon_n\} \\
C &= \{\mathrm{diag}(\theta_1, \ldots, \theta_n, -\theta_1, \ldots, -\theta_n) \mid \theta_i > \theta_{i+1} > 0, \theta_i \in \mathbb{R}\} \\
\rho &= n\epsilon_1 + (n-1)\epsilon_2 + \cdots + \epsilon_n
\end{aligned}$$

and the corresponding Dynkin diagram is

C_n

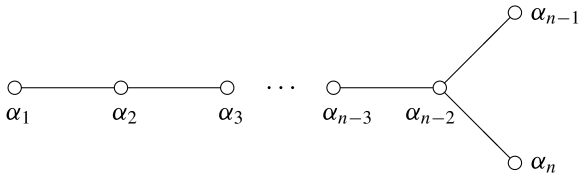


6.4.3.3 *$SO(E_{2n})$* For $SO(E_{2n})$ with $\mathfrak{t} = \{\mathrm{diag}(i\theta_1, \ldots, i\theta_n, -i\theta_1, \ldots, -i\theta_n) \mid \theta_i \in \mathbb{R}\}$, i.e., the D_n root system,

$$\begin{aligned}
\Pi &= \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\alpha_n = \epsilon_{n-1} + \epsilon_n\} \\
C &= \{\mathrm{diag}(\theta_1, \ldots, \theta_n, -\theta_1, \ldots, -\theta_n, 0) \mid \theta_i > \theta_{i+1}, \theta_{n-1} > |\theta_n|, \theta_i \in \mathbb{R}\} \\
\rho &= n\epsilon_1 + (n-1)\epsilon_2 + \cdots + \epsilon_{n-1}
\end{aligned}$$

and the corresponding Dynkin diagram is

D_n



6.4.3.4 $SO(E_{2n+1})$ For $SO(E_{2n+1})$ with

$$\mathfrak{t} = \{\mathrm{diag}(i\theta_1, \ldots, i\theta_n, -i\theta_1, \ldots, -i\theta_n, 0) \mid \theta_i \in \mathbb{R}\},$$

i.e., the B_n root system,

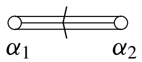
$$\Pi = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\alpha_n = \epsilon_n\}$$
$$C = \{\mathrm{diag}(\theta_1, \ldots, \theta_n, -\theta_1, \ldots, -\theta_n, 0) \mid \theta_i > \theta_{i+1} > 0, \theta_i \in \mathbb{R}\}$$
$$\rho = \frac{1}{2}\left((2n-1)\epsilon_1 + (2n-3)\epsilon_2 + \cdots + \epsilon_n\right)$$

and the corresponding Dynkin diagram is

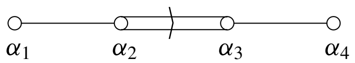
B_n

It is an important fact from the theory of Lie algebras that there are only five other simple Lie algebras over $\mathbb{C}$. They are called the exceptional Lie algebras and there is a simple compact group corresponding to each one. The corresponding Dynkin diagrams are given below (see [56] or [70] for details).

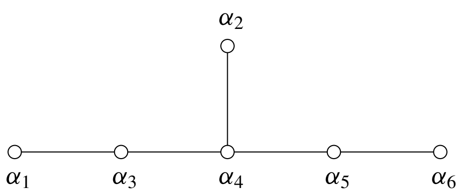
G_2



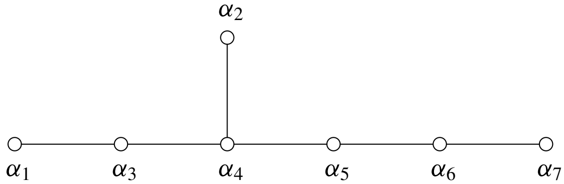
F_4



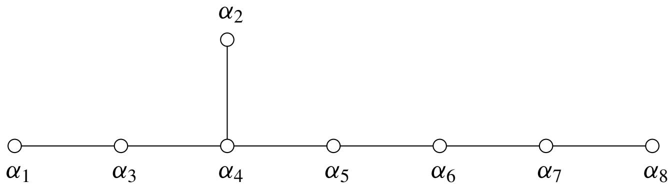
E_6



E_7



E_8



6.4.4 The Weyl Group as a Reflection Group

Definition 6.40. Let G be compact Lie group with a Cartan subalgebra $\mathfrak{t}$.
(a) For $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$, define $r_\alpha : (i\mathfrak{t})^* \to (i\mathfrak{t})^*$ by

$$r_\alpha(\lambda) = \lambda - 2\frac{B(\lambda, \alpha)}{B(\alpha, \alpha)}\alpha = \lambda - \lambda(h_\alpha)\alpha$$

and $r_{h_\alpha} : i\mathfrak{t} \to i\mathfrak{t}$ by

$$r_{h_\alpha}(H) = H - 2\frac{B(H, h_\alpha)}{B(h_\alpha, h_\alpha)}h_\alpha = H - \alpha(H)h_\alpha.$$

(b) Write $W(\Delta(\mathfrak{g}_{\mathbb{C}}))$ for the group generated by $\{r_\alpha \mid \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})\}$ and write $W(\Delta(\mathfrak{g}_{\mathbb{C}})^\vee)$ for the group generated by $\{r_{h_\alpha} \mid \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})\}$.

As usual, the action of $W(\Delta(\mathfrak{g}_{\mathbb{C}}))$ and $W(\Delta(\mathfrak{g}_{\mathbb{C}})^\vee)$ on $(i\mathfrak{t})^*$ and $i\mathfrak{t}$, respectively, is extended to an action on $\mathfrak{t}^*$ and $\mathfrak{t}$, respectively, by $\mathbb{C}$-linearity. Also observe that r_α acts trivially on $(\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{t})^*$ and acts on $(i\mathfrak{t}')^*$ as the reflection across the hyperplane perpendicular to α. Similarly, r_{h_α} acts trivially on $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{t}$ and acts on $i\mathfrak{t}'$ as the reflection across the hyperplane perpendicular to h_α (Exercise 6.28). In particular, $r_\alpha^2 = I$ and $r_{h_\alpha}^2 = I$.

Lemma 6.41. *Let G be compact Lie group with a maximal torus T.*
(a) *For $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$, there exists $w_\alpha \in N(T)$, so that the action of w_α on $(i\mathfrak{t})^*$ is given by r_α and the action of w_α on $i\mathfrak{t}$ is given by r_{h_α}.*
(b) *For $\alpha, \beta \in \Delta(\mathfrak{g}_{\mathbb{C}})$, $r_\alpha(\beta) \in \Delta(\mathfrak{g}_{\mathbb{C}})$ and $r_{h_\alpha}\left(h_\beta\right) = h_{r_\alpha(\beta)}$.*

Proof. Using Theorem 6.20, choose a standard $\mathfrak{su}(2)$-triple, $\{\mathcal{I}_\alpha, \mathcal{J}_\alpha, \mathcal{K}_\alpha\}$, and a standard $\mathfrak{sl}(2, \mathbb{C})$-triple, $\{E_\alpha, H_\alpha, F_\alpha\}$, corresponding to α and let $\varphi_\alpha : SU(2) \to G$

be the corresponding homomorphism. Let $w = \exp(\frac{\pi}{2}J) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SU(2)$, where $J = -E + F \in \mathfrak{su}(2)$. Thus $d\varphi_\alpha(J) = \mathcal{J}_\alpha = -E_\alpha + F_\alpha$. Define $w_\alpha \in G$ by $w_\alpha = \varphi_\alpha(w)$. For $H \in i\mathfrak{t}$, calculate

$$\operatorname{ad}(d\varphi_\alpha(\frac{\pi}{2}J))H = \frac{\pi}{2}\operatorname{ad}(-E_\alpha + F_\alpha)H = \alpha(H)\frac{\pi}{2}[E_\alpha + F_\alpha].$$

In particular if $B(H, h_\alpha) = 0$, then $\alpha(H) = 0$, so that $\operatorname{ad}(d\varphi_\alpha(\frac{\pi}{2}J))H = 0$. Thus, if $B(H, h_\alpha) = 0$,

$$\begin{aligned}\operatorname{Ad}(w_\alpha)H &= \operatorname{Ad}(\varphi_\alpha(\exp(\frac{\pi}{2}J)))H = \operatorname{Ad}(\exp(d\varphi_\alpha(\frac{\pi}{2}J)))H \\ &= e^{\operatorname{ad}(d\varphi_\alpha(\frac{\pi}{2}J))}H = H.\end{aligned}$$

On the other hand, consider the case of $H = h_\alpha$. Since $c_{\varphi_\alpha(w)} \circ \varphi_\alpha = \varphi_\alpha \circ c_w$, the differentials satisfy $\operatorname{Ad}(w_\alpha) \circ d\varphi_\alpha = d\varphi_\alpha \circ \operatorname{Ad}(w)$. Observing that $wHw^{-1} = -H$ in $\mathfrak{sl}(2, \mathbb{C})$, where $d\varphi_\alpha(H) = h_\alpha$, it follows that

$$\begin{aligned}\operatorname{Ad}(w_\alpha)h_\alpha &= \operatorname{Ad}(w_\alpha)\varphi_\alpha(H) \\ &= d\varphi_\alpha\,(\operatorname{Ad}(w)H) = -d\varphi_\alpha(H) = -h_\alpha.\end{aligned}$$

Thus $\operatorname{Ad}(w_\alpha)$ preserves $\mathfrak{t}$ and acts on $i\mathfrak{t}$ as the reflection across the hyperplane perpendicular to h_α. In other words, $\operatorname{Ad}(w_\alpha)$ acts as r_{h_α} on $i\mathfrak{t}$. Since $T = \exp\mathfrak{t}$ and $c_{w_\alpha}(\exp H) = \exp(\operatorname{Ad}(w_\alpha)H)$, this also shows that $w_\alpha \in N(T)$. To finish part (a), calculate

$$(w_\alpha\lambda)(H) = \lambda(w_\alpha^{-1}H) = B(u_\lambda, \operatorname{Ad}(w_\alpha)^{-1}H) = B(\operatorname{Ad}(w_\alpha)u_\lambda, H)$$

for $\lambda \in (i\mathfrak{t})^*$. Thus for $\lambda = \alpha$, $\operatorname{Ad}(w_\alpha)u_\alpha = -u_\alpha$, so $w_\alpha\alpha = -\alpha$. For $\lambda \in \alpha^\perp$, $u_\lambda \in h_\alpha^\perp$, so $\operatorname{Ad}(w_\alpha)u_\lambda = u_\lambda$ and $w_\alpha\lambda = \lambda$. In particular, $\operatorname{Ad}(w_\alpha)$ acts on $(i\mathfrak{t})^*$ by r_α. Part (b) now follows from Theorem 6.36. ☐

Lemma 6.42. *Let G be compact Lie group with a Cartan subalgebra $\mathfrak{t}$.*
(a) *There is a one-to-one correspondence between*

$$\{\textit{systems of simple roots}\} \longleftrightarrow \{\textit{Weyl chambers of } (i\mathfrak{t})^*\}.$$

The bijection maps a simple system Π to the Weyl chamber $C(\Pi)$ and maps a Weyl chamber C to the simple system $\Pi(C)$.
(b) *There is a one-to-one correspondence between*

$$\{\textit{systems of simple roots}\} \longleftrightarrow \{\textit{Weyl chambers of } i\mathfrak{t}\}.$$

The bijection maps a simple system Π to the Weyl chamber $C^\vee(\Pi)$ and maps a Weyl chamber $C^\vee$ to the simple system $\Pi(C^\vee)$.
(c) *If Π is a simple system with $\alpha, \beta \in \Pi$, then $B(\alpha, \beta) \leq 0$.*

Proof. Suppose $\Pi = \{\alpha_1, \dots, \alpha_l\}$ is a simple system. From Equation 6.39, recall that $\rho \in (i\mathfrak{t})^*$ satisfies $B(\rho, \alpha_j) = \frac{\|\alpha\|^2}{2} > 0$, so that $\rho \in C(\Pi)$. For any $\lambda \in C(\Pi)$, examining the map $t \to B(t\lambda + (1-t)\rho, \alpha)$ quickly shows that the line segment joining λ to ρ lies in $C(\Pi)$, so that $C(\Pi)$ is connected. Moreover, $B(\lambda, \alpha) > 0$ for $\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C})$ and $B(\lambda, \beta) < 0$ for $\beta \in \Delta^-(\mathfrak{g}_\mathbb{C})$, so that $C(\Pi) \subseteq (i\mathfrak{t})^* \backslash \left(\cup_{\alpha \in \Delta(\mathfrak{g}_\mathbb{C})} \alpha^\perp\right)$. In particular, $C(\Pi) \subseteq C$ for some Weyl chamber C of $(i\mathfrak{t})^*$. As the sign of $B(\gamma, \alpha_j)$ is constant for $\gamma \in C$, the fact that $\rho \in C$ forces $\alpha_j(C) > 0$. Thus, $C \subseteq C(\Pi)$, so that $C(\Pi) = C$ is a Weyl chamber and the first half of part (a) is done.

Next, let C be a Weyl chamber of $(i\mathfrak{t})^*$ and fix $\lambda \in C$. If $\alpha = \beta_1 + \beta_2$ for C-positive roots α, β_i, then $B(\alpha, \lambda) > B(\beta_i, \lambda)$. Since $\{B(\lambda, \alpha) \mid \alpha \in \Delta(\mathfrak{g}_\mathbb{C})$ is C-positive$\}$ is a finite (nonempty) subset of positive real numbers, it is easy to see that $\Pi(C)$ is nonempty. It now follows from the definition of $\Pi(C)$ that any $\beta \in \Delta(\mathfrak{g}_\mathbb{C})$ may be written as $\beta = \sum_{\alpha \in \Pi} k_\alpha \alpha$ with either $\{k_\alpha \mid \alpha \in \Pi\} \subseteq \mathbb{Z}_{\geq 0}$ or $\{k_\alpha \mid \alpha \in \Pi\} \subseteq \mathbb{Z}_{\leq 0}$ depending on whether β is C-positive or C-negative. Since $\Delta(\mathfrak{g}_\mathbb{C})$ spans $(i\mathfrak{t})^*$, it only remains to see $\Pi(C)$ is an independent set.

Let $\alpha, \beta \in \Pi(C)$ be distinct and, without loss of generality, assume $B(\alpha, \alpha) \leq B(\beta, \beta)$. Positivity implies $\alpha \neq -\beta$ so that the proof of Corollary 6.22 (c.f. Exercise 6.20) shows that $\beta(h_\alpha) = 2\frac{B(\alpha,\beta)}{B(\beta,\beta)} \in \pm\{0, 1\}$. If $B(\alpha, \beta) > 0$, then $r_\alpha(\beta) = \beta - \alpha \in \Delta(\mathfrak{g}_\mathbb{C})$ by Lemma 6.41. If $\beta - \alpha$ is C-positive, then $\beta = (\beta - \alpha) + \alpha$. If $\beta - \alpha$ is C-negative, then $\alpha = -(\beta - \alpha) + \beta$. Either violates the assumption that α, β are indecomposable. Thus $\alpha, \beta \in \Pi(C)$ implies that $B(\alpha, \beta) \leq 0$ and will finish part (c) once part (a) is complete.

To see that $\Pi(C)$ is independent, suppose $\sum_{\alpha \in I_1} c_\alpha \alpha = \sum_{\beta \in I_2} c_\beta \beta$ with $c_\alpha, c_\beta \geq 0$ and $I_1 \coprod I_2 = C(\Pi)$. Using the fact that $B(\alpha, \beta) \leq 0$, calculate

$$0 \leq \left\| \sum_{\alpha \in I_1} c_\alpha \alpha \right\|^2 = B(\sum_{\alpha \in I_1} c_\alpha \alpha, \sum_{\beta \in I_2} c_\beta \beta) = \sum_{\alpha \in I_1,\, \beta \in I_2} c_\alpha c_b B(\alpha, \beta) \leq 0.$$

Thus $0 = \sum_{\alpha \in I_1} c_\alpha \alpha$. Choosing any $\gamma \in C$, $0 = \sum_{\alpha \in I_1} c_\alpha B(\gamma, \alpha)$. Since $B(\gamma, \alpha) > 0$, $c_\alpha = 0$. Similarly $c_\beta = 0$ and part (a) is finished. As the proof of part (b) can either be done is a similar fashion or derived easily from part (a), it is omitted. $\square$

Theorem 6.43. *Let G be compact Lie group with a maximal torus T.*
(a) *The action of $W(G)$ on $i\mathfrak{t}$ establishes an isomorphism $W(G) \cong W(\Delta(\mathfrak{g}_\mathbb{C})^\vee)$.*
(b) *The action of $W(G)$ on $(i\mathfrak{t})^*$ establishes an isomorphism $W(G) \cong W(\Delta(\mathfrak{g}_\mathbb{C}))$.*
(c) *$W(G)$ acts simply transitively on the set of Weyl chambers.*

Proof. Using the faithful action of $W = W(G)$ on $i\mathfrak{t}$ via Ad from Theorem 6.36, identify W with the corresponding transformation group on $i\mathfrak{t}$ for the duration of this proof. Then Lemma 6.41 shows $r_{h_\alpha} \in W$ for each $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$, so that $W(\Delta(\mathfrak{g}_\mathbb{C})^\vee) \subseteq W$. It remains to show that $W \subseteq W(\Delta(\mathfrak{g}_\mathbb{C})^\vee)$ to finish the proof of part (a).

Reduce to the case where $\mathfrak{g}$ is semisimple. Fix a Weyl chamber C of $i\mathfrak{t}$ and fix $H \in C$. If $w \in W$, then W preserves $\{h_\alpha \mid \alpha \in \Delta(\mathfrak{g}_\mathbb{C})\}$. Since w leaves the Killing form invariant, w preserves $\{h_\alpha^\perp \mid \alpha \in \Delta(\mathfrak{g}_\mathbb{C})\}$, so that wC is also a Weyl chamber.

Let Ξ be the union of all intersections of hyperplanes of the form $h_\alpha^\perp$ for distinct $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$. As this is a finite union of subspaces of codimension at least 2, $(i\mathfrak{t})\backslash\Xi$

is path connected (Exercise 6.31). Thus there exists a piecewise linear path $\gamma(t) : [0, 1] \to i\mathfrak{t}$ from H to wH that does not intersect Ξ. Modifying $\gamma(t)$ if necessary (Exercise 6.31), there is a partition $\{s_i\}_{i=1}^N$ of $[0, 1]$, Weyl chambers C_i with $C_0 = C$ and $C_N = wC$, and roots α_i, so that $\gamma(s_{i-1}, s_i) \subseteq C_i$, $1 \le i \le N$, and $\gamma(s_i) \in h_{\alpha_i}^\perp$, $1 \le i \le N-1$.

As $\gamma(t)$ does not intersect Ξ, there is an entire ball, B_i, around $\gamma(s_i)$ in $h_{\alpha_i}^\perp$ (of codimension 1 in $i\mathfrak{t}$) lying on the boundary of both C_{i-1} and C_i, $1 \le i \le N-1$. For small nonzero ε, it follows that $B_i + \varepsilon h_\alpha$ lies in C_{i-1} or C_i, depending on the sign of ε. Since $r_{h_{\alpha_i}}(B_i + \varepsilon h_\alpha) = B_i - \varepsilon h_\alpha$ and since $r_{h_{\alpha_i}}$ preserves Weyl chambers, it follows that $r_{h_{\alpha_i}} C_{i-1} = C_i$. In particular, $r_{h_{\alpha_1}} \cdots r_{h_{\alpha_N}} wC = C$.

Now suppose $w_0 \in N(T)$ satisfies $\mathrm{Ad}(w_0)C = C$. To finish part (a), it suffices to show that $w_0 \in T$, so that w_0 acts by the identity operator on $i\mathfrak{t}$. Let $\Pi = \Pi(C)$ and define ρ as in Equation 6.39. By Lemma 6.42, it follows that $w\Pi = \Pi$, so that $\mathrm{Ad}(w_0)\rho = \rho$. Thus $c_{w_0} e^{itu_\rho} = e^{itu_\rho}$, $t \in \mathbb{R}$.

Choose a maximal torus S' of $Z_G(w_0)^0$ containing w_0. Note that S' is also a maximal torus of G by the Maximal Torus Theorem. Since $e^{i\mathbb{R}u_\rho}$ is in turn contained in some (other) maximal torus of $Z_G(w_0)^0$, Corollary 5.10 shows that there exists $g \in Z_G(w_0)^0$, so $c_g e^{i\mathbb{R}u_\rho} \subseteq S'$. Let S be the maximal torus of G given by $S = c_{g^{-1}} S'$. Then $w_0 \in S$ and $e^{i\mathbb{R}u_\rho} \subseteq S$ (c.f. Exercise 5.12). By definition of ρ and a simple a system, the root space decomposition of $\mathfrak{g}_\mathbb{C}$ shows $\mathfrak{z}_{\mathfrak{g}_\mathbb{C}}(u_\rho) = \mathfrak{t}$ so $\mathfrak{z}_\mathfrak{g}(iu_\rho) = \mathfrak{t}$. But since $\mathfrak{s} \subseteq \mathfrak{z}_\mathfrak{g}(iu_\rho) = \mathfrak{t}$, maximality implies $\mathfrak{s} = \mathfrak{t}$, and so $S = T$. Thus $w_0 \in T$, as desired.

Part (b) is done in a similar fashion to part (a). Part (c) is a corollary of the proof of part (a). □

6.4.5 Exercises

Exercise 6.27 For each compact classical group G in §6.4.2, verify that the Weyl group and its action on $\mathfrak{t}$ and $(i\mathfrak{t})^*$ is correctly calculated.

Exercise 6.28 Let G be compact Lie group with semisimple Lie algebra and $\mathfrak{t}$ a Cartan subalgebra. For $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$, show that r_α is the reflection of $(i\mathfrak{t})^*$ across the hyperplane perpendicular to α and r_{h_α} is the reflection of $i\mathfrak{t}$ across the hyperplane perpendicular to h_α.

Exercise 6.29 Let G be a compact connected Lie group with a maximal torus T. Theorem 6.36 shows that the conjugacy classes of G are parametrized by the W-orbits in T. In fact, more is true. Show that there is a one-to-one correspondence between continuous class functions on G and continuous W-invariant functions on T (c.f. Exercise 7.10).

Exercise 6.30 For each compact classical Lie group in §6.4.3, verify that the given system of simple roots and corresponding Weyl chamber is correct.

Exercise 6.31 Suppose G is compact Lie group with semisimple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{t}$.

(1) If Ξ is the union of all intersections of distinct hyperplanes of the form $h_\alpha^\perp$ for $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$, show $(i\mathfrak{t})\backslash\Xi$ is path connected.
(2) Suppose $\gamma(t) : [0, 1] \to i\mathfrak{t}$ is a piecewise linear path that does not intersect Ξ with $\gamma(0)$ and $\gamma(1)$ elements of (different) Weyl chambers. Show there is a piecewise linear path $\gamma'(t) : [0, 1] \to i\mathfrak{t}$ that does not intersect Ξ, satisfies $\gamma'(0) = \gamma(0)$ and $\gamma'(1) = \gamma(1)$, $\gamma(s_{i-1}, s_i) \subseteq C_i$, $1 \le i \le N$, and $\gamma(s_i) \in h_{\alpha_i}^\perp$, $1 \le i \le N-1$, for some partition $\{s_i\}_{i=0}^N$ of $[0, 1]$, some Weyl chambers C_i, and some roots α_i.

Exercise 6.32 Let G be compact Lie group with semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. Fix a basis of $(i\mathfrak{t})^*$. With respect to this basis, the *lexicographic order* on $(i\mathfrak{t})^*$ is defined by setting $\alpha > \beta$ if the first nonzero coordinate (with respect to the given basis) of $\alpha - \beta$ is positive.
(1) Let $\Pi = \{\alpha \in \Delta(\mathfrak{g}_\mathbb{C}) \mid \alpha > 0$ and $\alpha \ne \beta_1 + \beta_2$ for any $\beta_i \in \Delta(\mathfrak{g}_\mathbb{C})$ with $\beta_i > 0\}$. Show Π is a simple base of $\Delta(\mathfrak{g}_\mathbb{C})$ with $\Delta^+(\mathfrak{g}_\mathbb{C}) = \{\alpha \in \Delta(\mathfrak{g}_\mathbb{C}) \mid \alpha > 0\}$ and $\Delta^-(\mathfrak{g}_\mathbb{C}) = \{\alpha \in \Delta(\mathfrak{g}_\mathbb{C}) \mid \alpha < 0\}$.
(2) Show that all simple systems arise in this fashion.
(3) Show that there is a unique $\delta \in \Delta^+(\mathfrak{g}_\mathbb{C})$, so that $\delta > \beta$, $\beta \in \Delta^+(\mathfrak{g}_\mathbb{C})\backslash\{\delta\}$. The root δ is called the *highest root*. For the classical compact Lie groups, show δ is given by the following table:

G	$SU(n)$	$Sp(n)$	$SO(E_{2n})$	$SO(E_{2n+1})$
δ	$\epsilon_1 - \epsilon_n$	$2\epsilon_1$	$\epsilon_1 + \epsilon_2$	$\epsilon_1 + \epsilon_2$.

(4) Show that $B(\delta, \beta) \ge 0$ for all $\beta \in \Delta^+(\mathfrak{g}_\mathbb{C})$.
(5) For $G = SO(E_{2n+1})$, $n \ge 2$, show that there is another root besides δ satisfying the condition in part (4).

Exercise 6.33 Let G be compact Lie group with semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. Fix a simple system $\Pi = \{\alpha_i\}$ of $\Delta(\mathfrak{g}_\mathbb{C})$. For any $\beta \in \Delta^+(\mathfrak{g}_\mathbb{C})$, show that β can be written as $\beta = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_N}$, where $\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_k} \in \Delta^+(\mathfrak{g}_\mathbb{C})$ for $1 \le k \le N$.

Exercise 6.34 Let G be compact Lie group with a Cartan subalgebra $\mathfrak{t}$. Fix a simple system Π of $\Delta(\mathfrak{g}_\mathbb{C})$.
(1) For $\alpha \in \Pi$ and $\beta \in \Delta^+(\mathfrak{g}_\mathbb{C})\backslash\{\alpha\}$, write $r_\alpha\beta = \beta - 2\frac{B(\beta,\alpha)}{B(\alpha,\alpha)}\alpha$ to show that $\beta \in \Delta^+(\mathfrak{g}_\mathbb{C})$. Conclude that $r_\alpha(\Delta^+(\mathfrak{g}_\mathbb{C})\backslash\{\alpha\}) = \Delta^+(\mathfrak{g}_\mathbb{C})\backslash\{\alpha\}$.
(2) Let

$$\rho' = \frac{1}{2} \sum_{\beta\in\Delta^+(\mathfrak{g}_\mathbb{C})} \beta$$

and conclude from part (1) that $r_\alpha\rho' = \rho' - \alpha$. Use the definition of r_α to show that $\rho' = \rho$.

Exercise 6.35 Let G be compact Lie group with semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. Fix a simple system $\Pi = \{\alpha_i\}$ of $\Delta(\mathfrak{g}_\mathbb{C})$ and let $W(\Delta(\mathfrak{g}_\mathbb{C}))'$

be the subgroup of $W(\Delta(\mathfrak{g}_\mathbb{C}))$ generated $\{r_{\alpha_i}\}$.
(1) Given any $\beta \in \Delta(\mathfrak{g}_\mathbb{C})$, choose $x \in (\pm\beta)^\perp$ not lying on any other root hyperplane. For all sufficiently small $\varepsilon > 0$, show that $x + \varepsilon\beta$ lies in a Weyl chamber C' and that $\beta \in \Pi' = \Pi(C')$.
(2) Write ρ_Π for the element of $(i\mathfrak{t})^*$ satisfying $2\frac{B(\rho_\Pi,\alpha_i)}{B(\alpha_i,\alpha_i)} = 1$ from Equation 6.39 (c.f. Exercise 6.34) and choose $w \in W(\Delta(\mathfrak{g}_\mathbb{C}))'$ so that $B(w\rho_{\Pi'}, \rho_\Pi)$ is maximal. By examining $B(r_{\alpha_i} w\rho_{\Pi'}, \rho_\Pi)$, show that $w\rho_{\Pi'} \in C(\Pi)$. Conclude that $w\beta \in \Pi$.
(3) Show that $r_\beta = w^{-1} r_{w\beta} w$. Conclude that $W(\Delta(\mathfrak{g}_\mathbb{C}))' = W(\Delta(\mathfrak{g}_\mathbb{C}))$.

Exercise 6.36 Let G be compact Lie group with semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. Fix a simple system $\Pi = \{\alpha_i\}$ of $\Delta(\mathfrak{g}_\mathbb{C})$ and recall Exercise 6.35. For $w \in W(\Delta(\mathfrak{g}_\mathbb{C}))$, let $n(w) = \left|\{\beta \in \Delta^+(\mathfrak{g}_\mathbb{C}) \mid w\beta \in \Delta^-(\mathfrak{g}_\mathbb{C})\}\right|$. For $w \neq I$, write $w = r_{\alpha_1} \cdots r_{\alpha_N}$ with N as small as possible. Then $r_{\alpha_1} \cdots r_{\alpha_N}$ is called a *reduced* expression for w. The *length* of w, with respect to Π, is defined by $l(w) = N$. For $w = I$, $l(I) = 0$.
(1) Use Exercise 6.34 to show

$$n(wr_{\alpha_i}) = \begin{cases} n(w) - 1 & \text{if } w\alpha_i \in \Delta^-(\mathfrak{g}_\mathbb{C}) \\ n(w) + 1 & \text{if } w\alpha_i \in \Delta^+(\mathfrak{g}_\mathbb{C}). \end{cases}$$

Conclude that $n(w) \leq l(w)$.
(2) Use Theorem 6.43 and induction on the length to show that $n(w) = l(w)$.

Exercise 6.37 (Chevalley's Lemma) Let G be compact Lie group with semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. Fix $\lambda \in (i\mathfrak{t})^*$ and let $W_\lambda = \{w \in W(\Delta(\mathfrak{g}_\mathbb{C})) \mid w\lambda = \lambda\}$. Choose a Weyl chamber C, so that $\lambda \in \overline{C}$ and let $\Pi = \Pi(C)$.
(1) If $\beta \in \Delta(\mathfrak{g}_\mathbb{C})$ with $B(\lambda, \beta) > 0$, show that $\beta \in \Delta^+(\mathfrak{g}_\mathbb{C})$.
(2) If $\alpha \in \Pi$ and $w \in W_\lambda$ with $w\alpha \in \Delta^-(\mathfrak{g}_\mathbb{C})$, show $B(\lambda, \alpha) \leq 0$.
(3) *Chevalley's Lemma* states W_λ is generated by $W(\lambda) = \{r_\alpha \mid B(\lambda, \alpha) = 0\}$. Use Exercise 6.36 to prove this result. To this end, argue by contradiction and let $w \in W_\lambda \backslash \langle W(\lambda) \rangle$ be of minimal length.
(4) Show that the only reflections in $W(\Delta(\mathfrak{g}_\mathbb{C}))$ are of the form r_α for $\alpha \in W\Delta(\mathfrak{g}_\mathbb{C})$.
(5) If $W_\lambda \neq \{I\}$, show that there exists $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$ so $\lambda \in \alpha^\perp$.

Exercise 6.38 Let G be compact Lie group with semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. Fix a simple system $\Pi = \{\alpha_i\}$ of $\Delta(\mathfrak{g}_\mathbb{C})$. For $w \in W(\Delta(\mathfrak{g}_\mathbb{C}))$, let $\operatorname{sgn}(w) = (-1)^{l(w)}$ (c.f. Exercise 6.36). Show that $\operatorname{sgn}(w) = \det(w)$.

Exercise 6.39 Let G be compact Lie group with semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. Fix a Weyl chamber C and $H \in (i\mathfrak{t})^*$.
(1) Suppose $H \in \overline{C} \cap w\overline{C}$ for $w \in W(\Delta(\mathfrak{g}_\mathbb{C}))$. Show that $wH = H$.
(2) Let $H \in (i\mathfrak{t})^*$ be arbitrary. Show that $\overline{C}$ is a *fundamental chamber* for the action of $W(\Delta(\mathfrak{g}_\mathbb{C}))$, i.e., show that the Weyl group orbit of H intersects $\overline{C}$ in exactly one point.

Exercise 6.40 Let G be compact Lie group with semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. Fix a simple system Π of $\Delta(\mathfrak{g}_\mathbb{C})$.

(1) Show that there is a unique $w_0 \in W(\Delta(\mathfrak{g}_{\mathbb{C}}))$, so that $w_0\Pi = -\Pi$.
(2) Show that $w_0 = -I \in W(\Delta(\mathfrak{g}_{\mathbb{C}}))$ for G equal to $SU(2)$, $SO(E_{2n+1})$, $Sp(n)$, and $SO(E_{4n})$.
(3) Show that $w_0 \neq -I$, so $-I \notin W(\Delta(\mathfrak{g}_{\mathbb{C}}))$ for G equal to $SU(n)$ ($n \geq 3$) and $SO(E_{4n+2})$.

Exercise 6.41 Let G be compact Lie group with simple Lie algebra $\mathfrak{g}$ and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. Fix a simple system $\Pi = \{\alpha_i\}$ of $\Delta(\mathfrak{g}_{\mathbb{C}})$.
(1) If α_i and α_j are joined by a single edge in the Dynkin diagram, show that there exists $w \in W(\Delta(\mathfrak{g}_{\mathbb{C}}))$, so that $\omega\alpha_i = \alpha_j$.
(2) If G is a classical compact Lie group, i.e., G is $SU(n)$, $Sp(n)$, $SO(E_{2n})$, or $SO(E_{2n+1})$, show the set of roots of a fixed length constitutes a single Weyl group orbit.

Exercise 6.42 Let G be compact Lie group with semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. Fix a simple system $\Pi = \{\alpha_i\}$ of $\Delta(\mathfrak{g}_{\mathbb{C}})$ and let $\pi_i \in (i\mathfrak{t})^*$ be defined by the relation $2\frac{B(\pi_i,\alpha_j)}{B(\alpha_j,\alpha_j)} = \delta_{i,j}$.
(1) Show that $\{\alpha_i\}$ is a $\mathbb{Z}$-basis for the root lattice R and $\{\pi_i\}$ is a $\mathbb{Z}$-basis for the weight lattice P.
(2) Show the matrix $\left(B(\alpha_i, \alpha_j)\right)$ is positive definite. Conclude $\det\left(2\frac{B(\alpha_i,\alpha_j)}{B(\alpha_j,\alpha_j)}\right) > 0$.
(3) It is well known from the study of free Abelian groups ([3]) that there exists a $\mathbb{Z}$-basis $\{\lambda_i\}$ of P and $k_i \in \mathbb{Z}$, so that $\{k_i\lambda_i\}$ is a basis for R. Thus there is a change of basis matrix from the basis $\{\lambda_i\}$ to $\{\pi_i\}$ with integral entries and determinant ± 1. Show that $|P/R| = \det\left(2\frac{B(\alpha_i,\alpha_j)}{B(\alpha_j,\alpha_j)}\right)$. The matrix $\left(2\frac{B(\alpha_i,\alpha_j)}{B(\alpha_j,\alpha_j)}\right)$ is called the *Cartan matrix* of $\mathfrak{g}_{\mathbb{C}}$.

Exercise 6.43 Let G be a compact Lie group with semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. Fix a simple system $\Pi = \{\alpha_i\}$ of $\Delta(\mathfrak{g}_{\mathbb{C}})$. For each $\beta \in \Delta(\mathfrak{g}_{\mathbb{C}})$, choose a standard $\mathfrak{sl}(2,\mathbb{C})$-triple associated to β, $\{E_\beta, H_\beta, F_\beta\}$. Let $h = \sum_{\beta\in\Delta^+(\mathfrak{g}_{\mathbb{C}})} H_\beta$ and define $k_{\alpha_i} \in \mathbb{Z}_{>0}$, so $h = \sum_{\alpha_i\in\Pi} k_{\alpha_i} H_{\alpha_i}$. Set $e = \sum_i \sqrt{k_{\alpha_i}} E_{\alpha_i}$, $f = \sum_i k_{\alpha_i} F_{\alpha_i}$, and $\mathfrak{s} = \mathrm{span}_{\mathbb{C}}\{e, h, f\}$.
(1) Show $\frac{B(h,h_{\alpha_i})}{B(h_{\alpha_i},h_{\alpha_i})} = 1$ (c.f. Exercise 6.34).
(2) Show that $\mathfrak{s} \cong \mathfrak{sl}(2,\mathbb{C})$. The subalgebra $\mathfrak{s}$ is called the *principal three-dimensional subalgebra* of $\mathfrak{g}_{\mathbb{C}}$.

Exercise 6.44 Let G be a compact Lie group with semisimple Lie algebra $\mathfrak{g}$ and let T be a maximal torus of G. Fix a Weyl chamber C of $i\mathfrak{t}$ and let $N_G(C) = \{g \in G \mid \mathrm{Ad}(g)C = C\}$. Show that the inclusion map of $N_G(C) \to G$ induces an isomorphism $N_G(C)/T \cong G^0/G$.