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Lie Groups

An Approach through Invariants and Representations

Springer

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To the ladies of my life,
Liliana, Silvana, Michela, Martina

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Introduction

The subject of Lie groups, introduced by Sophus Lie in the second half of the nineteenth century, has been one of the important mathematical themes of the last century. Lie groups formalize the concept of continuous symmetry, and thus are a part of the foundations of mathematics. They also have several applications in physics, notably quantum mechanics and relativity theory. Finally, they link with many branches of mathematics, from analysis to number theory, passing through topology, algebraic geometry, and so on.

This book gives an introduction to at least the main ideas of the theory. Usually, there are two principal aspects to be discussed. The first is the description of the groups, their properties and classifications; the second is the study of their representations.

The problem that one faces when introducing representation theory is that the material tends to grow out of control quickly. My greatest difficulty has been to try to understand when to stop. The reason lies in the fact that one may represent almost any class of algebraic if not even mathematical objects. In fact it is clear that even the specialists only master part of the material.

There are of course many good introductory books on this topic. Most of them however favor only one aspect of the theory. I have tried instead to present the basic methods of Lie groups, Lie algebras, algebraic groups, representation theory, some combinatorics and basic functional analysis, which then can be used by the various specialists. I have tried to balance general theory with many precise concrete examples.

This book started as a set of lecture notes taken by G. Boffi for a course given at Brandeis University. These notes were published as a "Primer in Invariant Theory" [Pr]. Later, H. Kraft and I revised these notes, which have been in use and are available on Kraft's home page [KrP]. In these notes, we present classical invariant theory in modern language. Later, E. Rogora and I presented the combinatorial approach to representations of the symmetric and general linear groups [PrR]. In past years, while teaching introductory courses on representation theory, I became convinced that it would be useful to expand the material in these various expositions to give an idea of the connection with more standard classical topics, such as the theory of Young symmetrizers and Clifford algebras, and also not to restrict to classical groups but to include general semisimple groups as well.

The reader will see that I have constantly drawn inspiration from the book of H. Weyl, *Classical Groups* [W]. On the other hand it would be absurd and quite impossible to *update* this classic.

In his book Weyl stressed the relationship between representations and invariants. In the last 30 years there has been a renewed interest in classical methods of invariant theory, motivated by problems of geometry, in particular due to the ideas of Grothendieck and Mumford on moduli spaces. The reader will see that I do not treat geometric invariant theory at all. In fact I decided that this would have deeply changed the nature of the book, which tries to always remain at a relatively elementary level, at least in the use of techniques outside of algebra. Geometric invariant theory is deeply embedded in algebraic geometry and algebraic groups, and several good introductions to this topic are available.

I have tried to explain in detail all the constructions which belong to invariant theory and algebra, introducing and using only the essential notions of differential geometry, algebraic geometry, measure theory, and functional analysis which are necessary for the treatment here. In particular, I have tried to restrict the use of algebraic geometry and keep it to a minimum, nevertheless referring to standard books for some basic material on this subject which would have taken me too long to discuss in this text. While it is possible to avoid algebraic geometry completely, I feel it would be a mistake to do so since the methods that algebraic geometry introduces in the theory are very powerful. In general, my point of view is that some of the interesting special objects under consideration may be treated by more direct and elementary methods, which I have tried to do whenever possible since a direct approach often reveals some special features which may be lost in a general theory. A similar, although less serious, problem occurs in the few discussions of homotopy theory which are needed to understand simply connected groups.

I have tried to give an idea of how 19th-century algebraists thought of the subject. The main difficulty we have in understanding their methods is in the fact that the notion of representation appears only at a later stage, while we usually start with it.

The book is organized into topics, some of which can be the subject of an entire graduate course. The organization is as follows.

The first chapter establishes the language of group actions and representations with some simple examples from abstract group theory. The second chapter is a quick look into the theory of symmetric functions, which was one of the starting points of the entire theory. First, I discuss some very classical topics, such as the resultant and the Bézoutiant. Next I introduce Schur functions and the Cauchy identity. These ideas will play a role much later in the character theory of the symmetric and the linear group.

Chapter 3 presents again a very classical topic, that of the theory of algebraic forms, à la Capelli [Ca].

In Chapter 4, I change gears completely. Taking as pretext the theory of polarizations of Capelli, I systematically introduce Lie groups and Lie algebras and start to prove some of the basic structure theorems. The general theory is completed in

Chapter 5 in which universal enveloping algebras and free Lie algebras are discussed. Later, in Chapter 10 I treat semisimple algebras and groups. I complete the proof of the correspondence between Lie groups and Lie algebras via Ado's theorem. The rest of the chapter is devoted to Cartan–Weyl theory, leading to the classification of complex semisimple groups and the associated classification of connected compact groups.

Chapter 5 is quite elementary. I decided to include it since the use of tensor algebra and tensor notation plays such an important role in the treatment as to deserve some lengthy discussion. In this chapter I also discuss Clifford algebras and the spin group. This topic is completed in Chapter 11.

Chapter 6 is a short introduction to general methods of noncommutative algebra, such as Wedderburn's theorem and the double centralizer theorem. This theory is basic to the representation theory to be developed in the next chapters.

Chapter 7 is a quick introduction to algebraic groups. In this chapter I make fair use of notions from algebraic geometry, and I try to at least clarify the statements used, referring to standard books for the proofs. In fact it is impossible, without a rather long detour, to actually develop in detail the facts used. I hope that the interested reader who does not have a background in algebraic geometry can still follow the reasoning developed here.

I have tried to stress throughout the book the parallel theory of reductive algebraic and compact Lie groups. A full understanding of this connection is gained slowly, first through some classical examples, then by the Cartan decomposition and Tannaka duality in Chapter 8. This theory is completed in Chapter 10, where I associate, to a semisimple Lie algebra, its compact form. After this the final classification theorems are proved.

Chapter 8 is essentially dedicated to matrix coefficients and the Peter–Weyl theorem. Some elementary functional analysis is used here. I end the chapter with basic properties of Hopf algebras, which are used to make the link between compact and reductive groups.

Chapter 9 is dedicated to tensor symmetry, Young symmetrizers, Schur–Weyl duality and their applications to representation theory.

Chapter 10 is a short course giving the structure and classification of semisimple Lie algebras and their representations via the usual method of root systems. It also contains the corresponding theory of adjoint and simply connected algebraic groups and their compact forms.

Chapter 11 is the study of the relationship between invariants and the representation theory of classical groups. It also contains a fairly detailed discussion of spinors and terminates with the analytic proof of Weyl's character formula.

The last four chapters are complements to the theory. In Chapter 12 we discuss the combinatorial theory of tableaux to lead to Schützenberger's proof of the Littlewood–Richardson rule.

Chapter 13 treats the combinatorial approach to invariants and representations for classical groups. This is done via the theory of standard monomials, which is developed in a characteristic-free way, for some classical representations.

Chapter 14 is a very short glimpse into the geometric theory, and finally Chapter 15 is a return to the past, to where it all started: the theory of binary forms.

Many topics could not find a place in this treatment. First, I had to restrict the discussion of algebraic groups to a minimum. In particular I chose giving proofs only in characteristic 0 when the general proof is more complicated. I could not elaborate on the center of the universal enveloping algebra, Verma modules and all the ideas relating to finite and infinite-dimensional representations. Nor could I treat the conjugation action on the group and the Lie algebra which contains so many deep ideas and results. Of course I did not even begin to consider the theory of real semisimple groups. In fact, the topics which relate to this subject are so numerous that this presentation here is just an invitation to the theory. The theory is quite active and there is even a journal entirely dedicated to its developments.

Finally, I will add that this book has some overlaps with several books, as is unavoidable when treating foundational material.

I certainly followed the path already taken by others in many of the basic proofs which seem to have reached a degree of perfection and upon which it is not possible to improve.

The names of the mathematicians who have given important contributions to Lie theory are many, and I have limited to a minimum the discussion of its history. The interested reader can now consult several sources like [Bor2], [GW].

I wish finally to thank Laura Stevens for carefully reading through a preliminary version and helping me to correct several mistakes, and Alessandro D'Andrea, Figà Talamanca and Paolo Papi for useful suggestions, and Ann Kostant and Martin Stock for the very careful and complex editing of the final text.

Claudio Procesi
Università di Roma La Sapienza
July 2006

Conventional Notations

When we introduce a new symbol or definition we will use the convenient symbol $:=$ which means that the term introduced on its left is defined by the expression on its right.

A typical example could be $P := \{x \in \mathbb{N} \mid 2 \text{ divides } x\}$, which stands for *P is by definition the set of all natural numbers x such that 2 divides x*.

The symbol $\pi : A \to B$ denotes a mapping called π from the set A to the set B.

Most of our work will be for algebras over the field of real or complex numbers. Sometimes we will take a more combinatorial point of view and analyze some properties over the integers. Associative algebras will implicitly be assumed to have a unit element. When we discuss matrices over a ring A we always identify A with the scalar matrices (constant multiples of the identity matrix).

We use the standard notations:

$$\mathbb{N},\ \mathbb{Z},\ \mathbb{Q},\ \mathbb{R},\ \mathbb{C}$$

for the natural numbers (including 0), the integers, rational, real and complex numbers.