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Standard Monomials

While in the previous chapter standard tableaux had a purely combinatorial meaning, in the present chapter they will acquire a more algebro-geometric interpretation. This allows one to develop some invariant theory and representation theory in a characteristic-free way. The theory is at the same time a special case and a generalization of the results of Chapter 10, §6. In fact there are full generalizations of this theory to all semisimple groups in all characteristics, which we do not discuss (cf. [L-S], [Lit], [Lit2]).

1 Standard Monomials

1.1 Standard Monomials

We start with a somewhat axiomatic approach. Suppose that we are given: a commutative algebra R over a ring A, and a set $S := \{s_1, \dots, s_N\}$ of elements of R together with a partial ordering of S.

Definition.

(1) An ordered product $s_{i_1} s_{i_2} \dots s_{i_k}$ of elements of S is said to be standard (or to be a *standard monomial*) if the elements appear in nondecreasing order (with respect to the given partial ordering).

(2) We say that R has a standard monomial theory for S if the standard monomials form a basis of R over A.

Suppose that R has a standard monomial theory for S; given $s, t \in S$ which are not comparable in the given partial order. By axiom (2) we have a unique expression, called a *straightening law*:

$$st = \sum_i \alpha_i M_i, \quad \alpha_i \in A, \quad M_i \text{ standard.} \tag{1.1.1}$$

We devise now a possible algorithm to replace any monomial $s_{i_1}s_{i_2}\dots s_{i_k}$ with a linear combination of standard ones. If, in the monomial, we find a product st with $s > t$, we replace st with ts. If instead s, t are not comparable we replace st with the right-hand side of 1.1.1.

(3) We say that R *has a straightening algorithm* if the previous replacement algorithm always stops after finitely many steps (giving the expression of the given product in terms of standard monomials).

Our prime example will be the following:

We let $A = \mathbb{Z}$, $R := \mathbb{Z}[x_{ij}]$, $i = 1, \dots, n$; $j = 1 \dots m$, the polynomial ring in nm variables, and S be the set of determinants of all square minors of the $m \times n$ matrix with entries the x_{ij}.

Combinatorially it is useful to describe a determinant of a $k \times k$ minor as two sequences

$$(i_k\, i_{k-1} \dots i_1 | j_1\, j_2 \dots j_k), \quad \text{determinant of a minor} \tag{1.1.2}$$

where the i_t are the indices of the rows and the j_s the indices of the columns. It is customary to write the i in decreasing order and the j in increasing order.

In this notation a variable $x_{i,j}$ is denoted by $(i|j)$, e.g.,

$$(2|3) = x_{23}, \quad (2\,1|1\,3) := x_{11}x_{23} - x_{21}x_{13}.$$

The partial ordering will be defined as follows:

$$(i_h\, i_{h-1} \dots i_1 | j_1\, j_2 \dots j_h) \le (u_k\, u_{k-1} \dots u_1 | v_1\, v_2 \dots v_k) \text{ iff } h \le k,$$

$$i_s \ge u_s;\ j_t \ge v_t,\ \forall s, t \le h.$$

In other words, if we display the two determinants as rows of a bi-tableau, the left- and right-hand parts of the bi-tableau are semistandard tableaux. It is customary to call such a bi-tableau *standard*.[125]

$$\begin{array}{c} u_k \dots u_h\ u_{h-1} \dots u_1 | v_1\, v_2 \dots v_h \dots v_k \\ i_h\ i_{h-1} \dots\ i_1\ | j_1\, j_2 \dots\ j_h. \end{array}$$

Let us give the full partially ordered set of the 9 minors of a 2×3 matrix; see the figure on p. 501

In the next sections we will show that $\mathbb{Z}[x_{ij}]$ has a standard monomial theory with respect to this partially ordered set of minors and we will give explicitly the straightening algorithm.

[125] Now we are using the English notation. The left tableau is in fact a mirror of a semistandard tableau.

$$
\begin{tikzcd}
 & & (2\,1|1\,2) \arrow[d] & \\
 & & (2\,1|1\,3) \arrow[dl] \arrow[dr] & \\
 & (1|1) \arrow[dl] \arrow[dr] & & (2\,1|2\,3) \arrow[dl] \\
(2|1) \arrow[dr] & & (1|2) \arrow[dl] \arrow[dr] & \\
 & (2|2) \arrow[dr] & & (1|3) \arrow[dl] \\
 & & (2|3) &
\end{tikzcd}
$$

2 Plücker Coordinates

2.1 Combinatorial Approach

We start with a very simple combinatorial approach to which we will soon give a deeper geometrical meaning.

Denote by $M_{n,m}$ the space of $n \times m$ matrices. Assume $n \leq m$. We denote by $x_1, x_2, \ldots, x_m$ the columns of a matrix in $M_{n,m}$. Let $A := \mathbb{Z}[x_{ij}]$ be the ring of polynomial functions on $M_{n,m}$ with integer coefficients. We may wish to consider an element in A as a function of the columns and then we will write it as $f(x_1, x_2, \ldots, x_m)$. Consider the *generic matrix* $X := (x_{ij}),\ i = 1, \ldots, n;\ j = 1, \ldots, m$ of indeterminates. We use the following notation: Given n integers $i_1, i_2, \ldots, i_n$ chosen from the numbers $1, 2, \ldots, m$ we use the symbol:

(2.1.1) $\qquad [i_1, i_2, \ldots, i_n] \qquad$ Plücker coordinate

to denote the determinant of the maximal minor of X which has as columns the columns of indices $i_1, i_2, \ldots, i_n$. We call such a polynomial a *Plücker coordinate.*

The first properties of these symbols are:

S1) $[i_1, i_2, \ldots, i_n] = 0$ if and only if 2 indices coincide.
S2) $[i_1, i_2, \ldots, i_n]$ is *antisymmetric* (under permutation of the indices).
S3) $[i_1, i_2, \ldots, i_n]$ is multilinear as a function of the vector variables.

We are now going to show that the Plücker coordinates satisfy some basic quadratic equations. Assume $m \geq 2n$ and consider the product:

(2.1.2) $\qquad f(x_1, x_2, \ldots, x_{2n}) := [1, 2, \ldots, n][n+1, n+2, \ldots, 2n].$

Select now an index $k \leq n$ and the $n+1$ variables $x_k, x_{k+1}, \dots, x_n, x_{n+1}, x_{n+2}, \dots, x_{n+k}$.

Next alternate the function f in these variables:[126]

$$\sum_{\sigma \in S_{n+1}} \epsilon_\sigma f(x_1, \dots, x_{k-1}, x_{\sigma(k)}, x_{\sigma(k+1)}, \dots, x_{\sigma(n)},$$
$$x_{\sigma(n+1)}, \dots, x_{\sigma(n+k)}, x_{n+k+1}, \dots, x_{2n}).$$

The result is a multilinear and alternating expression in the $n+1$ vector variables

$$x_k, x_{k+1}, \dots, x_n, x_{n+1}, x_{n+2}, \dots, x_{n+k}.$$

This is necessarily 0 since the vector variables are n-dimensional.

We have thus already found a *quadratic relation* among Plücker coordinates. We need to simplify it and expand it.

The symmetric group S_{2n} acts on the space of $2n$-tuples of vectors x_i by permuting the indices. Then we have an induced action on functions by

$$(\sigma g)(x_1, x_2, \dots, x_{2n}) := g(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(2n)}).$$

The function $[1, 2, \dots, n][n+1, n+2, \dots, 2n]$ is alternating with respect to the subgroup $S_n \times S_n$ acting separately on the first n and last n indices.

Given $k \leq n$, consider the symmetric group S_{n+1} (subgroup of S_{2n}), permuting only the indices $k, k+1, \dots, n+k$. With respect to the action of this subgroup, the function $[1, 2, \dots, n][n+1, n+2, \dots, 2n]$ is alternating with respect to the subgroup $S_{n-k+1} \times S_k$ of the permutations which permute separately the variables $k, k+1, \dots, n$ and $n+1, n+2, \dots, n+k$.

Thus if $g \in S_{n+1}, h \in S_{n-k+1} \times S_k$, we have

$$ghf(x_1, x_2, \dots, x_{2n}) = \epsilon_h g f(x_1, x_2, \dots, x_{2n}).$$

We deduce that, if $g_1, g_2, \dots, g_N$ are representatives of the left cosets $g(S_{n-k+1} \times S_k)$,

$$0 = \sum_{i=1}^{N} \epsilon_{g_i} g_i f(x_1, x_2, \dots, x_{2n}). \tag{2.1.3}$$

As representatives of the cosets we may choose some canonical elements. Remark that two elements $a, b \in S_{n+1}$ are in the same left coset with respect to $S_{n-k+1} \times S_k$ if and only if they transform the numbers $k, k+1, \dots, n$ and $n+1, n+2, \dots, n+k$ into the same sets of elements. Therefore we can choose as representatives for right cosets the following permutations:

(i) Choose a number h and select h elements out of $k, k+1, \dots, n$ and another h out of $n+1, n+2, \dots, n+k$. Then exchange in order the first set of h elements with the second. Call this permutation an exchange. Its sign is $(-1)^h$.

[126] In this chapter ϵ_σ denotes the sign of a permutation.

(ii) A better choice may be the permutation obtained by composing such an exchange with a reordering of the indices in each Plücker coordinate. This is an *inverse shuffle* (inverse of the operation performed on a deck of cards by a single shuffle).

The inverse of a shuffle is a permutation σ such that

$$\sigma(k) < \sigma(k+1) < \cdots < \sigma(n) \quad \text{and} \quad \sigma(n+1) < \sigma(n+2) < \cdots < \sigma(n+k).$$

Thus the basic relation is: the sum (with signs) of all exchanges, or inverse shuffles, in the polynomial $f(x_1, x_2, \dots, x_{2n})$, of the variables $x_k, x_{k+1}, \dots, x_n$, with the variables $x_{n+1}, x_{n+2}, \dots, x_{n+k}$ equal to 0.

The simplest example is the *Klein quadric*, where $n = 2, m = 4$. It is the equation of the set of lines in 3-dimensional projective space. We start to use the combinatorial display of a product of Plücker coordinates as a tableau. In this case we write

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} := [a, b][c, d], \qquad \text{product of two Plücker coordinates.}$$

$$0 = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$$

which expresses the fact that the variety of lines in $\mathbb{P}^3$ is a quadric in $\mathbb{P}^5$.

We can now choose any indices $i_1, i_2, \dots, i_n; j_1, j_2, \dots, j_n$ and substitute in the basic relation 2.1.3 for the vector variables x_h, $h = 1, \dots, n$, the variable x_{i_h} and for $x_{n+h}, h = 1, \dots, n$ the variables x_{j_h}. The resulting relation will be denoted symbolically by

$$\sum \epsilon \begin{vmatrix} i_1, i_2, \dots, \underline{i_k, \dots, i_n} \\ \underline{j_1, j_2, \dots, j_k}, \dots, j_n \end{vmatrix} \cong 0 \tag{2.1.4}$$

where the symbol should remind us that we should sum over all exchanges of the underlined indices with the sign of the exchange, and the two-line tableau represents the product of the two corresponding Plücker coordinates.

We want to work in a formal way and consider the polynomial ring in the symbols $|i_1, i_2, \dots, i_n|$ as independent variables only subject to the symmetry conditions S1, S2. The expressions 2.1.4 are to be thought of as quadratic polynomials in this polynomial ring.

When we substitute for the symbol $|i_1, i_2, \dots, i_n|$ the corresponding Plücker coordinate $[i_1, i_2, \dots, i_n]$, the quadratic polynomials 2.1.4 vanish, i.e., they are *quadratic equations.*

Remark. If some of the indices i coincide with indices j, it is possible that several terms of the quadratic relation vanish or cancel each other.

Let us thus define a ring A as the polynomial ring $\mathbb{Z}[\,|i_1, i_2, \dots, i_n|\,]$ modulo the ideal J generated by the quadratic polynomials 2.1.4. The previous discussion shows that we have a homomorphism:

$$j : A = \mathbb{Z}[\,|i_1, i_2, \dots, i_n|\,]/J \to \mathbb{Z}[\,[i_1, i_2, \dots, i_n]\,]. \tag{2.1.5}$$

One of our goals is to prove:

Theorem. *The map j is an isomorphism.*

2.2 Straightening Algorithm

Before we can prove Theorem 2.1, we need to draw a first consequence of the quadratic relations. For the moment when we speak of a Plücker coordinate $[i_1, i_2, \dots, i_n]$ we will mean only the class of $|i_1, i_2, \dots, i_n|$ in A. Of course with Theorem 2.1 this use will be consistent with our previous one.

Consider a product of m Plücker coordinates

$$[i_{11}, i_{12}, \dots, i_{1k}, \dots, i_{1n}][i_{21}, i_{22}, \dots, i_{2k}, \dots, i_{2n}] \dots$$
$$\times [i_{m1}, i_{m2}, \dots, i_{mk}, \dots, i_{mn}]$$

and display it as an m-row tableau:

$$(2.2.1) \qquad \begin{vmatrix} i_{11}\ i_{12}\ \dots\ i_{1k}\ \dots\ i_{1n} \\ i_{21}\ i_{22}\ \dots\ i_{2k}\ \dots\ i_{2n} \\ \dots \\ \dots \\ i_{m1}\ i_{m2}\ \dots\ i_{mk}\ \dots\ i_{mn} \end{vmatrix}$$

Due to the antisymmetry properties of the coordinates let us assume that the indices in each row are strictly increasing; otherwise the product is either 0, or up to sign, equals the one in which each row has been reordered.

Definition. We say that a rectangular tableau is *standard* if its rows are strictly increasing and its columns are non-decreasing (i.e., $i_{hk} < i_{h\,k+1}$ and $i_{hk} \leq i_{h+1\,k}$). The corresponding monomial is then called a *standard monomial.*

It is convenient, for what follows, to associate to a tableau the word obtained by sequentially reading the numbers on each row:

$$(2.2.2) \qquad i_{11}\, i_{12} \dots i_{1k} \dots i_{1n}, i_{21}\, i_{22} \dots i_{2k} \dots\ i_{2n} \dots\dots i_{m1}\, i_{m2} \dots i_{mk}\ \dots\ i_{mn}$$

and order these words lexicographically. It is then clear that if the rows of a tableaux T are not strictly increasing, the tableaux T' obtained from T by reordering the rows in an increasing way is strictly smaller than T in the lexicographic order.

The main algorithm is given by:

Lemma. *A product T of two Plücker coordinates*

$$T := \begin{vmatrix} i_1, i_2, \dots, i_k, \dots, i_n \\ j_1, j_2, \dots, j_k, \dots, j_n \end{vmatrix}$$

can be expressed through the quadratic relations 2.1.4 as a linear combination with integer coefficients of standard tableaux with 2 rows, preceding T in the lexicographic order and filled with the same indices $i_1, i_2, \dots, i_k, \dots, i_n, j_1, j_2, \dots, j_k, \dots, j_n$.

Proof. We may assume first that the 2 rows are strictly increasing. Next, if the tableau is not standard, there is a position k for which $i_k > j_k$, and hence

$$j_1 < j_2 < \cdots < j_k < i_k < \cdots < i_n.$$

We call such a position a *violation* of the standard form. We then apply the corresponding quadratic equation. In this equation, every inverse shuffle different from the identity, replaces some of the indices $i_k < \cdots < i_n$ with indices from $j_1 < j_2 < \cdots < j_k$. It produces thus a tableau which is strictly lower lexicographically than T. Thus, if T is not standard it can be expressed, via the relations 2.1.4, as a linear combination of lexicographically smaller tableaux. We say that we have applied a step of a *straightening algorithm.*

Take the resulting expression, if it is a linear combination of standard tableaux we stop. Otherwise we repeat the algorithm to all the non-standard tableaux which appear. Each non-standard tableau is replaced with a linear combination of strictly smaller tableaux. Since the two-line tableaux filled with the indices $i_1, i_2, \ldots, i_k, \ldots, i_n, j_1, j_2, \ldots, j_k, \ldots, j_n$ are a finite number, totally ordered lexicographically, the straightening algorithm must terminate after a finite number of steps, giving an expression with only standard two-row tableaux. □

We can now pass to the general case:

Theorem. *Any rectangular tableau with m rows is a linear combination with integer coefficients of standard tableaux. The standard form can be obtained by a repeated application of the straightening algorithm to pairs of consecutive rows.*

Proof. The proof is essentially obvious. We first reorder each row, and then inspect the tableau for a possible violation in two consecutive rows. If there is no violation, the tableau is standard. Otherwise we replace the two given rows with a sum of two-row tableaux which are strictly lower than these two rows, and then we repeat the algorithm. The same reasoning of the lemma shows that the algorithm stops after a finite number of steps. □

2.3 Remarks

Some remarks on the previous algorithm are in order. First, we can express the same ideas in the language of §1.1. On the set S of $\binom{m}{n}$ symbols $|i_1\, i_2\, \ldots\, i_n|$ where $1 \leq i_1 < i_2 < \cdots < i_n \leq m$ we consider the partial ordering (the *Bruhat order*) given by

(2.3.1) $$|i_1\, i_2\, \ldots\, i_n| \leq |j_1\, j_2\, \ldots\, j_n| \text{ if and only if } i_k \leq j_k,\ \forall k = 1, \ldots, n.$$

Observe that $|i_1\, i_2\, \ldots\, i_n| \leq |j_1\, j_2\, \ldots\, j_n|$ if and only if the tableau:

$$\begin{vmatrix} i_1\, i_2\, \ldots\, i_n \\ j_1\, j_2\, \ldots\, j_n \end{vmatrix}$$

is standard. In this language, a standard monomial is a product

$$[i_{11}, i_{12}, \dots, i_{1k}, \dots, i_{1n}][i_{21}, i_{22}, \dots, i_{2k}, \dots, i_{2n}] \cdots [i_{m1}, i_{m2}, \dots, i_{mk}, \dots, i_{mn}]$$

in which the coordinates appearing are increasing from left to right in the order 2.3.1. This means that the associated tableau of 2.2 is standard.

If $a = |i_1\ i_2\ \dots\ i_n|$, $b = |j_1\ j_2\ \dots\ j_n|$ and the product ab is not standard, then we can apply a quadratic equation and obtain $ab = \sum_i \lambda_i a_i b_i$ with λ_i coefficients and a_i, b_i obtained from a, b by the shuffle procedure of Lemma 2.2. The proof we have given shows that this is indeed a straightening algorithm in the sense of 1.1. The proof of that lemma shows in fact that $a < a_i$, $b > b_i$. It is useful to axiomatize the setting.

Definition. Suppose we have a commutative algebra R over a commutative ring A, a finite set $S \subset R$, and a partial ordering in S for which R has a standard monomial theory and a straightening algorithm.

We say that R is a *quadratic Hodge algebra* over S if wherever $a, b \in S$ are not comparable,

$$ab = \sum_i \lambda_i a_i b_i \tag{2.3.2}$$

with $\lambda_i \in A$ and $a < a_i$, $b > b_i$.

Notice that the quadratic relations 2.3.2 give the straightening law for R. The fact that the straightening algorithm terminates after a finite number of steps is clear from the condition $a < a_i$, $b > b_i$.

Our main goal is a theorem which includes Theorem 2.1:

Theorem. *The standard tableaux form a $\mathbb{Z}$-basis of A and A is a quadratic Hodge algebra isomorphic under the map j (cf. 2.1.5), to the ring $\mathbb{Z}[[i_1, i_2, \dots, i_n]] \subset \mathbb{Z}[x_{ij}]$.*

Proof. Since the standard monomials span A linearly and since by construction j is clearly surjective, it suffices to show that the standard monomials are linearly independent in the ring $\mathbb{Z}[[i_1, i_2, \dots, i_n]]$. This point can be achieved in several different ways, we will follow first a combinatorial and then, in §3.6, a more complete geometric approach through Schubert cells.

The algebraic combinatorial proof starts as follows:

Remark that, in a standard tableau, each index i can appear only in the first i columns.

Let us define a tableau to be k-canonical if, for each $i \leq k$, the indices i which appear are all in the i^{th} column. Of course a tableau (with n columns and h rows) is n-canonical if and only if the i^{th} column is filled with i for each i, i.e., it is of type $|1\ 2\ 3\ \dots\ n-1\ n|^h$.

Suppose we are given a standard tableau T which is k-canonical. Let $p = p(T)$ be the minimum index (greater than k) which appears in T in a column $j < p$. Set

$m_p(T)$ to be the minimum of such column indices. The entries to the left of p, in the corresponding row, are then the indices $1\ 2\ 3\ \ldots\ j-1$.

Given an index j, let us consider the set $\mathcal{T}^k_{p,j,h}$ of k-canonical standard tableaux for which p is the minimum index (greater than k) which appears in T in a column $j < p$. $m_p(T) = j$ and in the j^{th} column p occurs exactly h times (necessarily in h consecutive rows). In other words, reading the j^{th} column from top to bottom, one finds first a sequence of j's and then h occurrences of p. What comes after is not relevant for the discussion.

The main combinatorial remark we make is that if we substitute p with j in all these positions, we see that we have a map which to distinct tableaux associates distinct k-canonical tableaux T', with either $p(T') > p(T)$ or $p(T') = p(T)$ and $m_p(T') > m_p(T)$.

To prove the injectivity of this map it is enough to observe that if a tableau T is transformed into a tableau T', then the tableau T is obtained from T' by substituting p for the last h occurrences of j (which are in the j^{th} column).

The next remark is that if we substitute the variable x_i with $x_i + \lambda x_j$, $(i \neq j)$ in a Plücker coordinate $[i_1, i_2, \ldots, i_n]$, then the result of the substitution is $[i_1, i_2, \ldots, i_n]$ if i does not appear among the indices $i_1, i_2, \ldots, i_n$ or if both indices i, j appear.

If instead $i = i_k$, the result of the substitution is

$$[i_1, i_2, \ldots, i_n] + \lambda[i_1, i_2, \ldots, i_{k-1}, j, i_{k+1}, \ldots, i_n].$$

Suppose we make the same substitution in a tableau, i.e., in a product of Plücker coordinates; then by expanding the product of the transformed coordinates we obtain a polynomial in λ of degree equal to the number of entries i which appear in rows of T where j does not appear. The leading coefficient of this polynomial is the tableau obtained from T by substituting j for all the entries i which appear in rows of T where j does not appear.

After these preliminary remarks we can give a proof of the linear independence of the standard monomials in the Plücker coordinates.

Let us assume by contradiction that

$$0 = f(x_1, \ldots, x_m) = \sum_i c_i T_i \tag{2.3.3}$$

is a dependence relation among (distinct) standard tableaux. We may assume it is homogeneous of some degree k.

At least one of the T_i must be different from a power $|1\ 2\ 3\ \ldots\ n-1\ n|^h$, since such a relation is not valid.

Then let p be the minimum index which appears in one of the T_i in a column $j < p$, and let j be the minimum of these column indices. Also let h be the maximum number of such occurrences of p and assume that the tableaux T_i, $i \leq k$ are the ones for which this happens. This implies that if in the relation 2.3.3 we substitute x_p with $x_p + \lambda x_j$, where λ is a parameter, we get a new relation which can be written as a polynomial in λ of degree h. Since this is identically 0, each coefficient must be zero. Its leading coefficient is

$$\sum_{i=1}^{k} c_i T_i' \tag{2.3.4}$$

where T_i' is obtained from T_i replacing the h indices p appearing in the j column with j.

According to our previous combinatorial remark the tableaux T_i' are distinct and thus 2.3.4 is a new relation. We are thus in an inductive procedure which terminates with a relation of type

$$0 = |1\ 2\ 3\ \dots\ n-1\ n|^k,$$

which is a contradiction. $\square$

3 The Grassmann Variety and Its Schubert Cells

In this section we discuss in a very concrete way what we have already done quickly but in general in Chapter 10 on parabolic subgroups. The reader should compare the two.

3.1 Grassmann Varieties

The theory of Schubert cells has several interesting features. We start now with an elementary treatment. Let us start with an m-dimensional vector space V over a field F and consider $\bigwedge^n V$ for some $n \leq m$.

Proposition.

(1) *Given n vectors $v_1, v_2, \dots, v_n \in V$, the decomposable vector*

$$v_1 \wedge v_2 \wedge \cdots \wedge v_n \neq 0$$

if and only if the vectors are linearly independent.

(2) *Given n linearly independent vectors $v_1, v_2, \dots, v_n \in V$ and a vector v:*

$$v \wedge v_1 \wedge v_2 \wedge \cdots \wedge v_n = 0$$

if and only if v lies in the subspace spanned by the vectors v_i.

(3) *If $v_1, v_2, \dots, v_n$ and $w_1, w_2, \dots, w_n$ are both linearly independent sets of vectors, then*

$$w_1 \wedge w_2 \wedge \cdots \wedge w_n = \alpha v_1 \wedge v_2 \wedge \cdots \wedge v_n, \quad 0 \neq \alpha \in F$$

if and only if the two sets span the same n-dimensional subspace W of V.

Proof. Clearly (2) is a consequence of (1). As for (1), if the $v_i's$ are linearly independent they may be completed to a basis, and then the statement follows from the fact that $v_1 \wedge v_2 \wedge \cdots \wedge v_n$ is one of the basis elements of $\bigwedge^n V$.

If, conversely, one of the v_i is a linear combination of the others, we replace this expression in the product and have a sum of products with a repeated vector, which is then 0.

For (3), assume first that both sets span the same subspace. By hypothesis $w_i = \sum_j c_{ij} v_j$ with $C = (c_{ij})$ an invertible matrix, hence

$$w_1 \wedge w_2 \wedge \cdots \wedge w_n = \det(C) v_1 \wedge v_2 \wedge \cdots \wedge v_n.$$

Conversely by (2) we see that

$$W := \{v \in V | v \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_n = 0\}. \qquad \square$$

We have an immediate geometric corollary. Given an n-dimensional subspace $W \subset V$ with basis $v_1, v_2, \ldots, v_n$, the nonzero vector $w := v_1 \wedge v_2 \wedge \cdots \wedge v_n$ determines a point in the projective space $\mathbb{P}(\bigwedge^n(V))$ (whose points are the lines in $\bigwedge^n(V)$).

Part (3) shows that this point is independent of the basis chosen but depends only on the subspace W, thus we can indicate it by the symbol $[W]$.

Part (2) shows that the subspace W is recovered by the point $[W]$. We get:

Corollary. *The map $W \to [W]$ is a 1-1 correspondence between the set of all n-dimensional subspaces of V and the points in $\mathbb{P}(\bigwedge^n V)$ corresponding to decomposable elements.*

Definition. We denote by $Gr_n(V)$ the set of n-dimensional subspaces of V or its image in $\mathbb{P}(\bigwedge^n(V))$ and call it the *Grassmann variety*.

In order to understand the construction we will be more explicit. Consider the set $S_{n,m}$ of n-tuples $v_1, v_2, \ldots, v_n$ of linearly independent vectors in V.

(3.1.1) $$S_{n,m} := \{(v_1, v_2, \ldots, v_n) \in V^n \mid v_1 \wedge v_2 \wedge \cdots \wedge v_n \neq 0\}.$$

To a given basis $e_1, \ldots, e_m$ of V, we associate the basis $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$ of $\bigwedge^n V$ where $(i_1 < i_2 < \cdots < i_n)$.

Represent in coordinates an n-tuple $v_1, v_2, \ldots, v_n$ of vectors in V as the rows of an $n \times m$ matrix X (of rank n if the vectors are linearly independent).

In the basis $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$ of $\bigwedge^n V$ the coordinates of $v_1 \wedge v_2 \wedge \cdots \wedge v_n$ are then the determinants of the maximal minors of X.

Explicitly, let us denote by $X[i_1, i_2, \dots, i_n]$ or just $[i_1, i_2, \dots, i_n]$ the determinant of the maximal minor of X extracted from the columns $i_1\, i_2 \dots\, i_n$. Then

$$(3.1.2) \qquad v_1 \wedge v_2 \wedge \cdots \wedge v_n = \sum_{1 \le i_1 < i_2 \dots < i_n \le m} X[i_1, i_2, \dots, i_n] e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}.$$

$S_{n,m}$ can be identified with the open set of $n \times m$ matrices of maximal rank, $S_{n,m}$ is called the (algebraic) Stiefel manifold.[127]

Let us indicate by $W(X)$ the subspace of V spanned by the rows of X. The group $Gl(n, F)$ acts by left multiplication on $S_{n,m}$ and if $A \in Gl(n, F)$, $X \in S_{n,m}$, we have:

$$W(X) = W(Y), \text{ if and only if, } Y = AX, \; A \in Gl(n, F)$$

$$Y[i_1, i_2, \dots, i_n] = \det(A) X[i_1, i_2, \dots, i_n].$$

In particular $Gr_n(V)$ can be identified with the set of orbits of $Gl(n, F)$ acting by left multiplication on $S_{n,m}$. We want to understand the nature of $Gr_n(V)$ as variety. We need:

Lemma. *Given a map between two affine spaces $\pi : F^k \to F^{k+h}$, of the form*

$$\pi(x_1, x_2, \dots, x_k) = (x_1, x_2, \dots, x_k, p_1, \dots, p_h)$$

with $p_i = p_i(x_1, x_2, \dots, x_k)$ polynomials, its image is a closed subvariety of F^{k+h} and π is an isomorphism of F^k onto its image.[128]

Proof. The image is the closed subvariety given by the equations

$$x_{k+i} - p_i(x_1, x_2, \dots, x_k) = 0.$$

The inverse of the map π is the projection

$$(x_1, x_2, \dots, x_k, \dots, x_{k+h}) \to (x_1, x_2, \dots, x_k). \qquad \square$$

In order to understand the next theorem let us give a general definition. Suppose we are given an algebraic group G acting on an algebraic variety V and a map $\rho : V \to W$ which is constant on the G-orbits.

We say that ρ *is a principal G-bundle locally trivial in the Zariski topology*[129] if there is a covering of W by Zariski open sets U_i in such a way that for each U_i we have a G-equivariant isomorphism $\phi_i : G \times U_i \to \rho^{-1}(U_i)$ so that the following diagram is commutative:

$$\begin{array}{ccc} G \times U_i & \xrightarrow{\phi_i} & \rho^{-1}(U_i) \\ {\scriptstyle p_2}\downarrow & & {\scriptstyle \rho}\downarrow \\ U_i & \xrightarrow{1} & U_i \end{array} , \qquad p_2(g, u) := u.$$

[127] The usual Stiefel manifold is, over $\mathbb{C}$, the set of n-tuples $v_1, v_2, \dots, v_n$ of orthonormal vectors in $\mathbb{C}^m$. It is homotopic to $S_{n,m}$.

[128] π is the graph of a polynomial map.

[129] Usually the bundles one encounters are locally trivial only in more refined topologies.

We can now state and prove the main result of this section:

Theorem.

(1) The Grassmann variety $Gr_n(V)$ is a smooth projective subvariety of $\mathbb{P}(\bigwedge^n(V))$.
(2) The map $X \to W[X]$ from $S_{n,m}$ to $Gr_n(V)$ is a principal $Gl(n, F)$ bundle (locally trivial in the Zariski topology).

Proof. In order to prove that a subset S of projective space is a subvariety one has to show that intersecting S with each of the open affine subspaces U_i, where the i^{th} coordinate is nonzero, one obtains a Zariski closed set $S_i := S \cap U_i$ in U_i. To prove furthermore that S is smooth one has to check that each S_i is smooth.

The proof will in fact show something more. Consider the affine open set U of $\mathbb{P}(\bigwedge^n(V))$ where one of the projective coordinates is not 0 and intersect it with $Gr_n(V)$. We claim that $U \cap Gr_n(V)$ is closed in U and isomorphic to an $n(m-n)$-dimensional affine space and that on this open set the bundle of (2) is trivial.

To prove this let us assume for simplicity of notation that U is the open set where the coordinate of $e_1 \wedge e_2 \wedge \ldots \wedge e_n$ is not 0. We use in this set the affine coordinates obtained by setting the corresponding projective coordinate equal to 1.

The condition that $W(X) \in U$ is clearly $X[1, 2, \ldots, n] \neq 0$, i.e., that the submatrix A of X formed from the first n columns is invertible. Since we have selected this particular coordinate it is useful to display the elements of $S_{n,m}$ in block form as $X = (A\ T)$, (A, T being respectively $n \times n$, $n \times m - n$ matrices).

Consider the matrix $Y = A^{-1}X = (1_n\ Z)$ with Z an $n \times m - n$ matrix and $T = AZ$. It follows that the map $i : Gl(n, F) \times M_{n,m}(F) \to S_{n,m}$ given by $i(A, Z) = (A\ AZ)$ is an isomorphism of varieties to the open set $S^0_{n,m}$ of $n \times m$ matrices X such that $W(X) \in U$. Its inverse is $j : S^0_{n,m} \to Gl(n, F) \times M_{n,m}(F)$ given by $j(A\ T) = (A, A^{-1}T)$.

Thus we have that the set of matrices of type $(1_n\ Z)$ is a set of representatives for the $Gl(n, F)$-orbits of matrices X with $W(X) \in U$. In other words, in a vector space W such that $[W] \in U$, there is a unique basis which in matrix form is of type $(1_n\ Z)$. i, j also give the required trivialization of the bundle.

Let us now understand in affine coordinates the map from the space of $n \times (m-n)$ matrices to $U \cap Gr_n(V)$. It is given by computing the determinants of the maximal minors of $X = (1_n\ Z)$. A simple computation shows that:

(3.1.3)

$$X[1\ 2 \ldots i-1\ n+k\ i+1 \ldots n] = \begin{vmatrix} 1 & 0 & \ldots & 0 & z_{1\,k} & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & z_{2\,k} & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & \ldots & 1 & z_{i-1\,k} & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & z_{i\,k} & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & z_{i+1\,k} & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & \ldots & 0 & z_{n\,k} & 0 & 0 & \ldots & 1 \end{vmatrix}$$

This determinant is z_{ik}. Thus Z maps to a point in U in which $n \times (m - n)$ of the coordinates are, up to sign, the coordinates z_{ik}. The remaining coordinates are instead polynomials in these variables. Now we can invoke the previous lemma and conclude that $Gr_n(V) \cap U$ is closed in U and it is isomorphic to the affine space $F^{n(m-n)}$. □

3.2 Schubert Cells

We now display a matrix $X \in S_{n,m}$ as a sequence $(w_1, w_2, \dots, w_m)$ of *column vectors* so that if A is an invertible matrix, $AX = (Aw_1, Aw_2, \dots, Aw_m)$.

If $i_1 < i_2 < \cdots < i_k$ are indices, the property that the corresponding columns in X are linearly independent is invariant in the $Gl(n, F)$-orbit, and depends only on the space $W(X)$ spanned by the rows. In particular we will consider the sequence $i_1 < i_2 < \dots < i_n$ defined inductively in the following way: w_{i_1} is the first nonzero column and inductively $w_{i_{k+1}}$ is the first column vector which is linearly independent from $w_{i_1}, w_{i_2}, \dots, w_{i_k}$.

For an n-dimensional subspace W we will set $s(W)$ to be the sequence thus constructed from a matrix X for which $W = W(X)$. We set

(3.2.1) $$C_{i_1,i_2,\dots,i_n} = \{W \in Gr_n(V) | \, s(W) = i_1, i_2, \dots, i_n\}, \quad \text{a Schübert cell.}$$

$C_{i_1,i_2,\dots,i_n}$ is contained in the open set $U_{i_1,i_2,\dots,i_n}$ of $Gr_n(V)$ where the Plücker coordinate $[i_1, i_2, \dots, i_n]$ is not zero. In 3.1, we have seen that this open set can be identified to the set of $n \times (m - n)$ matrices X for which the submatrix extracted from the columns $i_1 < i_2 < \dots < i_n$ is the identity matrix. We wish thus to represent our set $C_{i_1,i_2,\dots,i_n}$ by these matrices.

By definition, we have now that the columns $i_1, i_2, \dots, i_n$ are the columns of the identity matrix, the columns before i_1 are 0 and the columns between i_k, i_{k+1} are vectors in which all coordinates greater that k are 0. We will refer to such a matrix as a *canonical representative*. For example, $n = 4, m = 11, i_1 = 2, i_2 = 6, i_3 = 9$, $i_4 = 11$. Then a canonical representative is

(3.2.2) $$\begin{vmatrix} 0 & 1 & a_1 & a_2 & a_3 & 0 & b_{11} & b_{12} & 0 & c_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & b_{33} & b_{34} & 0 & c_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & c_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}.$$

Thus $C_{i_1,i_2,\dots,i_n}$ is an affine subspace of $U_{i_1,i_2,\dots,i_n}$ given by the vanishing of certain coordinates. Precisely the free parameters appearing in the columns between i_k, i_{k+1} are displayed in a $k \times (i_{k+1} - i_k - 1)$ matrix, and the ones in the columns after i_n in an $n \times (m - i_n)$ matrix. Thus:

Proposition. *$C_{i_1,i_2,\dots,i_n}$ is a closed subspace of the open set $U_{i_1,i_2,\dots,i_n}$ of the Grassmann variety called a Schübert cell. Its dimension is*

$$\dim(C_{i_1,i_2,\dots,i_n}) = \sum_{k=1}^{n-1} k(i_{k+1} - i_k - 1) + n(m - i_n)$$

$$= nm - \frac{n(n-1)}{2} - \sum_{j=1}^{n} i_j. \tag{3.2.3}$$

3.3 Plücker equations

Let us make an important remark. By definition of the indices $i_1, i_2, \dots, i_n$ associated to a matrix X, we have that, given a number $j < i_k$, the submatrix formed by the first j columns has rank at most $k - 1$. This implies immediately that if we give indices $j_1, j_2, \dots, j_n$ for which the corresponding Plücker coordinate is nonzero, then $i_1, i_2, \dots, i_n \leq j_1, j_2, \dots, j_n$. In other words:

Proposition. *$C_{i_1,i_2,\dots,i_n}$ is the subset of $Gr_n(V)$ where $i_1, i_2, \dots, i_n$ is nonzero and all Plücker coordinates $[j_1, j_2, \dots, j_n]$ which are not greater than or equal to $[i_1, i_2, \dots, i_n]$ vanish.*

Proof. We have just shown one implication. We must see that if at a point of the Grassmann variety all Plücker coordinates $[j_1, j_2, \dots, j_n]$ which are not greater than or equal to $[i_1, i_2, \dots, i_n]$ vanish and $[i_1, i_2, \dots, i_n]$ is nonzero, then this point is in the cell $C_{i_1,i_2,\dots,i_n}$. Take as representative the matrix X which has the identity in the columns $i_1, i_2, \dots, i_n$. We must show that if $i_k < i < i_{k+1}$ the entries $x_{i,j}$, $j > k$, of this matrix are 0. We can compute this entry up to sign as the Plücker coordinate $[i_1, i_2, \dots i_{j-1}, i, i_{j+1}, \dots, i_n]$ (like in 3.1.3). Finally, reordering, we see that this coordinate is $[i_1, i_2, \dots, i_k, i, i_{k+1}, \dots, i_{j-1}, i_{j+1}, \dots, i_n]$ which is strictly less than $[i_1, i_2, \dots, i_n]$, hence 0 by hypothesis. □

We have thus decomposed the Grassmann variety into cells, indexed by the elements $[i_1, i_2, \dots, i_n]$. We have already seen that this set of indices has a natural total ordering and we wish to understand this order in a geometric fashion. Let us indicate by $P_{n,m}$ this partially ordered set. Let us visualize $P_{2,5}$ (see the diagram on p. 514):

First, let us make a simple remark based on the following:

Definition. In a partially ordered set P we will say that 2 elements a, b are *adjacent* if

$$a < b, \text{ and if } a \leq c \leq b, \text{ then } a = c, \text{ or } c = b.$$

Remark. The elements adjacent to $i_1, i_2, \dots, i_n$ are obtained by selecting any index i_k such that $i_k + 1 < i_{k+1}$ and replacing it by $i_k + 1$ (if $k = n$ the condition is $i_k < m$).

Proof. The proof is a simple exercise left to the reader. □

$$
\begin{tikzcd}
{[4,5]} \arrow[dr] & & & \\
& {[3,5]} \arrow[dl] \arrow[dr] & & \\
{[3,4]} \arrow[dr] & & {[2,5]} \arrow[dl] \arrow[dr] & \\
& {[2,4]} \arrow[dl] \arrow[dr] & & {[1,5]} \arrow[dl] \\
{[2,3]} \arrow[dr] & & {[1|4]} \arrow[dl] & \\
& {[1,3]} \arrow[dl] & & \\
{[1,2]} & & &
\end{tikzcd}
$$

3.4 Flags

There is a geometric meaning of the Schubert cells related to the relative position with respect to a *standard flag*.

Definition. A flag in a vector space V is an increasing sequence of subspaces:

$$F_1 \subset F_2 \subset \cdots \subset F_k.$$

A complete flag in an m-dimensional space V is a flag

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_{m-1} \subset F_m = V \tag{3.4.1}$$

with $\dim(F_i) = i, \ i = 1, \dots, m$.

Sometimes it is better to use a projective language, so that F_i gives rise to an $i-1$-dimensional linear subspace in the projective space $\mathbb{P}(V)$.

A complete flag in an m-dimensional projective space is a sequence: $\pi_0 \subset \pi_1 \subset \pi_2 \ldots \subset \pi_m$ with π_i a linear subspace of dimension i.[130]

We fix as standard flag $F_1 \subset F_2 \subset \cdots \subset F_k$ with F_i the set of vectors with the first $m-i$ coordinates equal to 0, spanned by the last i vectors of the basis $e_1, \dots, e_m$.

Given a space $W \in C_{i_1, i_2, \dots, i_n}$ let $v_1, \dots, v_n$ be the corresponding normalized basis as rows of an $n \times m$ matrix X for which the submatrix extracted from the columns

[130] The term flag comes from a simple drawing in 3-dimensional projective space. The base of a flagpole is a point, the pole is a line, and the flag is a plane.

$i_1, i_2, \dots, i_n$ is the identity matrix. Therefore a linear combination $\sum_{k=1}^n c_k v_k$ has the number c_k as the i_k coordinate $1 \le k \le n$. Thus for any i we see that

$$W \cap F_i = \left\{ \sum_{k=1}^n c_k v_k | c_k = 0, \text{ if } i_k < m - i \right\}. \tag{3.4.2}$$

We deduce that for every $1 \le i \le m$,

$$d_i := \dim(F_i \cap W) = n - k \text{ if and only if } i_k < m - i \le i_{k+1}.$$

In other words, the sequence of d_i's is completely determined, and determines the numbers $\underline{i} := i_1 < i_2 < \cdots < i_n$. Let us denote by $\underline{d}[\underline{i}]$ the sequence thus defined; it has the properties:

$$d_m = n, \ d_1 \le 1, d_i \le d_{i+1} \le d_i + 1.$$

The numbers $m - i_k + 1$ are the ones in which the sequence jumps by 1. For the example given in (3.2.2) we have the sequence

$$1, 1, 2, 2, 2, 3, 3, 3, 3, 4, 4.$$

We observe that given two sequences

$$\underline{i} := i_1 < i_2 < \dots < i_n, \quad \underline{j} := j_1 < j_2 < \dots < j_n,$$

we have

$$\underline{i} \le \underline{j} \text{ iff } \underline{d}[\underline{i}] \le \underline{d}[\underline{j}].$$

3.5 B-orbits

We pass now to a second fact:

Definition. Let:

$$S_{i_1, i_2, \dots, i_n} := \{W \mid \dim(F_i \cap W) \le d_i[\underline{i}], \ \forall i\}.$$

From the previous remarks:

$$C_{i_1, i_2, \dots, i_n} := \{W \mid \dim(F_i \cap W) = d_i[\underline{i}], \ \forall i\}, \quad S_{\underline{i}} = \cup_{\underline{j} \ge \underline{i}} C_{\underline{j}}.$$

We need now to interpret these notions in a group-theoretic way.

We define T to be the subgroup of $GL(m, F)$ of diagonal matrices. Let $I_{i_1, i_2, \dots, i_n}$ be the $n \times m$ matrix with the identity matrix in the columns $i_1, i_2, \dots, i_n$ and 0 in the other columns. We call this the *center* of the Schubert cell.

Lemma. *The $\binom{m}{n}$ decomposable vectors associated to the matrices $I_{i_1, i_2, \dots, i_n}$ are the vectors $e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_n}$. These are a basis of weight vectors for the group T acting on $\bigwedge^n F^m$. The corresponding points in projective space $P(\bigwedge^n F^m)$ are the fixed points of the action of T, and the corresponding subspaces are the only T-stable subspaces of F^m.*

Proof. Given an action of a group G on a vector space, the fixed points in the corresponding projective space are the stable 1-dimensional subspaces. If the space has a basis of weight vectors of distinct weights, any G-stable subspace is spanned by a subset of these vectors. The lemma follows. $\square$

Remark. When $F = \mathbb{C}$, the space $\bigwedge^n \mathbb{C}^m$ is an irreducible representation of $SL(m, \mathbb{C})$ and a fundamental representation. It has a basis of weight vectors of distinct weights and they are one in orbit under the symmetric group. A representation with this property is called *minuscule*. For general Lie groups few fundamental representations are minuscule.[131]

We define B to be the subgroup of $GL(m, F)$ which stabilizes the standard flag. A matrix $X \in B$ if and only if, for each i, Xe_i is a linear combination of the elements e_j with $j \geq i$. This means that B is the group of lower triangular matrices, usually denoted by B^-. From the definitions we have clearly that the sets $C_{i_1,i_2,\dots,i_n}$, $S_{i_1,i_2,\dots,i_n}$ are stable under the action of B. In fact we have:

Theorem. *$C_{i_1,i_2,\dots,i_n}$ is a B-orbit.*

Proof. Represent the elements of $C_{i_1,i_2,\dots,i_n}$ by their matrices whose rows are the canonical basis. Consider, for any such matrix X, an associated matrix $\tilde{X}$ which has the i_k row equal to the k^{th} row of X and otherwise the rows of the identity matrix. For instance, if X is the matrix of 3.2.2 we have

$$\tilde{X} = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & a_1 & a_2 & a_3 & 0 & b_{11} & b_{12} & 0 & c_{11} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & b_{33} & b_{34} & 0 & c_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & c_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}. \tag{3.5.1}$$

We have

$$X = I_{i_1,i_2,\dots,i_n}\tilde{X},$$

and $\tilde{X}^t \in B$. This implies the theorem. $\square$

Finally we have:

[131] In this chapter the minuscule property is heavily used to build the standard monomial theory. Nevertheless there is a rather general standard monomial theory due to Lakshmibai–Seshadri (cf [L-S]) and Littelmann for all irreducible representations of semisimple algebraic groups (cf. [Lit], [Lit2]).

Proposition. *$S_{i_1,i_2,\dots,i_n}$ is the Zariski closure of $C_{i_1,i_2,\dots,i_n}$.*

Proof. $S_{i_1,i_2,\dots,i_n}$ is defined by the vanishing of all Plücker coordinates not greater or equal to $i_1, i_2, \dots, i_n$, hence it is closed and contains $C_{i_1,i_2,\dots,i_n}$.

Since $C_{i_1,i_2,\dots,i_n}$ is a B-orbit, its closure is a union of B orbits and hence a union of Schubert cells. To prove the theorem it is enough, by 3.3, to show that, if for some k we have $i_k + 1 < i_{k+1}$, then $I_{i_1,i_2,\dots,i_{k-1},i_k+1,i_{k+1},i_n}$ is in the closure of $C_{i_1,i_2,\dots,i_n}$.

For this consider the matrix $I_{i_1,i_2,\dots,i_n}(b)$ which differs from $I_{i_1,i_2,\dots,i_n}$ only in the $i_k + 1$ column. This column has 0 in all entries except b in the k row.

The space defined by this matrix lies in $C_{i_1,i_2,\dots,i_n}$ and equals the one defined by the matrix obtained from $I_{i_1,i_2,\dots,i_n}(b)$ dividing the k row by b.

This last matrix equals $I_{i_1,i_2,\dots,i_{k-1},i_k+1,i_{k+1},i_n}$ except in the i_k column, which has 0 in all entries except b^{-1} in the k row. The limit as $b \to \infty$ of this matrix tends to $I_{i_1,i_2,\dots,i_{k-1},i_k+1,i_{k+1},i_n}$. For example,

$$W\left(\begin{vmatrix} 0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&b&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&1 \end{vmatrix}\right) = W\left(\begin{vmatrix} 0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&b^{-1}&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&1 \end{vmatrix}\right);$$

$$\lim_{b\to\infty} W\left(\begin{vmatrix} 0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&b^{-1}&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&1 \end{vmatrix}\right) = W\left(\begin{vmatrix} 0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&1 \end{vmatrix}\right).$$

□

3.6 Standard Monomials

We want to apply to standard monomials the theory developed in the previous sections. We have seen that the Schubert variety $S_{i_1,i_2,\dots,i_n} = S_{\underline{i}}$ is the intersection of the Grassmann variety with the subspace where the coordinates $\underline{j}$ which are not greater than or equal to $\underline{i}$ vanish.

Definition. We say that a standard monomial is standard on $S_{\underline{i}}$ if it is a product of Plücker coordinates greater or equal than $\underline{i}$.[132]

Theorem. *The monomials that are standard on $S_{\underline{i}}$ are a basis of the projective coordinate ring of $S_{\underline{i}}$.*

Proof. The monomials that are not standard on $S_{\underline{i}}$ vanish on this variety, hence it is enough to show that the monomials standard on $S_{\underline{i}}$, restricted to this variety, are linearly independent. Assume by contradiction that some linear combination $\sum_{k=1}^{a} c_k T_k$ vanishes on $S_{\underline{i}}$, and assume that the degree of this relation is minimal.

[132] This definition and the corresponding approach to standard monomials is due to Seshadri ([Seh]).

Let us consider, for each monomial T_k, its minimal coordinate p_k and write $T_k = p_k T_k'$; then select, among the Plücker coordinates p_k, a maximal coordinate $p_{\underline{j}}$ and decompose the sum as

$$\sum_{k=1}^{b} c_k p_k T_k' + p_{\underline{j}}\left(\sum_{k=b+1}^{a} c_k T_k'\right),$$

where the sum $\sum_{k=1}^{b} c_k p_k T_k'$ collects all terms which start from a coordinate p_k different from p_j. By hypothesis $\underline{i} \leq \underline{j}$. Restricting the relation to $S_{\underline{j}}$, all the standard monomials which contain coordinates not greater than $\underline{j}$ vanish, so, by choice of $\underline{j}$, we have that $p_{\underline{j}}\left(\sum_{k=m+1}^{n} c_k T_k'\right)$ vanishes on $S_{\underline{j}}$. Since $S_{\underline{j}}$ is irreducible and $p_{\underline{j}}$ is nonzero on $S_{\underline{j}}$, we must have that $\left(\sum_{k=m+1}^{n} c_k T_k'\right)$ vanishes on $S_{\underline{j}}$. This relation has a lower degree and we reach a contradiction by induction. $\square$

Of course this theorem is more precise than the standard monomial theorem for the Grassmann variety.

4 Double Tableaux

4.1 Double Tableaux

We return now to the polynomial ring $\mathbb{Z}[x_{i,j}] : 1 \leq i \leq n;\ 1 \leq j \leq m$ of §1.1, which we think of as polynomial functions on the space of $n \times m$ matrices.

In this ring we will study the relations among the special polynomials obtained as determinants of minors of the matrix X. We use the notations (1.1.2) of Section 1.1.

Consider the Grassmann variety $Gr_n(m+n)$ and in it the open set A where the Plücker coordinate extracted from the last n columns is nonzero. In §2 we have seen that this open set can be identified with the space $M_{n,m}$ of $n \times m$ matrices. The identification is defined by associating to a matrix X the space spanned by the rows of $(X\ 1_n)$.

Remark. In more intrinsic terms, given two vector spaces V and W, we identify $\hom(V, W)$ with an open set of the Grassmannian in $V \oplus W$ by associating to a map $f : V \to W$ its graph $\Gamma(f) \subset V \oplus W$. The fact that the first projection of $\Gamma(f)$ to V is an isomorphism is expressed by the nonvanishing of the corresponding Plücker coordinate.

The point 0 thus corresponds to the unique 0-dimensional Schubert cell, which is also the only closed Schubert cell. Thus every Schubert cell has a nonempty intersection with this open set.[133]

[133] One can consider that in the Grassman variety we can construct two different cellular decompositions using the two opposite Borel subgroups B^+, B^-. Thus here we are considering the intersection of the open cell relative to B^- with the cells relative to B^+.

We use as coordinates in X the variables x_{ij} but we display them as

$$X' := \begin{vmatrix} x_{n1} & x_{n2} & \dots & x_{n,m-1} & x_{nm} \\ x_{n-1,1} & x_{n-1,2} & \dots & x_{n-1,m-1} & x_{n-1,m} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ x_{11} & x_{12} & \dots & x_{1,m-1} & x_{1m} \end{vmatrix}.$$

Let us compute a Plücker coordinate $[i_1, i_2, \dots, i_n]$ for $X'\ 1_n$. We must distinguish, among the indices i_k appearing, the ones $\leq m$, say $i_1, i_2, \dots, i_h$ and the ones bigger than m, that is $i_{h+t} = m + j_t$ where $t = 1, \dots, n-h$; $1 \leq j_t \leq n$.

The last $n-h$ columns of the submatrix of $(X'\ 1_n)$ extracted from the columns $i_1, i_2, \dots, i_n$ are thus the columns of indices $j_1, j_2, \dots, j_{n-h}$ of the identity matrix.

Let first Y be an $n \times (n-1)$ matrix and e_i the i^{th} column of the identity matrix.

The determinant $\det(Ye_i)$ of the $n \times n$ matrix, obtained from Y by adding e_i as the last column, equals $(-1)^{n+i}\det(Y_i)$, where Y_i is the $(n-1) \times (n-1)$ matrix extracted from Y by deleting the i^{th} row. When we repeat this construction we erase successive rows.

In our case, therefore, we obtain that $[i_1, i_2, \dots, i_h, m + j_1, \dots, m + j_{n-h}]$ equals, up to sign, the determinant $(u_1, u_2, \dots, u_h | i_1, i_2, \dots, i_h)$ of X, where the indices $u_1, u_2, \dots, u_h$ are complementary, in $1, 2, \dots, n$, to the indices $n+1-j_1$, $n+1-j_2, \dots, n+1-j_{n-h}$.

We have defined a bijective map between the set of Plücker coordinates $[i_1, i_2, \dots, i_n]$ in $1, 2, \dots, n+m$ distinct from the last coordinate and the minors of the $n \times m$ matrix.

4.2 Straightening Law

Since the Plücker coordinates are naturally partially ordered, we want to understand the same ordering transported to the minors. It is enough to do it for adjacent elements. We must distinguish various cases.

Suppose thus that we are given a coordinate $[i_1, i_2, \dots, i_h, m+j_1, \dots, m+j_{n-h}]$ corresponding to the minor $(v_h, \dots, v_2, v_1 | i_1, i_2, \dots, i_s, \dots, i_h)$, and consider

$$\begin{aligned}&[i_1, i_2, \dots, i_s, \dots, i_h, m + j_1, \dots, m + j_{n-h}] \\ &\quad \leq [i_1, i_2, \dots, i_s + 1, \dots, i_h, m + j_1, \dots, m + j_{n-h}].\end{aligned}$$

This gives

$$(v_h, \dots, v_2, v_1 | i_1, i_2, \dots, i_s, \dots, i_h) \leq (v_h, \dots, v_2, v_1 | i_1, i_2, \dots, i_s + 1, \dots, i_h).$$

Similarly

$$\begin{aligned}&[i_1, \dots, i_h, m + j_1, \dots, m + j_s, \dots, m + j_{n-h}] \\ &\quad \leq [i_1, \dots, i_h, m + j_1, \dots, m + j_s + 1, \dots, m + j_{n-h}]\end{aligned}$$

gives, for $v_t := n - j_s - 1$,

$$(v_h, \dots, v_t, \dots, v_2, v_1 | i_1, i_2, \dots, i_h)$$
$$\leq (v_h, \dots, v_t + 1, \dots, v_2, v_1 | i_1, i_2, \dots, i_h).$$

Finally we have the case in which the number of indices $\leq m$ decrease, i.e.,

$$[i_1, \dots, i_{h-1}, i_h = m, m + j_1, \dots, m + j_{n-h}]$$
$$\leq [i_1, \dots, i_{h-1}, m + 1, m + j_1, \dots, m + j_{n-h}].$$

This gives $n = v_h,\ j_1 > 1$ and

$$(n, v_{h-1}, \dots, v_2, v_1 | i_1, i_2, \dots, i_{h-1}, m) \leq (v_{h-1}, \dots, v_2, v_1 | i_1, i_2, \dots, i_{h-1}).$$

In particular we see that a $k \times k$ determinant can be less than an $h \times h$ determinant only if $k \geq h$.

The formal implication is that a standard product of Plücker coordinates, interpreted (up to sign) as a product of determinants of minors, appears as a double tableau, in which the shape of the left side is the reflection of the shape on the right. The columns are non-decreasing. The rows are strictly increasing in the right tableau and strictly decreasing in the left. As example, let $n = 3, m = 5$, and consider a tableau:

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 4 & 7 \\ 2 & 4 & 8 \\ 2 & 6 & 8 \\ 3 & 7 & 8 \end{vmatrix}.$$

To this corresponds the double tableau:

$$\begin{array}{rrr|lll} 3 & 2 & 1 & 1 & 2 & 3 \\ 3 & 2 & 1 & 1 & 2 & 4 \\ & 3 & 1 & 1 & 4 & \\ & 3 & 2 & 2 & 4 & \\ & & 2 & 2 & & \\ & & 3 & 3 & & \end{array}$$

We will call such a tableau a *double standard tableau.*[134]

Of course, together with the notion of double standard tableau we also have that of double tableau or *bi-tableau*, which can be either thought of as a product of determinants of minors of decreasing sizes or as a pair of tableaux, called left (or row) and right (or column) tableau of the same size.

[134] The theory of double standard tableaux was introduced by Doubilet, Rota and Stein [DRS]. The treatment here is due to Seshadri [L-S].

If one takes the second point of view, which is useful when analyzing formally the straightening laws, one may think that the space of one-line tableaux of size k is a vector space M_k with basis the symbols $(v_h, \dots, v_2, v_1 | i_1, i_2, \dots, i_h)$. The right indices run between 1 amd m, while the left indices run between 1 and n. The symbols are assumed to be separately antisymmetric in the left and right indices. In particular, when two indices on the right or on the left are equal the symbol is 0.

For a partition $\lambda := m_1 \geq m_2 \geq \cdots \geq m_t$ the tableaux of shape λ can be thought of as the tensor product $M_{m_1} \otimes M_{m_2} \otimes \dots \otimes M_{m_t}$. When we evaluate a formal tableau as a product of determinants we have a map with nontrivial kernel (the space spanned by the straightening laws).

We now want to interpret the theory of tableaux in terms of representation theory. For this we want to think of the space of $n \times m$ matrices as $\hom(V, W) = W \otimes V^*$ where V is m-dimensional and W is n-dimensional (as free $\mathbb{Z}$-modules if we work over $\mathbb{Z}$). The algebra R of polynomial functions on $\hom(V, W)$ is the symmetric algebra on $W^* \otimes V$.

$$R = S(W^* \otimes V). \tag{4.2.1}$$

The two linear groups $GL(V)$, $GL(W)$ act on the space of matrices and on R.

Over $\mathbb{Z}$ we no longer have the decomposition 6.3.2 of Chapter 9 so our theory is a replacement, and in a way, also a refinement of that decomposition.

In matrix notations the action of an element $(A, B) \in GL(m) \times GL(n)$ on an $n \times m$ matrix Y is BYA^{-1}. If e_i, $i = 1, \dots, n$, is a basis of W and f_j, $j = 1, \dots, m$, one of V under the identification $R = S(W^* \otimes V) = \mathbb{Z}[x_{ij}]$, the element $e^i \otimes f_j$ corresponds to x_{ij}:

$$\langle e^i \otimes f_j | X \rangle := \langle e^i | X f_j \rangle = \Big\langle e^i \,\Big|\, \sum_h x_{hj} e_h \Big\rangle = x_{ij}.$$

Geometrically we can think as follows. On the Grassmannian $G_{m,m+n}$ acts the linear group $GL(m+n)$. The action is induced by the action on $n \times (m+n)$ matrices Y by YC^{-1}, $C \in GL(m+n)$.

The space of $n \times m$ matrices is identified with the cell $(X\ 1_n)$ and is stable under the diagonal subgroup $GL(m) \times GL(n)$. Thus if $C = \begin{vmatrix} A & 0 \\ 0 & B \end{vmatrix}$ we have

$$(X\ 1_n)C^{-1} = (XA^{-1}\ B^{-1}) \equiv (BXA^{-1}\ 1_n). \tag{4.2.2}$$

If now we want to understand the dual action on polynomials, we can use the standard dual form $(gf)(u) = f(g^{-1}u)$ for the action on a vector space as follows:

Remark. The transforms of the coordinate functions x_{ij} under A, B are the entries of $B^{-1}XA$, where $X = (x_{ij})$ is the matrix having as entries the variables x_{ij}.

Let us study the subspace $M_k \subset R$ of the ring of polynomials spanned by the determinants of $(v_k, \dots, v_2, v_1 | i_1, i_2, \dots, i_s, \dots, i_k)$, i.e., the $k \times k$ minors.

Given an element $A \in \hom(V, W)$, it induces a map $\wedge^k A : \bigwedge^k V \to \bigwedge^k W$. Thus the formula , $i_k(\phi \otimes u)(A) := \langle \phi | \wedge^k Au \rangle$ defines a map

$$i_k : \hom\left(\bigwedge^k V, \bigwedge^k W\right)^* = \bigwedge^k W^* \otimes \bigwedge^k V \to R = S(V^* \otimes W).$$

It is clear that $i_k(f_{v_k} \wedge \dots f_{v_2} \wedge f_{v_1} | e_{i_1} \wedge e_{i_2} \dots \wedge e_{i_k}) = (v_k, \dots, v_2, v_1 | i_1, i_2, \dots, i_s, \dots, i_k)$ and thus M_k is the image of i_k.

Lemma. *M_h is isomorphic to* $\hom(\bigwedge^h V, \bigwedge^h W)^* = \bigwedge^h V \otimes (\bigwedge^h W)^*$ *in a* $GL(V) \times GL(W)$ *equivariant way.*

Proof. Left to the reader as in Chapter 9, 7.1. □

The action of the two linear groups on rows and columns induces, in particular, an action of the two groups of diagonal matrices, and a double tableau is clearly a weight vector under both groups. Its weight (or double weight) is read off from the row and column indices appearing.

We may encode the number of appearances of each index on the row and column tableaux as two sequences

$$1^{h_1} 2^{h_2} \dots n^{h_n};\ 1^{k_1} 2^{k_2} \dots m^{k_m}.$$

When one wants to stress the combinatorial point of view one calls these two sequences the *content* of the double tableau.

According to the definition of the action of a group on functions, we see that the weight of a diagonal matrix in $GL(n)$ with entries b_i acting on rows is $\prod_{i=1}^n b^{-h_i}$ while the weight of a diagonal matrix in $GL(m)$ with entries a_i acting on columns is $\prod_{i=1}^m a^{k_i}$.

We come now to the main theorem:

Theorem. *The double standard tableaux are a $\mathbb{Z}$-basis of $\mathbb{Z}[x_{i,j}]$.*

Proof. The standard monomials in the Plücker coordinates are a basis of $\mathbb{Z}[[i_1, i_2, \dots, i_n]]$, so we have that the double standard tableaux span the polynomial algebra $\mathbb{Z}[x_{i,j}]$ over $\mathbb{Z}$.

We need to show that they are linearly independent. One could give a proof in the same spirit as for the ordinary Plücker coordinates, or one can argue as follows.

We have identified the space of $n \times m$ matrices with the open set of the Grassmann variety where the Plücker coordinate $p = [m+1, m+2, \dots, m+n]$ is nonzero.

There are several remarks to be made:

(1) The coordinate p is the maximal element of the ordered set of coordinates, so that if T is a standard monomial, so is Tp.
(2) Since a $\mathbb{Z}$-basis of $\mathbb{Z}[[i_1, i_2, \dots, i_n]]$ is given by the tableaux Tp^k, where T is a standard tableau not containing p, we have that these tableaux not containing p are a basis over the polynomial ring $\mathbb{Z}[p]$.
(3) The algebra $\mathbb{Z}[x_{i,j}]$ equals the quotient algebra $\mathbb{Z}[[i_1, i_2, \dots, i_n]]/(p-1)$.

From (2) and (3) it follows that the image in $\mathbb{Z}[x_{i,j}]$ of the standard monomials which do not end with p are a $\mathbb{Z}$-basis. But the images of these monomials are the double standard tableaux and the theorem follows. To finish the proof, it remains to check (1), (2), (3).

Points (1) and (2) are clear.

Point (3) is a general fact on projective varieties. If $W \subset P^n$ is a projective variety and A is its homogeneous coordinate ring, the coordinate ring of the affine part of W where a coordinate x is not zero is $A/(x-1)$. □

4.3 Quadratic Relations

We need to analyze now the straightening algorithm for double tableaux. To begin, we consider a basic quadratic relation for a two-line tableau. We have thus to understand the quadratic relation 2.1.4 for a product of two Plücker coordinates $|i_1, \dots, i_n||j_1, \dots, j_n|$ in terms of double tableaux. We may assume without loss of generality that the two coordinates give a double tableau with two rows of length $a \geq b$. There are two possibilities for the point $i_k > j_k$ where the violation occurs: either the two indices i_k, j_k are both column indices or both row indices. Let us treat the first case, the other is similar. In this case all indices $j_1, \dots, j_k$ are column indices while among the $i_k, \dots, i_n$ there can be also row indices.

In each summand of 2.1.4 some top indices are exchanged with bottom indices, so we can separate the sum into two contributions, the first in which no row indices are exchanged and the second with the remaining terms. Thus in the first we have a sum of tableaux of type a, b, while in the second the possible types are $a+t, b-t$, $t > 0$.

Summarizing,

Proposition. *A straightening law on the column indices for a product*

$$T = (u_a \dots u_1 | i_1 \dots i_a)(v_b \dots v_1 | j_1 \dots j_b)$$

of 2 determinants of sizes $a \geq b$ is the sum of two terms $T_1 + T_2$, where T_2 is a sum of tableaux of types $a+t, b-t,\ t > 0$, and T_1 is the sum of the tableaux obtained from T by selecting an index i_k such that $i_k > j_k$ and performing all possible exchanges among $i_k \dots i_a$ and $j_1 \dots j_k$, while leaving fixed the row indices and summing with the sign of the exchange:

$$(4.3.1) \qquad \sum \epsilon \quad \begin{matrix} u_a, u_{a-1}, \dots, u_2, u_1 | i_1, i_2, \dots, \underline{i_k, \dots\dots, i_a} \\ v_b, \dots, v_2, v_1 | \underline{j_1, j_2, \dots, j_k}, \dots, j_b \end{matrix} \quad + T_2.$$

All the terms of the quadratic relation have the same double weight. There is a similar statement for row straightening.

For our future analysis, it is not necessary to make more explicit the terms T_2, which are in any case encoded formally in the identity 2.1.4.

Remark. (1) There is an important special case to be noticed: when the row indices $u_m, \ldots, u_2, u_1$ are all contained in the row indices $i_k \ldots i_1$. In this case the terms T_2 do not appear, since raising a row index creates a determinant with two equal rows.

(2) The shapes of tableaux appearing in the quadratic equations are closely connected with a special case of Pieri's formula (in characteristic 0).

$$a \geq b, \qquad \bigwedge^a V \otimes \bigwedge^b V = \bigoplus_{t=0}^b S_{a+t,b-t}(V).$$

Regarding (1) we define:

Definition 1. A tableau A is said to be extremal if, for every $i > 1$, the indices of the i^{th} row are contained in the indices of the $(i-1)^{st}$ row.

Let us take a double tableau $A|B$, where A, B represent the two tableaux of row and column indices. Let us apply sequentially straightening relations on the column indices. We see that again we have two contributions $A|B = T_1 + T_2$, where in T_1 we have tableaux of the same shape while in T_2 the shape has changed (we will see how in a moment).

Lemma. *The contribution from the first part of the sum is of type*

$$T_1 = \sum_C c_{B,C}\, A|C, \tag{4.3.2}$$

where the coefficients $c_{B|C}$ are independent of A. If A is an extremal tableau, $T_2 = 0$.

There is an analogous statement for row relations.

We can now use the previous straightening relations to transform a double tableau into a sum of double standard tableaux. For this we have to remark that, starting from a product of determinants of sizes $a_1 \geq a_2 \geq \cdots \geq a_i$ and applying a quadratic relation, we may replace two successive sizes $a \geq b$ with some $a+t, b-t$. In this way the product does not appear as a product of determinants of decreasing sizes. We have thus to reorder the terms of the product to make the sizes decreasing. To understand how the shapes of tableaux behave with respect to this operation, we give the following:

Definition 2. The dominance order for sequences of real numbers is

$$(a_1, \ldots, a_n) \geq (b_1, \ldots, b_n) \quad \text{iff} \quad \sum_{i=1}^h a_i \geq \sum_{i=1}^h b_i \qquad \forall h = 1, \ldots, n.$$

In particular we obtain a (partial) ordering on partitions.

Remark. If we take a vector $(b_1, \ldots, b_n)$ and construct $(a_1, \ldots, a_n)$ by reordering the entries in decreasing order, then $(a_1, \ldots, a_n) \geq (b_1, \ldots, b_n)$.

Corollary. *Given a double tableau of shape λ, by the straightening algorithm it is expressed as a linear combination of standard tableaux of shapes $\geq \lambda$ in the dominance order and of the same double weight.*

5 Representation Theory

5.1 U Invariants

Consider the root subgroups, which we denote by $a + \lambda b$, acting on matrices by adding to the a^{th} column the b^{th} column multiplied by λ. This is the result of the multiplication $X(1+\lambda e_{ba})$. A single determinant of a minor $D := (i_1, \dots, i_k | j_1, \dots, j_k)$ is transformed according to the following rule (cf. §2.3):

If a does not appear among the elements j_s, or if both a, b appear among these elements, D is left invariant.

If $a = j_s$ and b does not appear, D is transformed into $D + \lambda D'$ where D' is obtained from D by substituting a in the column indices with b.

Of course a similar analysis is valid for the row action.

This implies a combinatorial description of the group action of $G = GL(m) \times GL(n)$ on the space of tableaux. In particular we can apply it when the base ring is $\mathbb{Z}$ or a field F, so that the special linear group over F or $\mathbb{Z}$ is generated by the elements $a + \lambda b$. We have described the action of such an element on a single determinant, which then extends by multiplication and straightening algorithm.

An argument similar to the one performed in §2.3 shows that: Given a linear combination $C := \sum_i c_i T_i$ of double standard tableaux, apply to it the transformation $2 + \lambda 1$ and obtain a polynomial in λ. The degree k of this polynomial is the maximum of the number of occurrences of 2 in a tableau T_i as a column index not preceded by 1, i.e., 2 occurs on the first column.

Its leading term is of the form $\sum c_i T_i'$ where the sum extends to all the indices of tableaux T_i where 2 appears in the first column k times and T_i' is obtained from T_i by replacing 2 with 1 in these positions. It is clear that to distinct tableaux T_i correspond distinct tableaux T_i' and thus this leading coefficient is nonzero. It follows that:

The element C is invariant under $2 + \lambda 1$ if and only if in the column tableau, 2 appears only on the second column.

Let us indicate by $A^{1,2}$ this ring of invariant elements under $2 + \lambda 1$.

We can now repeat the argument using $3 + \lambda 1$ on the elements of $A^{1,2}$ and see that

An element $C \in A^{1,2}$ is invariant under $3+\lambda 1$ if and only if in the column tableau each occurrence of 3 is preceded by 1.

By induction we can define $A^{1,k}$ to be the ring of invariants under all the root subgroups $i + \lambda 1, \ i \leq k$.

$A^{1,k}$ is spanned by the elements such that in the column tableau no element $1 < i \leq k$ appears in the first column.

Now when $k = m$, all tableaux which span this space have only 1's on the first column of the right tableau.

Next we can repeat the argument on $A^{1,m}$ using the root subgroups $i + \lambda 2, \ i \leq k$. We thus define $A^{2,k}$ to be the ring of invariants under all the root subgroups $i + \lambda 1$ and all the root subgroups $i + \lambda 2, \ i \leq k$.

$A^{2,k}$ is spanned by the elements with 1 on the first column of the right tableau and no element $2 < i \leq k$ appearing on the second column.

In general, given $i < j \le m$ consider the subgroup $U_{i,j}$ of upper triangular matrices generated by the root subgroups

$$b + \lambda a,\ a \le i-1,\ b \le m;\ b + \lambda i,\ \ b \le j$$

and denote by $A^{i,j}$ the corresponding ring of invariants. Then:

Proposition 1. *$A^{i,j}$ is spanned by the elements in which the first $i-1$ columns of the right tableau are filled, respectively, with the numbers $1, 2, \dots, i-1$, while no number $i < k \le j$ is in the i column.*

Corollary 1. *The ring of polynomial invariants under the full group U^+ of upper triangular matrices, acting on the columns, is spanned by the double standard tableaux whose column tableau has the i^{th} column filled with i for all i. We call such a tableau right canonical.*

The main remark is that, given a shape λ, there is a *unique* canonical tableau of that given shape characterized by having 1 in the first column, 2 in the second, etc. We denote this canonical tableau by C_λ. For example, for $m = 5$,

$$C_{33211} := \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \\ 1 & & \\ 1 & & \end{matrix}\ , \qquad C_{54211} := \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & \\ 1 & 2 & & & \\ 1 & & & & \\ 1 & & & & \end{matrix}\ .$$

One could have proceeded similarly starting from the subgroups $m + \lambda i$ and getting:

Corollary 2. *The ring of polynomial invariants under the full group U^- of lower triangular matrices, acting on the columns, is spanned by the double standard tableaux whose column side has the property that each index $i < m$ appearing is followed by $i+1$. We call such a tableau anticanonical.*

Again, given a shape λ, there is a *unique* anticanonical tableau of that given shape, e.g., for $m = 5$,

$$\begin{matrix} 3 & 4 & 5 \\ 3 & 4 & 5 \\ 4 & 5 & \\ 5 & & \\ 5 & & \end{matrix}\ , \qquad \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & \\ 4 & 5 & & & \\ 5 & & & & \\ 5 & & & & \end{matrix}\ .$$

Observe that a tableau can be at the same time canonical and anticanonical if and only if all its rows have length m (e.g., for $m = 5$):

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{matrix}\ . \tag{5.1.1}$$

Of course we have a similar statement for the action on rows (the left action) except that the invariants under left action by U^- are left canonical, and under the left action by U^+ action are left anticanonical.

Now we will obtain several interesting corollaries.

Definition. For a partition λ define W^λ (resp. V_λ) to be the span of all double tableaux $A|C_\lambda$ of shape λ with left canonical tableau (resp. $C_\lambda|B$).

From Lemma 4.3, since a canonical tableau is extremal, we have:

Proposition 2. *W^λ has as basis the double standard tableaux $A|C_\lambda$.*
V_λ has as basis the double standard tableaux $C_\lambda|A$.

Theorem 1. *The invariants under the right U^+ action (resp. the left U^- action) decompose as*

$$\bigoplus_\lambda W^\lambda, \quad \textit{resp.} \ \bigoplus_\lambda V_\lambda.$$

If we act by right multiplication with a diagonal matrix t with entry a_i in the ii position, this multiplies the i^{th} column by a_i and thus transforms a double tableau T which is right canonical and of shape λ into $T\prod a_i^{k_i}$, where k_i is the length of the i^{th} column.[135]

If t with entries a_i is the diagonal part of an upper triangular matrix, we can think of $\prod a_i^{k_i}$ as a function on B^+ which is still a *character* denoted by λ. Thus the decomposition $\bigoplus_\lambda W^\lambda$ is a decomposition into weight spaces under the Borel subgroup of upper triangular matrices. We have proved:

Theorem 2. *W^λ is the space of functions which, under the right action of B^+, are weight vectors of character λ. W^λ is a $GL(n)$-submodule, (similar statement for V_λ).*

Proof. The left action by $GL(n)$ commutes with the right action and thus each W^λ is a $GL(n)$ submodule. □

Assume for instance $n \le m$. The $U^- \times U^+$ invariants are spanned by those tableaux which are canonical on the left and the right and will be called *bicanonical.* These tableaux are the polynomials in the determinants

$$d_k := (k, k-1, \dots, 1|1, 2, \dots, k).$$

A monomial $d_1^{h_1} d_2^{h_2} \dots d_n^{h_n}$ is a bicanonical tableau whose shape λ is determined by the sequence h_i and will be denoted by K_λ.

An argument similar to the previous analysis of U invariants shows that:

[135] We are skimming over a point. When we work over $\mathbb{Z}$ there are not enough diagonal matrices so we should really think that we compute our functions, which are defined over $\mathbb{Z}$, into any larger ring. The identities we are using are valid when we compute in any such extension.

Proposition 3.

(1) Any U^- fixed vector in W^λ is multiple of the bicanonical tableau K_λ of shape λ.
(2) If the base ring is an infinite field every U^- stable subspace of W^λ contains K_λ.
(3) W^λ is an indecomposable U^-- or $GL(n)$-module.
(4) $W^\lambda W^\mu = W^{\lambda+\mu}$ (Cartan multiplication).
(5) When we work over an infinite field F, the $GL(n)$-submodule L_λ generated by K_λ is irreducible and it is the unique irreducible submodule of V_λ.

Proof. (1) and (2) follow from the previous analysis. In fact given any U^- submodule M and an element $\sum_i c_i T_i \in M$, a linear combination of double standard tableaux, apply to it the transformation $2+\lambda 1$. By hypothesis, for all λ this element is in M and so its leading term is also in M. Repeat the argument with the other transformations $i + \lambda j$ as in the previous proof until we get the bicanonical tableau in M.

(3) follows from (2). For (4) we have to specify the meaning of $\lambda + \mu$. Its correct meaning is by interpreting the partitions as weights for the torus. Then it is clear that a product of two weight vectors has as weight the sum of the weights. Thus $W^\lambda W^\mu \subset W^{\lambda+\mu}$. To show equality we observe that a standard tableau of shape $\lambda + \mu$ can be written as the product of two standard tableaux of shapes λ and μ.

(5) If A is a minimal submodule of W^λ it is necessarily irreducible. By (1) it must contain K_λ hence L_λ and this suffices to prove the statement. □

Remark. The above proposition is basically the theory of the highest weight vector in this case. The reader is invited to complete the representation theory of the general linear group in characteristic 0 by this combinatorial approach (as an alternative to the one developed in Chapter 9).

In general the previous theorem is interpreted by saying that W^λ is an *induced representation* of a 1-dimensional representation of B^+. The geometric way of expressing this is by taking the 1-dimensional representation F_λ of B, given by the character λ, forming the line bundle $L_\lambda := G \times_{B^+} F_\lambda$ on the flag variety G/B^+ and interpreting:

$$W^\lambda = H^0(G/B^+, L_\lambda).$$

If the reader knows the meaning of these terms it should not be difficult to prove this statement in our case. One has just to identify the sections of the line bundle with the functions on G which are eigenvectors of B^+ of the appropriate character. But it would take us too far afield to introduce this language in detail to explain it here.

If A denotes the coordinate ring of the linear group $GL(n, F)$ we know that $A = F[x_{ij}][1/d]$, where d is the determinant, and we can extend the previous theorems to study A as a representation. It is enough to remark that d is also a U^+- invariant of weight d itself and every double standard tableau is uniquely a power of d times a double standard tableau of shape λ with $ht(\lambda) \leq n-1$. Thus we obtain (cf. Chapter 9, 8.1.3 and 8.2.1):

Theorem 3. *The space A^{U^+} of functions on $GL(n, F)$ right invariant under U^+ decomposes as*

$$A^{U^+} = \bigoplus_{\lambda} \bigoplus_{k \in \mathbb{Z}} W^{\lambda}[d^k], \quad ht(\lambda) \leq n - 1.$$

We also have:

Theorem 4. *Every rational irreducible $GL(n, F)$-module is of the type $L_\lambda[d^k]$. These modules are not isomorphic.*

Proof. Given a rational irreducible $GL(n, F)$-module M we can embed M into A. Since we have a filtration of A with factors isomorphic to $W^\lambda[d^k]$ we must have a nonzero morphism of M into one of these modules. Now $W^\lambda[d^k]$ contains a unique irreducible submodule $L_\lambda[d^k]$. Hence M is isomorphic to $L_\lambda[d^k]$. The fact that these modules are not isomorphic depends on the fact that each of them contains a unique (up to constant) U^+-invariant vector of weight $d^k\lambda$, and these weights are distinct. □

5.2 Good Filtrations

A one-row double tableau which is right canonical is the determinant of a $i \times i$ minor $u_i, \dots, u_1 | 1, 2, \dots, i$ extracted from the first i columns of X. Let W^i denote the space of these tableaux. As a representation of $GL(n) = GL(W)$, W^i is isomorphic to $\bigwedge^i(W)^*$.

Similarly, a one-row double tableau which is left canonical is the determinant of a $i \times i$ minor $i, \dots, 2, 1 | v_1, \dots, v_i$ extracted from the first i rows of X. Let V_i denote the space of these tableaux. As a representation of $GL(m) = GL(V)$, V_i is isomorphic to $\bigwedge^i(V)$.

If $\lambda = k_1 \geq k_2 \geq \dots \geq k_r$ the tableaux of shape λ can be viewed as the natural tensor product basis of $W^{k_1} \otimes W^{k_2} \dots \otimes W^{k_r}$.

The straightening laws for W^λ can be viewed as elements of this tensor product, and we will call the subspace spanned by these elements R_λ. Then

$$W^\lambda := W^{k_1} \otimes W^{k_2} \dots \otimes W^{k_r} / R^\lambda.$$

Similarly, on the rows

$$V_\lambda := V_{k_1} \otimes V_{k_2} \dots \otimes V_{k_r} / R_\lambda.$$

Quite often, when dealing with right or left canonical tableaux it is better to drop the C_λ completely and write the corresponding double tableau as a single tableau (since the row or column indices are completely determined).

We can now reinterpret the straightening algorithm as the existence of a *good filtration*[136] on the algebra R of functions on matrices.

[136] The theory of good filtrations has been developed by Donkin [Do] for semisimple algebraic groups and it is an essential tool for the characteristic free theory.

Theorem 1. *(1) Given a double tableau of shape λ by the straightening algorithm it is expressed as a linear combination of standard tableaux of shapes $\geq \lambda$ and of the same double weight.*

(2) Let S_λ, resp. A_λ, denote the linear span of all tableaux of shape $\geq \lambda$ (resp. of standard tableaux of shape λ). We have

$$S_\mu := \bigoplus\nolimits_{\lambda \geq \mu,\ |\lambda|=|\mu|} A_\lambda.$$

Denote by $S'_\mu := \bigoplus_{\lambda > \mu,\ |\lambda|=|\mu|} A_\lambda$ (which has as basis the double standard tableaux of shape $> \lambda$ in the dominant ordering).

(3) The space S_μ / S'_μ is a representation of $GL(V) \times GL(W)$ equipped with a natural basis indexed by double standard tableaux $A|B$ of shape μ. When we take an operator $X \in GL(V)$ we have $X(A|B) = \sum_C c_{B,C} A|C$ where C runs over the standard tableaux and the coefficients are independent of A, and similarly for $GL(W)$.

(4) As a $GL(V) \times GL(W)$-representation we have that

$$S_\lambda / S'_\lambda \cong W^\lambda \otimes V_\lambda.$$

Proof. The first fact is Corollary 4.3.

By definition, if $\lambda := k_1, k_2, \ldots, k_i$ is a partition, we have that $T_\lambda := M_{k_1} M_{k_2} \ldots M_{k_i}$ is the span of all double tableaux of shape λ. Thus $S_\mu = \sum_{\lambda \geq \mu,\ |\lambda|=|\mu|} T_\lambda$ by (1).

Parts (3) and (4) follow from Lemma 4.3.2. We establish a combinatorial linear isomorphism j_λ between $W^\lambda \otimes V_\lambda$ and S_λ / S'_λ by setting $j_\lambda(A \otimes B) := A|B$ where A is a standard row tableau (identified to $A|C_\lambda$), B a standard column tableau (identified to $C_\lambda|B$) and $A|B$ the corresponding double tableau. From (1), j_λ is an isomorphism of $GL(m) \times GL(n)$-modules. □

Before computing explicitly we relate our work to Cauchy's formula.

In 4.2 we have identified the subspace M_k of the ring of polynomials spanned by the determinants of the $k \times k$ minors with $\bigwedge^k W^* \otimes \bigwedge^k V$. The previous theorem implies in particular that the span of all tableaux of shapes $\geq \mu$ and some fixed degree p is a quotient of a direct sum of tensor products $\tilde{T}_\lambda := M_{k_1} \otimes M_{k_2} \otimes \ldots \otimes M_{k_i}$, where $\lambda \geq \mu,\ |\lambda| = p$, modulo a subspace which is generated by the straightening relations. In other words we can view the straightening laws as a combinatorial description of a set of generators for the kernel of the map $\bigoplus_{\lambda \geq \mu,\ |\lambda|=p} \tilde{T}_\lambda \to S_\mu$. Thus we have a combinatorial description by generators and relations of the group action on S_μ.

Revert for a moment to characteristic 0. Take a Schur functor associated to a partition λ and define

$$i_\lambda : \hom(V_\lambda, W_\lambda)^* = W_\lambda^* \otimes V_\lambda \to R = S(V^* \otimes W),\ i_\lambda(\phi \otimes u)(A) := \langle \phi | Au \rangle.$$

Set $M_\lambda = i_\lambda(W_\lambda^* \otimes V_\lambda)$. The map i_λ is $GL(V) \times GL(W)$-equivariant, $W_\lambda^* \otimes V_\lambda$ is irreducible and we identify $M_\lambda = W_\lambda^* \otimes V_\lambda$.

To connect with our present theory we shall compute the invariants

$$(W_\lambda^* \otimes V_\lambda)^{U^- \times U^+} = (W_\lambda^*)^{U^-} \otimes (V_\lambda)^{U^+}.$$

From the highest weight theory of Chapter 10 we know that V_λ has a unique U^+-fixed vector of weight λ (cf. Chapter 10, §5.2) while W_λ^* has a unique U^--fixed vector of weight $-\lambda$. It follows that the space $(W_\lambda^*)^{U^-} \otimes (V_\lambda)^{U^+}$ is formed by the multiples of the bicanonical tableau K_λ.

Theorem 2. *In characteristic* 0, *if* $\mu \vdash p$:

$$S_\mu = \bigoplus_{\substack{|\lambda| \leq \min(m,n), \\ \mu \leq \lambda, \ \lambda \vdash p}} W_\lambda^* \otimes V_\lambda$$

S_μ / S'_μ *is isomorphic to* $W_\mu^* \otimes V_\mu$.

Proof. We can apply the highest weight theory and remark that the highest weight of $W_\lambda^* \otimes V_\lambda$ under $U^- \times U^+$ is the bicanonical tableau of shape λ (since it is the only $U^- \times U^+$ invariant of the correct weight). Thus to identify the weights λ for which $W_\lambda^* \otimes V_\lambda \subset S_\mu$ it suffices to identify the bicanonical tableaux in S_μ. From the basis by standard tableaux we know that the $U^- \times U^+$ fixed vectors in S_μ are the linear combinations of the bicanonical tableaux K_λ for $|\lambda| \leq \min(m,n)$, $\mu \leq \lambda$, $\lambda \vdash p$. □

Over the integers or in positive characteristic we no longer have the direct sum decomposition. The group $GL(n)$ or $SL(n)$ is not linearly reductive and rational representations do not decompose into irreducibles. Nevertheless, often it is enough to use particularly well behaved filtrations. It turns out that the following is useful:

Definition. Given a polynomial representation P of $GL(m)$ a *good filtration* of P is a filtration by $GL(m)$ submodules such that the quotients are isomorphic to the modules V_λ.

5.3 $SL(n)$

Now we want to apply this theory to the special linear group.

We take double tableaux for an $n \times n$ matrix $X = (x_{ij})$. Call $A := F[x_{ij}]$ and observe that $d = \det(X) = (n, \dots, 1 | 1, \dots, n)$ is the first coordinate, so the double standard tableaux with at most $n-1$ columns are a basis of A over the polynomial ring $F[d]$. Hence, setting $d = 1$ in the quotient ring $A/(d-1)$ the double standard tableaux with at most $n-1$ columns are a basis over F. Moreover, d is invariant under the action of $SL(n) \times SL(n)$ and thus $A/(d-1)$ is an $SL(n) \times SL(n)$-module.

We leave to the reader to verify that $A/(d-1)$ is the coordinate ring of $SL(n)$ and its $SL(n) \times SL(n)$-module action corresponds to the let and right group actions, and that the image of the V_λ for λ with at most $n-1$ columns give a decomposition of $A/(d-1)^{U^+}$ (similarly for W^μ).

We want now to analyze the map $\overline{f}(g) := f(g^{-1})$ which exchanges left and right actions on standard tableaux.

For this remark that the inverse of a matrix X of determinant 1 is the adjugate $\bigwedge^{n-1} X$. More generally consider the pairing $\bigwedge^k F^n \times \bigwedge^{n-k} F^n \to \bigwedge^n F^n = F$ under which

$$\left\langle \bigwedge^k X(u_1 \wedge \cdots \wedge u_k) | \bigwedge^{n-k} X(v_1 \wedge \cdots \wedge v_{n-k}) \right\rangle$$
$$= \bigwedge^n X(u_1 \wedge \cdots \wedge u_k \wedge v_1 \wedge \cdots \wedge v_{n-k})$$
$$= u_1 \wedge \cdots \wedge u_k \wedge v_1 \wedge \cdots \wedge v_{n-k}.$$

If we write everything in matrix notation the pairing between basis elements of the two exterior powers is a diagonal $\binom{n}{k}$ matrix of signs ± 1 that we denote by J_k. We thus have:

Lemma. *There is an identification between* $(\bigwedge^k X^{-1})^t$ *and* $J_k \bigwedge^{n-k} X$.

Proof. From the previous pairing and compatibility of the product with the operators $\bigwedge X$ we have

$$\left(\bigwedge^k X\right)^t J_k \bigwedge^{n-k} X = 1_{\binom{n}{k}}.$$

Thus

$$\left(\bigwedge^k X^{-1}\right)^t = J_k \bigwedge^{n-k} X.$$

This implies that under the map $f \to \overline{f}$ a determinant $(i_1 \dots i_k | j_1 \dots j_k)$ of a k minor is transformed, up to sign, into the $n-k$ minor with complementary row and column indices. □

Corollary. $f \to \overline{f}$ *maps* V_λ *isomorphically into* W^μ *where, if* λ *has rows* $k_1, k_2, \dots, k_r$, *then* μ *has rows* $n - k_r, n - k_{r-1}, \dots, n - k_1$.

5.4 Branching Rules

Let us recover in a characteristic free way the branching rule from $GL(m)$ to $GL(m-1)$ of Chapter 9, §10. Here the branching will not give a decomposition but a good filtration.

Consider therefore the module V_λ for $GL(m)$, with its basis of semistandard tableau of shape λ, filled with the indices $1, \dots, m$. First, we can decompose $V_\lambda = \bigoplus_k V_\lambda^k$ where V_λ^k has as basis the semistandard tableau of shape λ where m appears k-times. Clearly each V_λ^k is $GL(m-1)$-stable. Now take a semistandard tableau of shape λ, in which m appears k times. Erase all the boxes where m appears. We obtain a semistandard tableau filled with the indices $1, \dots, m-1$ of some shape μ, obtained from λ by removing k boxes and at most one box in each row.[137] Let us denote by A_μ the space spanned by these tableaux. Thus we can further decompose

[137] In this chapter the indexing by diagrams is dual.

as $V_\lambda^k = \bigoplus_\mu A_\mu$. When we apply an element of $GL(m-1)$ to such a tableau, we see that we obtain in general a tableau of the same shape, but not semistandard. The straightening algorithm of such a tableau will consist of two terms, $T_1 + T_2$; in T_1 the index m is not moved, while in T_2 the index m is moved to some upper row in the tableau. It is easily seen that this implies that:

Theorem. *V_λ and each V_λ^k have a good filtration for $GL(m-1)$ in which the factors are the modules V_μ for the shapes μ obtained from λ by removing k boxes and at most one box in each row.*

This is the characteristic free analogue of the results of Chapter 9, §10.3. Of course, in characteristic 0, we can split the terms of the good filtration and obtain an actual decomposition.

5.5 $SL(n)$ Invariants

Theorem. *The ring generated by the Plücker coordinates $[i_1, \dots, i_n]$ extracted from an $n \times m$ matrix is the ring of invariants under the action of the special linear group on the columns.*

Proof. If an element is $SL(n)$ invariant, it is in particular both U^-- and U^+-invariant under left action. By the analysis in 5.1 its left tableau must be at the same time canonical and anticanonical. Hence by 5.1.1 it is the tableau defining a product of maximal minors involving all rows, i.e., Plücker coordinates. □

Classically this theorem is used to prove the projective normality of the Grassmann variety and the factoriality of the ring of Plücker coordinates, which is necessary for the definition of the Chow variety.

Let us digress on this application. Given an irreducible variety $V \subset \mathbb{P}^n$ of codimension $k+1$, a generic linear subspace of $\mathbb{P}^n$ of dimension k has empty intersection with V. The set of linear subspaces which have a nonempty intersection is (by a simple dimension count) a hypersurface, of some degree u, in the corresponding Grassmann variety. Thus it can be defined by a single equation, which is a polynomial of degree u in the Plücker coordinates. This in turn can finally be seen as a point in the (large) projective space of lines in the space of standard monomials of degree u in the Plücker coordinates. This is the Chow point associated to V, which is a way to parameterize projective varieties.

6 Characteristic Free Invariant Theory

6.1 Formal Invariants

We have been working in this chapter with varieties defined over $\mathbb{Z}$ without really formalizing this concept. If we have an affine variety V over an algebraically closed field k and a subring $A \subset k$ (in our case either $\mathbb{Z}$ or a finite field), we say that V is defined over A if there is an algebra $A[V]$ such that $k[V] = A[V] \otimes_A k$. Similarly, a map of two varieties $V \to W$ both defined over A is itself defined over A if its comorphism maps $A[W]$ to $A[V]$.

For an algebraic group G to be defined over A thus means that also its group structures Δ, S are defined over A.

When a variety is defined over A one can consider the set $V[A]$ of its A-rational points. Thinking of points as homomorphisms, these are the homomorphisms of $A[V]$ to A. Although the variety can be of large dimension, the set of its A-rational points can be quite small. In any case if V is a group, $V[A]$ is also a group.

As a very simple example, we take the multiplicative group defined over $\mathbb{Z}$, its coordinate ring being $\mathbb{Z}[x, x^{-1}]$. Its $\mathbb{Z}$-rational points are invertible integers, that is, only ± 1.

More generally, if B is any A algebra, the A-homomorphisms of $A[V]$ to B are considered as the *B-rational points* of V or *points with coefficients in B*. Of course one can define a new variety defined over B by the *base change* $B[V] := A[V] \otimes_A B$.[138]

This causes a problem in the definition of invariant. If a group G acts on a variety V and the group, the variety and the action are defined over A, one could consider the invariants just under the action of the A-rational points of G. These usually are not really the invariants one wants to analyze. In order to make the discussion complete, let us go back to the case of an algebraically closed field k, a variety V and a function $f(x)$ on V. Under the G-action we have the function $f(g^{-1}x)$ on $G \times V$; f is invariant if and only if this function is independent of g, equivalently, if $f(gv)$ is independent of g. In the language of comorphism we have the comorphisms

$$\mu : k[V] \to k[G] \otimes k[V], \quad \mu(f)(g, v) := f(gv).$$

So, to say that f is invariant, is equivalent to saying that $\mu(f) = 1 \otimes f$. Furthermore in the language of morphisms, a specific point $g_0 \in G$ corresponds to a morphism $\phi : k[V] \to k$ and the function (of x only) $f(g_0 x)$ is $\phi \otimes 1 \circ \mu(f)$.

Now we leave to the reader to verify the simple:

Proposition 1. *Let G, V and the action be defined over $A \subset k$. For an element $f \in A[V]$ the following are equivalent:*

(1) $\mu(f) = 1 \otimes f$.
(2) For every commutative algebra B the function $f \otimes 1 \in B[V]$ is invariant under the group $G[B]$ of B-rational points.
(3) The function $f \otimes 1 \in k[V]$ is invariant.

If f satisfies the previous properties, then it is called an absolute invariant or just an invariant.

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[138] Actually, to be precise we should extend our notions to the idea of *affine scheme*; otherwise there are some technical problems with this definition.

One suggestive way of thinking of condition (1) is the following. Since we have defined a rational point of G in an algebra B as a homomorphism $A[G] \to B$, we can in particular consider the identity map $A[G] \xrightarrow{1} A[G]$ as a point of G with coefficients in $A[G]$. This is by definition *the generic point* of G. Thus condition (1) means that f is invariant under the action of a generic group element. The action under any group element $\phi : A[G] \to B$ is obtained by *specializing* the generic action.

It may be useful to see when invariance only under the rational points over A implies invariance. We have:

Proposition 2. *If the points $G[A]$ are Zariski dense in G, then a function invariant under $G[A]$ is an invariant.*

Proof. We have $f(x) = f(gx)$ when $g \in G[A]$. Since for any given x the function $f(gx) - f(x)$ is a regular function on G, if it vanishes on a Zariski dense subset it is identically 0. □

Exercise. Prove that if F is an infinite field and k is its algebraic closure, the rational points $GL(n, F)$ are dense in the group $GL(n, k)$.

Prove the same statement for the groups which can be parameterized by a linear space through the Cayley transform (Chapter 4, §5.1).

A similar discussion applies when we say that a vector is a weight vector under a torus defined over $\mathbb{Z}$ or a finite field. We mean that it is an absolute weight vector under any base change. We leave it to the reader to repeat the formal definition.

6.2 Determinantal Varieties

Consider now the more general theory of standard tableaux on a Schubert variety. We have remarked at the beginning of §4.1 that every Schubert cell intersects the affine set A which we have identified with the space $M_{n,m}$ of $n \times m$ matrices. The intersection of a Schubert variety with A will be called an affine Schubert variety. It is indexed by a minor a of the matrix X and indicated by S_a. The proof given in 4.2 and the remarks on the connection between projective and affine coordinate rings give:

Proposition. *Given a minor a of X the ideal of the variety S_a is generated by the determinants of the minors b which are not greater than or equal to the minor a. Its affine coordinate ring has a basis formed by the standard monomials in the determinants of the remaining minors.*

There is a very remarkable special case of this proposition. Choose the $k \times k$ minor whose row and column indices are the first indices $1, 2, \ldots, k$. One easily verifies: A minor b is not greater than or equal to a if and only if it is a minor of rank $> k$. Thus S_a is the determinantal variety of matrices of rank at most k. We deduce:

Theorem. *The ideal I_k generated by the determinants of the $(k+1)\times(k+1)$ minors is prime (in the polynomial ring $A[x_{i,j}]$ over any integral domain A).*

The standard tableaux which contain at least one minor of rank $\geq k+1$ are a basis of the ideal I_k.

The standard tableaux formed by minors of rank at most k are a basis of the coordinate ring $A[x_{i,j}]/I_k$.

Proof. The only thing to be noticed is that a determinant of a minor of rank $s > k+1$ can be expanded, by the Laplace rule, as a linear combination of determinants of $(k+1)\times(k+1)$ minors. So these elements generate the ideal defined by the Plücker coordinates which are not greater than a. □

Over a field the variety defined is the determinantal variety of matrices of rank at most k.

6.3 Characteristic Free Invariant Theory

Now we give the characteristic free proof of the first fundamental theorem for the general linear group.

Let F be an infinite field.[139] We want to show the FFT of the linear group for vectors and forms with coefficients in F.

FFT Theorem. *The ring of polynomial functions on $M_{p,m}(F)\times M_{m,q}(F)$ which are $Gl(m,\mathbb{F})$-invariant is given by the polynomial functions on $M_{p,q}(F)$ composed with the product map, which has as image the determinantal variety of matrices of rank at most m.*

Let us first establish some notation. We display a matrix $A\in M_{p,m}(F)$ as p rows ϕ_i:

$$A := \begin{vmatrix} \phi_1 \\ \phi_2 \\ \dots \\ \phi_p \end{vmatrix}$$

and a matrix $B\in M_{m,q}(F)$ as q columns x_i:

$$B := \begin{vmatrix} x_1 & x_2 & \dots & x_p \end{vmatrix}.$$

The entries of the product are the *scalar products* $\overline{x}_{ij} := \langle \phi_i | x_j \rangle$.

The theory developed for the determinantal variety implies that the double standard tableaux in these elements $\overline{x}_{ij}$ with at most m columns are a basis of the ring A_m generated by these elements.

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[139] One could relax this by working with formal invariants.

Lemma. *Assume that an element $P := \sum c_i T_i \in A_m$, with T_i distinct double standard tableaux, vanishes when we compute it on the variety C_m formed by those pairs A, B of matrices for which the first m columns x_i of B are linearly dependent; then the column tableau of each T_i starts with the row $1, 2, \dots, m$.*

Similarly if P vanishes when we compute it on the variety R_m formed by those pairs A, B of matrices for which the first m rows ϕ_i of A are linearly dependent, then the row tableau of each T_i starts with the row $m, m-1, \dots, 1$.

Proof. First, it is clear that every double standard tableau with column tableau starting with the row $1, 2, \dots, m$ vanishes on C_m. If we split $P = P_0 + P_1$ with P_0 of the previous type, then also P_1 vanishes on C_m. We can thus assume $P = P_1$, and we must show that $P_1 = 0$.

Decompose $P = Q + R$ where R is the sum of all tableaux which do not contain 1 among the column indices. When we evaluate P in the subvariety of $M_{p,m}(F) \times M_{m,q}(F)$ where $x_1 = 0$, we get that R vanishes identically, hence Q vanishes on this variety. But this variety is just of type $M_{p,m}(F) \times M_{m,q-1}(F)$. We deduce that R is a relation on the double standard tableaux in the indices $1, \dots, p; 2, \dots, q$. Hence $R = 0$ and $P = Q$.

Next, by substituting $x_1 \to x_1 + \lambda x_2$ in P we have a polynomial vanishing identically on C_m. Hence its leading term vanishes on C_m. This leading term is a linear combination of double standard tableaux obtained from some of the T_i by substituting all 1's not followed by 2 with 2's.

Next we perform the substitutions $x_1 + \lambda x_3, \dots, x_1 + \lambda x_m$, and in a similar fashion we deduce a new leading term in which the 1's which are not followed by $2, 3, \dots, m$ have been replaced with larger indices.

Formally this step does not immediately produce a standard tableau, for instance if we have a row $1\,2\,3\,7 \dots$ and replace 1 by 4 we get $4\,2\,3\,7 \dots$, but this can be immediately rearranged up to sign to $2\,3\,4\,7 \dots$.

Since by hypothesis P does not contain any tableau with first row in the right side equal to $1, 2, 3, \dots, m$, at the end of this procedure we must get a nontrivial linear combination of double standard tableaux in which 1 does not appear in the column indices and vanishing on C_m. This, we have seen, is a contradiction. The proof for the rows is identical. □

At this point we are ready to prove the FFT.

Proof. We may assume $p \geq m, \; q \geq m$ and consider

$$d := (m, m-1, \dots, 1 | 1, 2, \dots, m).$$

Let $\mathcal{A}$ be the open set in the variety of matrices of rank $\leq m$ in $M_{p,q}(F)$ where $d \neq 0$. Similarly, let $\mathcal{B}$ be the open set of elements in $M_{p,m}(F) \times M_{m,q}(F)$ which, under multiplication, map to $\mathcal{A}$.

The space $\mathcal{B}$ can be described as pairs of matrices in block form, with multiplication:

$$\begin{vmatrix} A \\ B \end{vmatrix}, \begin{vmatrix} C & D \end{vmatrix} \xrightarrow{\pi} \begin{vmatrix} AC & AD \\ BC & BD \end{vmatrix}$$

and AC invertible.

The complement of $\mathcal{B}$ is formed by those pairs of matrices (A, B) in which either the first m columns x_i of B or the first m rows ϕ_j of A are linearly dependent, i.e., in the notations of the Lemma it is $C_m \cup R_m$.

Finally, setting $\mathcal{B}' := \left\{ \left(\begin{vmatrix} 1_m \\ B \end{vmatrix}, \begin{vmatrix} C & D \end{vmatrix} \right) \right\}$ with C invertible, we get that $\mathcal{B}$ is isomorphic to the product $GL(m, F) \times \mathcal{B}'$ by the map

$$\left(A, \left(\begin{vmatrix} 1_m \\ B \end{vmatrix}, \begin{vmatrix} C & D \end{vmatrix} \right) \right) \mapsto \left(\begin{vmatrix} A^{-1} \\ BA^{-1} \end{vmatrix}, \begin{vmatrix} AC & AD \end{vmatrix} \right).$$

By multiplication we get

$$\begin{vmatrix} 1_m \\ B \end{vmatrix} \begin{vmatrix} C & D \end{vmatrix} = \begin{vmatrix} C & D \\ BC & BD \end{vmatrix}.$$

This clearly implies that the matrices $\mathcal{B}'$ are isomorphic to $\mathcal{A}$ under multiplication and that they form a section of the quotient map π. It follows that the invariant functions on $\mathcal{B}$ are just the coordinates of $\mathcal{A}$. In other words, after inverting d, the ring of invariants is the ring of polynomial functions on $M_{p,q}(F)$ composed with the product map.

We want to use the theory of standard tableaux to show that this denominator can be eliminated. Let then f be a polynomial invariant. By hypothesis f can be multiplied by some power of d to get a polynomial coming from $M_{p,q}(F)$.

Now we take a minimal such power of d and will show that it is 1.

For this we remark that fd^h, when $h \geq 1$, vanishes on the complement of $\mathcal{B}$ and so on the complement of $\mathcal{A}$. Now we only have to show that a polynomial on the determinantal variety that vanishes on the complement of $\mathcal{A}$ is a multiple of d.

By the previous lemma applied to columns and rows we see that each first row of each double standard tableau T_i in the development of fd^h is $(m, m-1, \dots, 1|1, 2, \dots, m)$, i.e., d divides this polynomial, as desired. □

7 Representations of S_n

7.1 Symmetric Group

We want to recover now, and generalize in a characteristic free way, several points of the theory developed in Chapter 9.

Theorem 1. *If V is a finite-dimensional vector space over a field F with at least $m+1$ elements, then the centralizer of $G := GL(V)$ acting on $V^{\otimes m}$ is spanned by the symmetric group.*

Proof. We start from the identification of $\mathrm{End}_G V^{\otimes m}$ with the invariants $(V^{*\otimes m} \otimes V^{\otimes m})^G$. We can clearly restrict to $G = SL(V)$ getting the same invariants.

Now we claim that the elements of $(V^{*\otimes m} \otimes V^{\otimes m})^G$ are invariants for any extension of the field F, and so are multilinear invariants. Then we have that the multilinear invariants as described by Theorem 6.3 are spanned by the products $\prod_{i=1}^{m} \langle \alpha_{\sigma(i)} | x_i \rangle$, which corresponds to σ, and the theorem is proved.

To see that an invariant $u \in (V^{*\otimes m} \otimes V^{\otimes m})^{SL(V)}$ remains invariant over any extension field, it is enough to show that u is invariant under all the elementary transformations $1 + \lambda e_{ij}$, $\lambda \in SL(V)$, since these elements generate the group $SL(V)$.

If we write the condition of invariance $u(1+\lambda e_{ij}) = (1+\lambda e_{ij})u$, we see that it is a polynomial in λ of degree $\leq m$ and by hypothesis vanishes on F. By the assumption that F has at least $m + 1$ elements it follows that this polynomial is identically 0. $\square$

Next, we have seen in Corollary 4.3 that the space of double tableaux of given double weight has as basis the standard bi-tableaux of the same weight. We want to apply this idea to *multilinear tableaux.*

Let us start with a remark on tensor calculus.

Let V be an n-dimensional vector space. Consider $V^{*\otimes m}$, the space of multilinear functions on V. Let e_i, $i = 1, \dots, n$, be a basis of V and e^i the dual basis. The elements $e^{i_1} \otimes e^{i_2} \otimes \cdots \otimes e^{i_m}$ form an associated basis of $V^{*\otimes m}$. In functional notation $V^{*\otimes m}$ is the space of multilinear functions $f(x_1, \dots, x_m)$ in the arguments $x_i \in V$.

Writing $x_i := \sum x_{ji} e_j$ we have

$$\langle e^{i_1} \otimes e^{i_2} \otimes \cdots \otimes e^{i_m} | x_1 \otimes \ldots \otimes x_m \rangle = \prod_{h=1}^{m} x_{i_h h}. \tag{7.1.1}$$

Thus the space $V^{*\otimes m}$ is identified to the subspace of the polynomials in the variables x_{ij}, $i = 1, \dots n$; $j = 1, \dots, m$, which are *multilinear* in the right indices $1, 2, \dots, m$.

From the theory of double standard tableaux it follows immediately that:

Theorem 2. *$V^{*\otimes m}$ has as basis the double standard tableaux T of size m which are filled with all the indices $1, 2, \dots, m$ without repetitions in the column tableau and with the indices from $1, 2, \dots, n$ (with possible repetitions) in the row tableau.*

To these tableau we can apply the theory of §5.3. One should remark that on $V^{*\otimes m}$ we obviously do not have the full action of $GL(n) \times GL(m)$ but only of $GL(n) \times S_m$, where $S_m \subset GL(m)$ as permutation matrices.

Corollary. *(1) Given a multilinear double tableau of shape λ by the straightening algorithm it is expressed as a linear combination of multilinear standard tableaux of shapes $\geq \lambda$.*

(2) Let S_λ^0, resp. A_λ^0, denote the linear span of all multilinear double standard tableaux tableaux of shape $\geq \lambda$, resp. of multilinear double standard tableaux of shape λ. We have

$$S^0_\mu := \bigoplus_{\substack{\lambda\geq\mu,\\ |\lambda|=|\mu|}} A^0_\lambda.$$

Denote by $S^1_\mu := \bigoplus_{\lambda>\mu,\ |\lambda|=|\mu|} A^0_\lambda$ *(which has as basis the multilinear double standard tableaux of shape* $> \lambda$ *in the dominance ordering).*

(3) The space S^0_μ/S^1_μ *is a representation of* $GL(n) \times S_m$ *equipped with a natural basis indexed by double standard tableaux* $A|B$ *of shape* μ *and with* B *doubly standard (or multilinear).*

It is isomorphic to the tensor product $V_\lambda \otimes M_\lambda$ *with* V_λ *a representation of* $GL(n)$ *with basis the standard tableaux of shape* λ *and* M_λ *a representation of* S_m *with basis the multilinear standard tableaux of shape* λ.

The proof is similar to 5.2, and so we omit it. In both cases the straightening laws give combinatorial rules to determine the actions of the corresponding groups on the basis of standard diagrams.

7.2 The Group Algebra

Let us consider in $\mathbb{Z}[x_{ij}]$, $i, j = 1, \dots, n$, the space Σ_n spanned by the monomials of degree n multilinear in both the right and left indices.

These monomials have as basis the $n!$ monomials $\prod_{i=1}^n x_{\sigma(i)i} = \prod_{j=1}^n x_{j\sigma^{-1}(j)}$, $\sigma \in S_n$ and also the double standard tableaux which are multilinear or doubly standard both on the left and the right.

Proposition. *The map* $\phi : \mathbb{Z}[S_n] \to \Sigma_n$*, defined by* $\phi : \sigma \to \prod_{i=1}^n x_{\sigma(i)i}$, *is an* $S_n \times S_n$ *linear isomorphism, where on the group algebra* $\mathbb{Z}[S_n] \to \Sigma_n$ *we have the usual left and right actions, while on* Σ_n *we have the two actions on left and right indices.*

Proof. By construction it is an isomorphism of abelian groups and

$$\phi(abc^{-1}) = \prod_{i=1}^n x_{(abc^{-1})(i)i} = \prod_{i=1}^n x_{a(b(i))\,c(i)}. \qquad \square$$

As in the previous theory, we have a filtration by the shape of double standard tableaux (this time multilinear on both sides or *bimultilinear*) which is stable under the $S_n \times S_n$ action. The factors are tensor products $M^\lambda \otimes M_\lambda$. It corresponds, in a characteristic free way, to the decomposition of the group algebra in its simple ideals.

Corollary. *(1) Given a bimultilinear double tableau of shape* λ*, by the straightening algorithm it is expressed as a linear combination of bimultilinear standard tableaux of shapes* $\geq \lambda$.

(2) Let S^{00}_λ*, resp.* A^{00}_λ*, denote the linear span of all bimultilinear tableaux of shape* $\geq \lambda$*, resp. of bimultilinear standard tableaux of shape* λ*. We have*

$$S^{00}_\mu := \bigoplus_{\substack{\lambda\geq\mu,\\ |\lambda|=|\mu|}} A^{00}_\lambda.$$

Denote by $S^{11}_\mu := \bigoplus_{\lambda > \mu,\ |\lambda|=|\mu|} A^{00}_\lambda$ *(which has as basis the multilinear double standard tableaux of shape* $> \lambda$ *in the dominant ordering).*

(3) The space S^{00}_μ / S^{11}_μ *is a representation of* $S_n \times S_n$ *equipped with a natural basis indexed by the double doubly standard (or bimultilinear) tableaux* $A|B$ *of shape* μ.

It is isomorphic to the tensor product $M^\lambda \otimes M_\lambda$ *with* M^λ *a representation of* S_n *with basis the left multilinear standard tableaux of shape* λ *and* M_λ*, a representation of* S_n *with basis the right multilinear standard tableaux of shape* λ.

The proof is similar to 5.2, and so we omit it.

Again one could completely reconstruct the characteristic 0 theory from this approach.

7.3 Kostka Numbers

Let us consider in the tensor power $V^{*\otimes m}$ the tensors of some given weight $h_1, h_2, \dots, h_m$, $\sum h_i = m$, i.e., the span of the tensors $e^{i_1} \otimes e^{i_2} \otimes \cdots \otimes e^{i_m}$ in which the indices $i_1, i_2, \dots, i_m$ contain 1 h_1 times, 2 h_2 times, and so on. These tensors are just the S_m orbit of $(e^1)^{h_1} \otimes (e^2)^{h_2} \otimes \cdots (e^m)^{h_m}$ and, as a representation of S_m, they give the permutation representation on $S_m / S_{h_1} \times \dots \times S_{h_m}$. By the theory of standard tableaux this space has also a basis of double tableaux $A|B$ where A is standard and B semistandard of weight $\mu := h_1, h_2, \dots, h_m$. In characteristic 0 we thus obtain:

Theorem. *The multiplicity of the irreducible representation* M_λ *of* S_m *in the permutation representation on* $S_m / S_{h_1} \times \dots \times S_{h_m}$ *(Kostka number) is the number of semistandard tableaux* B *of shape* λ *and of weight* μ.

In positive characteristic, we replace the decomposition with a good filtration.

8 Second Fundamental Theorem for GL and S_m

8.1 Second Fundamental Theorem for the Linear Group

Given an m-dimensional vector space V over an infinite field F, the first fundamental theorem for the general linear group states that the ring of polynomial functions on $(V^*)^p \times V^q$ which are $GL(V)$-invariant is generated by the functions $\langle \alpha_i | v_j \rangle$.

Equivalently, the ring of polynomial functions on $M_{p,m} \times M_{m,q}$ which are $Gl(m, F)$-invariant is given by the polynomial functions on $M_{p,q}$ composed with the product map, which has as image the determinantal variety of matrices of rank at most m. Thus Theorem 6.2 can be interpreted as:

Theorem (Second fundamental theorem for the linear group). *Every relation among the invariants* $\langle \alpha_i | v_j \rangle$ *is in the ideal* I_m *generated by the determinants of the* $(m+1) \times (m+1)$ *minors of the matrix formed by the* $\langle \alpha_i | v_j \rangle$.

8.2 Second Fundamental Theorem for the Symmetric Group

We have seen that the space of $GL(V)$-endomorphisms of $V^{\otimes n}$ is spanned by the symmetric group S_n. We have a linear isomorphism between the space of operators on $V^{\otimes n}$ spanned by the permutations and the space of multilinear invariant functions.

To a permutation σ corresponds f_σ:

$$f_\sigma(\alpha_1, \alpha_2, \dots, \alpha_n, v_1, v_2, \dots, v_n) = \prod_{i=1}^{n} \langle \alpha_{\sigma i} | v_i \rangle.$$

More formally, f_σ is obtained by evaluating the variables x_{hk} in the invariants $\langle \alpha_h | v_k \rangle$ in the monomial $\prod_{i=1}^n x_{\sigma i, i}$. We want to analyze the relations among these invariants. We know that such relations are the intersection of the linear span of the given monomials with the determinantal ideal I_k (cf. §6.2).

Now the span of the multilinear monomials $\prod_{i=1}^n x_{\sigma i, i}$ is the span of the double tableaux with n boxes in which both the right and left tableau are filled with the n distinct integers $1, \dots, n$.

Theorem. *The intersection of the ideal I_k with the span of the multilinear monomials corresponds to the two-sided ideal of the algebra of the symmetric group S_n generated by the antisymmetrizer $\sum_{\sigma \in S_{k+1}} \epsilon_\sigma \sigma$ in $k+1$ elements.*

Proof. By the previous paragraph it is enough to remark that this antisymmetrizer corresponds to the polynomial

$$(k+1, k, \dots, 2, 1 | 1, 2, \dots, k, k+1) \prod_{j=k+2}^{m} (j|j),$$

and then apply the symmetric group on both sets of indices, and the straightening laws. □

8.3 More Standard Monomial Theory

We have seen in Chapter 11, §4 the two plethysm formulas 4.5.1, 4.5.2 for $S(S^2(V))$ and $S(\bigwedge^2(V))$. We want to now give a combinatorial interpretation of these
formulas.

We think of the first algebra over $\mathbb{Z}$ as the polynomial ring $\mathbb{Z}[x_{ij}]$ in a set of variables x_{ij} subject to the symmetry condition $x_{ij} = x_{ji}$, while the second algebra is the polynomial ring $\mathbb{Z}[y_{ij}]$ is a set of variables y_{ij}, $i \neq j$, subject to the skew symmetry condition $y_{ij} = -y_{ji}$.

In the first case we will display the determinant of a $k \times k$ minor extracted from the rows $i_1, i_2, \dots, i_k$ and columns $j_1, j_2, \dots, j_k$ as a two-row tableau:

$$(8.3.1) \qquad \begin{vmatrix} i_1, i_2, \dots, i_k \\ j_1, j_2, \dots, j_k \end{vmatrix}.$$

The main combinatorial identity is this:

Lemma. *If we fix any index a and consider the $k+1$ indices $i_a, i_{a+1}, \dots, i_k, j_1, j_2, \dots, j_a$, then alternating the two-row tableau in these indices produces* 0.

Proof. The proof is by decreasing induction on a. Since this is a formal identity we can work in $\mathbb{Q}[x_{ij}]$. It is convenient to rename these indices $u_{a+1}, u_{a+2}, \dots, u_{k+1}, u_1, u_2, \dots, u_a$.

We start by proving the result for the case $a = k$, which is the identity

$$\begin{vmatrix} i_1, i_2, \dots, i_{k-1}, s \\ j_1, j_2, \dots, j_{k-1}, j_k \end{vmatrix} = \sum_{p=1}^{k} \begin{vmatrix} i_1,\ i_2,\ i_3, \dots\dots, \ i_{k-1}, j_p \\ j_1, \dots, j_{p-1}, s, j_{p+1}, \dots, j_k \end{vmatrix}.$$

To prove this identity expand the determinants appearing on the right-hand side with respect to the last row:

$$\sum_{p=1}^{k} \begin{vmatrix} i_1,\ i_2,\ i_3, \dots\dots, \ i_{k-1}, j_p \\ j_1, \dots, j_{p-1}, s, j_{p+1}, \dots, j_k \end{vmatrix}$$

$$= \sum_{p=1}^{k} \left(\sum_{u=1}^{p-1} (-1)^{n+u} \begin{vmatrix} j_p \\ j_u \end{vmatrix} \begin{vmatrix} i_1, i_2, \dots\dots\dots\dots\dots\dots\dots, i_{k-2}, i_{k-1} \\ j_1, j_2, \dots, \check{j}_u, \dots, j_{p-1}, s, j_{p+1}, \dots, j_k \end{vmatrix} \right.$$

$$+ (-1)^{n+p} \begin{vmatrix} j_p \\ s \end{vmatrix} \begin{vmatrix} i_1, i_2, \dots\dots\dots\dots\dots, i_{k-1} \\ j_1, j_2, \dots, j_{p-1}, j_{p+1}, \dots, j_k \end{vmatrix}$$

$$\left. + \sum_{u=p+1}^{k} (-1)^{n+u} \begin{vmatrix} j_p \\ j_u \end{vmatrix} \begin{vmatrix} i_1, i_2, \dots\dots\dots\dots\dots\dots\dots\dots, i_{k-1} \\ j_1, j_2, \dots, j_{p-1}, s, j_{p+1}, \dots, \check{j}_u, \dots, j_k \end{vmatrix} \right).$$

The right-hand side of the above equals

$$\sum_{p=1}^{k} \left(\sum_{u=1}^{p-1} (-1)^{n+u} \begin{vmatrix} j_p \\ j_u \end{vmatrix} \begin{vmatrix} i_1, i_2, \dots\dots\dots\dots\dots\dots\dots, i_{k-1} \\ j_1, j_2, \dots, \check{j}_u, \dots, j_{p-1}, s, j_{p+1}, \dots, j_k \end{vmatrix} \right)$$

$$+ \sum_{u=1}^{k} \left(\sum_{p=u+1}^{k} (-1)^{n+p} \begin{vmatrix} j_u \\ j_p \end{vmatrix} \begin{vmatrix} i_1, i_2, \dots\dots\dots\dots\dots\dots\dots\dots, i_{k-1} \\ j_1, j_2, \dots, j_{u-1}, s, j_{u+1}, \dots, \check{j}_p, \dots, j_k \end{vmatrix} \right)$$

$$+ \sum_{p=1}^{k} (-1)^{n+p} \begin{vmatrix} j_p \\ s \end{vmatrix} \begin{vmatrix} i_1, i_2, \dots\dots\dots\dots\dots, i_{k-1} \\ j_1, j_2, \dots, j_{p-1}, j_{p+1}, \dots, j_k \end{vmatrix}.$$

The first two terms of the above expression cancel and the last is the expansion of $\begin{vmatrix} i_1, i_2, \dots, i_{k-1}, s \\ j_1, j_2, \dots\dots, j_k \end{vmatrix}$.

Suppose the lemma is proved for some $a+1$. We want to prove it for a. Compute

$$\sum_{\sigma \in S_{k+1}} \epsilon_\sigma \begin{vmatrix} i_1, i_2, \dots, i_a, u_{\sigma(a+1)}, u_{\sigma(a+2)}, \dots, u_{\sigma(k+1)} \\ u_{\sigma(1)}, u_{\sigma(2)}, \dots, u_{\sigma(a)}, \ j_{a+1}, \dots, j_{k-1}, \ j_k \end{vmatrix} = \sum_{\sigma \in S_{k+1}} \epsilon_\sigma$$

$$\times \left(\sum_{b=1}^{a} \begin{vmatrix} i_1, i_2, \dots, i_a, u_{\sigma(a+1)}, u_{\sigma(a+2)}, \dots, u_{\sigma(k)}, u_{\sigma(b)} \\ u_{\sigma(1)}, \dots, u_{\sigma(b-1)}, u_{\sigma(k+1)}, u_{\sigma(b+1)}, \dots, u_{\sigma(a)}, \ j_{a+1}, \dots, j_{k-1}, \ j_k \end{vmatrix} \right.$$

$$\left. + \sum_{b=a+1}^{k} \begin{vmatrix} i_1, i_2, \dots, i_a, u_{\sigma(a+1)}, u_{\sigma(a+2)}, \dots, u_{\sigma(k)}, j_b \\ u_{\sigma(1)}, \dots, u_{\sigma(a)}, \ j_{a+1}, \dots, j_{b-1}, u_{\sigma(k+1)}, j_{b+1}, j_{k-1}, \ j_k \end{vmatrix} \right)$$

$$= -a \sum_{\sigma \in S_{k+1}} \epsilon_\sigma \begin{vmatrix} i_1, i_2, \dots, i_a, u_{\sigma(a+1)}, u_{\sigma(a+2)}, \dots, u_{\sigma(k+1)} \\ u_{\sigma(1)}, u_{\sigma(2)}, \dots, u_{\sigma(a)}, \ j_{a+1}, \dots, j_{k-1}, \ j_k \end{vmatrix} + \sum_{b=a+1}^{k} (-1)^{k-b-1}$$

$$\times \sum_{\sigma \in S_{k+1}} \epsilon_\sigma \begin{vmatrix} i_1, i_2, \dots, i_a, j_b, u_{\sigma(a+1)}, u_{\sigma(a+2)}, \dots, u_{\sigma(k)} \\ u_{\sigma(1)}, \dots, u_{\sigma(a)}, u_{\sigma(k+1)}, \ j_{a+1}, \dots, j_{b-1}, j_{b+1}, j_{k-1}, \ j_k \end{vmatrix}.$$

By induction this last sum is 0 and we have

$$(1+a) \sum_{\sigma \in S_{k+1}} \epsilon_\sigma \begin{vmatrix} i_1, i_2, \dots, i_a, u_{\sigma(a+1)}, u_{\sigma(a+2)}, \dots, u_{\sigma(k+1)} \\ u_{\sigma(1)}, u_{\sigma(2)}, \dots, u_{\sigma(a)}, \ j_{a+1}, \dots, j_{k-1}, \ j_k \end{vmatrix} = 0. \qquad \square$$

Recall that in order to alternate a function which is already alternating on two sets of variables, it is sufficient to alternate it over the coset representatives of $S_{k+1}/S_{k+1-a} \times S_a$. We will use the previous relation in this form.

Let us take any product of minors of a symmetric matrix displayed as in 8.3.1. We obtain a tableau in which each type of row appearing appears an even number of times. In other words, the columns of the tableau are all even. We deduce:

Theorem. *In the case of symmetric variables $x_{i,j} = x_{j,i}$, the standard tableaux with even columns form a $\mathbb{Z}$-basis of $\mathbb{Z}[x_{ij}]$.*

Proof. A product of variables x_{ij} is a tableau (with just one column). We show first that every tableau is a linear combination of standard ones.

So we look at a violation of standardness in the tableau. This can occur in two different ways since a tableau is a product $d_1 d_2 \dots d_s$ of determinants of minors.

The first case is when the violation appears in two indices $i_a > j_a$ of a minor, displayed as $d_k = \begin{vmatrix} i_1, i_2, \dots, i_k \\ j_1, j_2, \dots, j_k \end{vmatrix}$. The identity in the previous lemma implies immediately that this violation can be removed by replacing the tableau with lexicographically smaller ones. The second case is when the violation occurs between a column index of some d_k and the corresponding row index of d_{k+1}. Here we can use the fact that by symmetry we can exchange the rows with the column indices in a minor and then we can apply the identity on double tableaux discussed in §4.3. The final result is to express the given tableau as a linear combination of tableaux which are either of strictly higher shape or lexicographically inferior to the given one. Thus this straightening algorithm terminates.

In order to prove that the standard tableaux so obtained are linearly independent, one could proceed as in the previous sections. Alternatively, we can observe that since standard tableaux of a given shape are, in characteristic 0, in correspondence with a basis of the corresponding linear representation of the linear group, the proposed basis is in each degree k (by the plethysm formula) of cardinality equal to the dimension of $S^k[S^2(V)]$, and so, being a set of linear generators, it must be a basis. $\square$

8.4 Pfaffians

For the symplectic case $\mathbb{Z}[y_{ij}]$, $i, j = 1, \dots, n$, subject to the skew symmetry, we define, for every sequence $1 \leq i_1 < i_2 < \cdots < i_{2k} \leq n$ formed by an even number of indices, the symbol $|i_1, i_2, \dots, i_{2k}|$ to denote the Pfaffian of the principal minor of the skew matrix $Y = (y_{ij})$. A variable y_{ij}, $i < j$ equals the Pfaffian $|ij|$.

A product of such Pfaffians can be displayed as a tableau with even rows, thus a product of variables y_{ij} is a tableau with two columns. The theorem in this case is:

Theorem 1. *The standard tableaux with even rows form a $\mathbb{Z}$-basis of $\mathbb{Z}[y_{ij}]$.*

The proof of this theorem is similar to the previous one. In order to show that every tableau is a linear combination of standard ones we need an identity between Pfaffians, which produces the straightening algorithm.

Lemma 1. *The a_i, b_j are indices among $1, \dots, n$:*

$$[a_1, \dots, a_p][b_1, \dots, b_m] - \sum_{h=1}^{p} [a_1, \dots, a_{h-1}, b_1, a_{h+1}, \dots a_p][a_h, b_2, \dots, b_m]$$
$$= \sum_{k=2}^{m} (-1)^{k-1} [b_2, \dots, \check{b}_k, \dots, b_m][b_k, b_1, a_1, \dots, a_p].$$

Proof. We use the development of a Pfaffian:

$$[a_1,\dots,a_p][b_1,\dots,b_m]-\sum_{h=1}^{p}[a_1,\dots,a_{h-1},b_1,a_{h+1},\dots a_p][a_h,b_2,\dots,b_m]$$
$$=[a_1,\dots,a_p]\left(\sum_{k=2}^{m}(-1)^k[b_1,b_k][b_2,\dots,\check{b}_k,\dots,b_m]\right)$$
$$-\sum_{h=1}^{p}[a_1,\dots,a_{h-1},b_1,a_{h+1},\dots a_p]\sum_{k=2}^{m}(-1)^k[a_h,b_k][b_2,\dots,\check{b}_k,\dots,b_m]$$
$$=\sum_{k=2}^{m}(-1)^k[b_2,\dots,\check{b}_k,\dots,b_m](-[b_k,b_1][a_1,\dots,a_p]$$
$$+(-1)^{h-1}[b_k,a_h][b_1,a_1,\dots,a_{h-1},a_{h+1},\dots,a_p])$$
$$=\sum_{k=2}^{m}(-1)^{k-1}[b_2,\dots,\check{b}_k,\dots,b_m][b_k,b_1,a_1,\dots,a_p] \qquad \square$$

We are ready now to state and prove the basic form of the straightening algorithm. First, we do it in a weak form over $\mathbb{Q}$.

Lemma 2. *a_i, b_j, c_l are indices from $1,\dots,n$:*

$$\sum_{\sigma\in S_{k+i+1}}\epsilon_\sigma[a_1,a_2,\dots,a_i,b_{\sigma(1)},\dots,b_{\sigma(k)}][b_{\sigma(k+1)},\dots,b_{\sigma(k+i+1)},c_1,\dots,c_t]$$

is a linear combination, with rational coefficients, of higher terms

$$[i_1,\dots,i_p][j_1,j_2,\dots,j_r]$$

with $p>i+k$.

Proof. We prove the statement by induction on i. If $i=0$, we can apply the previous lemma. Otherwise, assume the statement true for $i-1$ and use Lemma 1 to deduce:

$$\sum_{\sigma\in S_{k+i+1}}\epsilon_\sigma[a_1,\dots,a_i,b_{\sigma(1)},\dots,b_{\sigma(k)}][b_{\sigma(k+1)},\dots,b_{\sigma(k+i+1)},c_1,\dots,c_t]$$
$$=\sum_{\sigma\in S_{k+i+1}}\epsilon_\sigma\Bigg(\sum_{j=1}^{i}[a_1,\dots,a_{j-1},b_{\sigma(k+1)},\dots,a_{j+1},\dots,a_i,b_{\sigma(1)},\dots,b_{\sigma(k)}]$$
$$\times[a_j,b_{\sigma(k+2)},\dots b_{\sigma(k+i+1)},c_1,\dots,c_t]$$
$$+\sum_{u=1}^{k}[a_1,\dots,a_i,b_{\sigma(1)},\dots,b_{\sigma(u-1)},b_{\sigma(k+1)},b_{\sigma(u+1)},\dots,b_{\sigma(k)}]$$
$$\times[b_{\sigma(k)},b_{\sigma(k+2)},\dots,b_{\sigma(k+i+1)},c_1,\dots,c_t]\Bigg)+R$$
$$=R'-k\sum_{\sigma\in S_{k+i+1}}\epsilon_\sigma[a_1,\dots,a_i,b_{\sigma(1)},\dots,b_{\sigma(k)}]$$
$$\times[b_{\sigma(k+1)},\dots,b_{\sigma(k+i+1)},c_1,\dots,c_t]+R$$

where R are terms of *higher shape* given by Lemma 1, while R' are terms of *higher shape* given by induction. Thus

$$(1+k)\sum_{\sigma\in S_{k+i+1}}\epsilon_\sigma[a_1,\dots,a_i,b_{\sigma(1)},\dots,b_{\sigma(k)}][b_{\sigma(k+1)},\dots,b_{\sigma(k+i+1)},c_1,\dots,c_t]$$

is a sum of higher terms. □

Lemma 3. *The standard tableaux (products of Pfaffians) are a linear basis of* $\mathbb{Q}[y_{i,j}]$.

Proof. The fact that they span $\mathbb{Q}[y_{i,j}]$ comes from the fact that the previous lemma gives a straightening algorithm over $\mathbb{Q}$. The linear independence follows from the Plethysm formula and the fact that, from the representation theory of the linear group, we know that the number of standard tableaux of a given degree equals the dimension of the polynomial ring in that degree. □

We now restate, and prove Theorem 1 in a more precise form.

Theorem 2. *The standard tableaux with even rows form a* $\mathbb{Z}$*-basis of* $\mathbb{Z}[y_{ij}]$. *Moreover*

$$\sum_{\sigma\in S_{k+i+1}/S_k\times S_{i+1}}\epsilon_\sigma[a_1,a_2,\dots,a_i,b_{\sigma(1)},\dots,b_{\sigma(k)}] \times[b_{\sigma(k+1)},\dots,b_{\sigma(k+i+1)},c_1,\dots,c_t]$$

is a linear combination, with integral coefficients, of higher terms $[i_1,\dots,i_n]$ $[j_1,j_2,\dots j_r]$ *with* $n>i+k$ *and gives a straightening algorithm over* $\mathbb{Z}$.

Proof. The proof goes in two steps. In the first step we prove that, taking as coefficients an infinite field F, the given standard tableaux are linearly independent. For this we see that the proof of 2.3 applies with a little change. Namely, here the transformations $i+\lambda j$ are applied to the matrix of variables $Y=(y_{i,j})$ on rows and columns. Y transforms to a matrix whose Pfaffians are multilinear in the indices, so if i appears and j does not appear it creates a term in λ, and we can argue as in that section. Starting from a possible relation we get a relation of type $[1,2,\dots,k]^h=0$ which is not valid.

In the second step we see that, if the standard tableaux with even rows are not a $\mathbb{Z}$-basis of $\mathbb{Z}[y_{ij}]$, since they are a basis over $\mathbb{Q}$, we can specialize at some prime so that they become linearly dependent, contradicting the previous step.

As a final step, the straightening algorithm over $\mathbb{Q}$ in the end expresses a two-line tableau T as a sum of standard tableaux of the same and of higher shape. Since we have seen that the standard tableaux are a basis over $\mathbb{Z}$, this implies that the final step of the straightening algorithm must express T as an integral linear combination of tableaux. □

Exercise. Do the standard monomial theory for Pfaffians as a theory of Schubert cells for the variety of pure spinors. Interpret in this way the quadratic equations satisfied by pure spinors as the basic straightening laws (cf. Chapter 11, §7.2).

8.5 Invariant Theory

We are now going to deduce the first fundamental theorem for invariants of the orthogonal and symplectic group in all characteristics, using a method similar to the one of §6.3 for the linear group. For the second fundamental theorem the argument is like the one of §8.1.

We do first the symplectic group which is simpler.[140] For this we have to prove the usual:

Lemma 1. *If a polynomial in the skew product vanishes on the set where the first* $2n$ *elements are linearly dependent, it is a multiple of* $[1, 2, \dots, 2n]$.

The proof is similar to 5.1 and we omit it.

Theorem 1. *Over any field* F *the ring of invariants of* p *copies of the fundamental representation of* $Sp(2n, F)$ *is generated by the skew products.*

Proof. Take p copies of the fundamental representation of $Sp(2n, F)$. We may assume $p \geq 2n$ is even. We work geometrically and think of the invariants $[v_i, v_j]$ as the coordinates of a map π from $p \times 2n$ matrices to skew-symmetric $p \times p$ matrices, $\pi(T) := TJT^t$. The image is formed by the set D_{2n}^p of skew-symmetric $p \times p$ matrices of rank $\leq 2n$. The first step is thus to consider, in the variety D_{2n}^p of skew-symmetric $p \times p$ matrices of rank $\leq 2n$, the open set U where the Pfaffian $[1, 2, \dots, 2n]$ is different from 0.

The open set $\pi^{-1}(U)$ is the set of p-tuples of vectors $v_1, \dots, v_p$ with the property that the first $2n$ vectors are linearly independent. We claim that the map $\pi : \pi^{-1}(U) \to U$ is a locally trivial fibration. For each point in U there is a neighborhood W with $\pi^{-1}(W)$ equal to the product $Sp(2n, F) \times W$. In other words we want to find a section $s : W \to \pi^{-1}(W)$ so that $\pi s = 1$ and the map $Sp(2n, F) \times W \to \pi^{-1}(W)$, $(g, w) \mapsto g(s(w))$ is the required isomorphism.

In fact let A be the principal $2n \times 2n$ minor of a matrix X in U, an invertible skew-symmetric matrix. If $\pi(v_1, \dots, v_p) = X$, the entries of A are the elements $[v_i, v_j], i, j \leq 2n$.

We want to find the desired section and trivialization by interpreting the algorithm of finding a symplectic basis for the form $u^t A v$. We consider A as the matrix of a symplectic form in some basis $b_1, b_2, \dots, b_{2n}$.

First, let us analyze this algorithm which proceeds stepwise. There are two types of steps. In one type of step, we have determined $e_1, f_1, \dots, e_{i-1}, f_{i-1}$ as linear combinations of the b_i and we have to choose e_i. This is done by choosing any vector orthogonal to the previously determined ones, which in turn involves a solution of the linear system of equations $\sum_{j=1}^n x_j [b_j, e_i] = \sum_{j=1}^n x_j [b_j, f_i] = 0$. The linear system is of maximal rank but in order to solve it explicitly we have to choose an invertible maximal minor by whose determinant we have to divide. This choice depends on the initial value A_0 of A and thus the resulting formula is valid only in some open set. The choice of e_i has coordinates which are rational functions of the entries of A.

140 We correct a mistake in [DC] in the statement of the theorem.

The other type of step consists of completing e_i to f_i which is again the solution of a linear equation. The algorithm furnishes a rational function on some open set W containing any given matrix A_0, which associates to a skew matrix A a symplectic basis S written in terms of the given basis b_i, in other words a matrix $f(A)$ such that $f(A)Af(A)^t = J_{2n}$, the standard matrix of the symplectic form. The rows of $f(A)^{-1}$ define an explicit choice of vectors $v_i(A)$, depending on A through a rational function defined in a neighborhood of a given A_0, with matrix of skew products $[v_i(A), v_j(A)] = a_{i,j}$. Using the full matrix $X \in D^p_{2n}$ of skew products, of which A is a principal minor, we can complete this basis to a full p-tuple with skew products X. Since $v_k(A) = \sum_{j=1}^{2n} z_{k,j} v_j(A)$ can be solved from the identities, $x_{i,k} = [v_i(A), v_k(A)] = \sum_{j=1}^{2n} z_{k,j}[v_i(A), v_j(A)] = \sum_{j=1}^{2n} z_{k,i} a_{i,j}$.

Thus we have constructed a section $s(X) \in M_{p,2n}$ with $s(X)Js(X)^t = X$. From this the trivialization is $(X, Y) \to s(X)Y^{-1}$, $X \in U$, $Y \in Sp(2n, F)$.

Once we have proved the local triviality of the map, let us take a function on $\pi^{-1}(U)$ which is invariant under the symplectic group. On each open set $\pi^{-1}(W) = Sp(2n, F) \times W$ the function must necessarily come from a regular function on W. Since the regular functions on an algebraic variety have the *sheaf* property, i.e., a function which is locally regular is regular, we deduce that the invariant comes from a function on U.

At this point we know that if f is an invariant, after eventually multiplying it by a power of the Pfaffian $[1, 2, \dots, 2n]$ it lies in the subring generated by the skew products with basis the standard tableaux. We now have to do the cancellation. This is a consequence of Lemma 1 as in the case of the general linear group. □

We have already mentioned the fact that the orthogonal group is harder. First, we will work in characteristic $\neq 2$. In characteristic 2 there are various options in defining the orthogonal group, one being to define the orthogonal group by the equations $XX^t = 1$ as a *group scheme*, since in characteristic 2 these equations do not generate a radical ideal.

Apart from the problem of characteristic 2, the difference between the symplectic and the orthogonal group is the following. The map $X \to XJX^t$ from invertible $2n \times 2n$ matrices to invertible skew matrices is a fibration locally trivial in the Zariski topology as we have seen by the algorithm of constructing a symplectic basis. For the orthogonal group $O(V)$, $\dim(V) = n$ the map is $X \mapsto XX^t$, but the theory is not the same. In this case we start as before taking the open set U of matrices in which the determinant of the first principal minor $A := \begin{vmatrix} 1, 2, \dots, n \\ 1, 2, \dots, n \end{vmatrix}$ is invertible. We need some algorithm to construct some kind of standard basis for the space with a symmetric form given by a matrix A. In general we may try to find an orthogonal basis, otherwise a hyperbolic basis. In the first case, when we do the standard Gram–Schmidt orthogonalization, if we want to pass to an orthonormal basis (as we do) we have to extract some square roots. In the second case we still have to solve quadratic equations since we have to find isotropic vectors. In any case the formulas we will find when we want to find a section of the fibration π as in the previous case will also involve extracting square roots. The technical way of expressing this is that the fibration is locally trivial in the étale topology. In fact, apart from introducing a new technical notion the proof still works. We need to remark though that regular functions have the sheaf property also with respect to this topology. In fact in our case it is really some simple Galois theory. Alternatively we can work more geometrically.

Lemma 2. *The variety S_n^p of symmetric $p \times p$ matrices of rank $\leq n$ is smooth at the points S^0 of rank exactly n. The map $\pi : M_{p,n} \to S_n^p$, $X \mapsto XX^t$ is smooth on the points $\pi^{-1}(S^0)$.*

Proof. The group $GL(n, F)$ acts on both spaces by AX, AYA^t, $X \in M_{p,n}$, $Y \in S_n^p$ and the map π is equivariant. Since clearly any symmetric matrix of rank n can be transformed, using the action of $GL(n, F)$, to the open set U where the determinant of the first principal minor $A := \begin{vmatrix} 1, 2, \dots, n \\ 1, 2, \dots, n \end{vmatrix}$ is invertible, it is enough to show that U is smooth and that the map $\pi^{-1}(U) \to U$, $X \mapsto XX^t$ is smooth.

Let

$$X = \begin{vmatrix} A & B \\ B^t & C \end{vmatrix} \in U, \quad \det(A) \neq 0, \quad \operatorname{rank}(X) = n.$$

Next we claim that U projects isomorphically to the pairs A, B with $\det(A) \neq 0$ and B an $n \times (n - p)$ matrix. In fact the entries of the matrix C are determined and are of the form $f_{i,j}(A, B)/\det(A)$ with $f_{i,j}(A, B)$ polynomials.

To prove this, take the $(n + 1) \times (n + 1)$ minor where to A we add the row i and the column j. By hypothesis its determinant is 0, but this determinant is $\det(A)c_{i,j} + f_{i,j}(A, B)$ (where $f_{i,j}(A, B)$ are the remaining terms of the expansion of the determinant on the last column).

Using this isomorphism we can see that π is a smooth map. In fact compute the differential at a point X by the method explained in Chapter 8, §7.3, substituting $(X + Y)(X + Y)^t$ collecting linear terms $XY^t + YX^t$. Write both $X = \begin{vmatrix} T \\ V \end{vmatrix}$, $Y = \begin{vmatrix} U \\ W \end{vmatrix}$ in block form with U, W square $n \times n$ matrices. Now the linear terms of the differential read:

$$\begin{vmatrix} TU^t + UT^t & TW^t + UV^t \\ VU^t + WT^t & VW^t + UV^t \end{vmatrix} \to TU^t + UT^t, TW^t + UV^t$$

at a point in which T is invertible. Let the target matrix be a pair (C, D) with C symmetric.

If the characteristic is different from 2, we can solve $TU^t + UT^t = C, \quad TW^t + UV^t = D$ setting $U := C(T^t)^{-1}/2$, $W^t = T^{-1}(UV^t - D)$. Thus $d\pi$ is surjective and π is smooth. □

Lemma 3. *Let f be a regular function on $\pi^{-1}(U)$ which is invariant under the orthogonal group. Then f comes from a function on U.*

Proof. Let us consider an invariant function f on the open set $\pi^{-1}(U)$. Let R be the ring $F[U][f]$ in which we add f to the ring $k[U]$. We need to prove in fact that $f \in k[U]$. The ring $F[U][f]$ is a coordinate ring of some variety Y so that we have a factorization of the map $\pi : \pi^{-1}(U) \to Y \xrightarrow{\rho} U$. Since f is an invariant, f is constant on the fibers of π which are all orbits. Thus it follows that the map ρ is bijective. At this point we can conclude as follows. The map ρ is separable and bijective, and U is smooth, hence normal, so, by ZMT, ρ is an isomorphism. In other words f is a function on U. $\square$

Lemma 4. *If a polynomial in the scalar product vanishes on the set where the first n elements are linearly dependent, it is a multiple of* $\begin{vmatrix} 1, 2, \dots, n \\ 1, 2, \dots, n \end{vmatrix}$.

The proof is similar to 5.1 and we omit it.

Theorem 2. *Over any field F of characteristic not 2, the ring of invariants of p copies of the fundamental representation of $O(n, F)$ is generated by the scalar products.*

Proof. Let f be an invariant. From the previous lemmas we know that, after eventually multiplying f by a power of the determinant $\begin{vmatrix} 1, 2, \dots, n \\ 1, 2, \dots, n \end{vmatrix}$, it lies in the subring generated by the scalar products with basis the standard tableaux. We have to do the cancellation as in §5.1, and this is achieved by the previous lemma.

Recently M. Domokos and P. E. Frenkel, in the paper "Mod 2 indecomposable orthogonal invariants" (to appear in *Advances in Mathematics*), have shown that in characteristic 2 there are other indecomposable invariants of degree higher than two. So in this case the invariant theory has to be deeply modified.