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Algebraic Groups

Summary. In this chapter we want to have a first look into algebraic groups. We will use the necessary techniques from elementary algebraic geometry, referring to standard textbooks. Our aim is to introduce a few techniques from algebraic geometry commonly used in representation theory.

1 Algebraic Groups

1.1 Algebraic Varieties

There are several reasons to introduce algebraic geometric methods in representation theory. One which we will not pursue at all is related to the classification of finite simple groups. A standard textbook on this subject is ([Ca]). Our point of view is based instead on the fact that in a precise sense, compact Lie groups can be extended to make them algebraic, and representations should be found inside algebraic functions. We start to explain these ideas in this chapter.

We start from an algebraically closed field k, usually the complex numbers.

Recall that an *affine variety* is a subset V, of some space k^m, defined as the vanishing locus of polynomial equations (in $k[x_1, \dots, x_m]$).

The set I_V of polynomials vanishing on V is its *defining ideal.*

A *regular algebraic function* (or just algebraic function) on V is the restriction to V of a polynomial function on k^m. The set $k[V]$ of these functions is the *coordinate ring* of V; it is isomorphic to the algebra $k[x_1, \dots, x_m]/I_V = k[V]$.

Besides being a finitely generated commutative algebra over k, the ring $k[V]$, being made of functions, does not have any nilpotent elements. $k[V]$ can have zero divisors: this happens when the variety V is not *irreducible*, for instance the variety given by $xy = 0$ in k^2, which consists of two lines.

The notion of subvariety of a variety is clear. Since the subvarieties naturally form the closed sets of a topology (the Zariski topology) one often speaks of a Zariski closed subset rather than a subvariety. The Zariski topology is a rather weak topology. It is clear that a Zariski closed set is also closed in the complex topology. More important is that if V is an irreducible variety and $W \subset V$ a proper subvariety, the open set $V - W$ is dense in V, and also in the complex topology.

Finally, given two affine varieties $V \subset k^n$, $W \subset k^m$ a *regular map* or *morphism* between the two varieties, $f : V \to W$ is a map which, in coordinates, is given by regular functions $(f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$.

Remark 1. A morphism $f : V \to W$ induces a *comorphism* of coordinate rings

$$f^* : k[W] \to k[V], \quad f^*(g) := g \circ f.$$

Remark 2. One should free the concept of affine variety from its embedding, i.e., verify that embedding an affine variety V in a space k^m is the same as choosing m elements in $k[V]$ which generate this algebra over k.

The main fact that one uses at the beginning of the theory is the Hilbert Nullstellensatz, which implies that given an affine algebraic variety V over an algebraically closed field k, with coordinate ring $k[V]$, there is a bijective correspondence between the three sets:

1. Points of V.
2. Homomorphisms, $\phi : k[V] \to k$.
3. Maximal ideals of $k[V]$.

For the basic affine space k^n this means that the maximal ideals of $k[x_1, \dots, x_n]$ are all of the type $(x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$, $(a_1, \dots, a_n) \in k^n$.

This general fact allows one to pass from the language of affine varieties to that of *reduced affine algebras*, that is commutative algebras A, finitely generated over k and without nilpotent elements. In the end one can state that the category of affine varieties is antiisomorphic to that of reduced affine algebras. In this sense one translates from algebra to geometry and conversely.

Affine varieties do not by any means exhaust all varieties. In fact in the theory of algebraic groups one has to use systematically (see §2) at least one other class of varieties, *projective varieties*. These are defined by passing to the projective space $P^n(k)$ of lines of k^{n+1} (i.e., 1-dimensional subspaces of k^{n+1}). In this space now the coordinates $(x_0, x_1, x_2, \dots, x_n)$ must represent a line (through the given point and 0) so they are not all 0 and *homogeneous* in the sense that if $a \neq 0$, $(x_0, x_1, x_2, \dots, x_n)$ and $(ax_0, ax_1, ax_2, \dots, ax_n)$ represent the same point. Then the projective varieties are the subsets V of $P^n(k)$ defined by the vanishing of systems of homogeneous equations.

A system of homogeneous equations of course also defines an affine variety in k^{n+1}: it is the union of 0 and all the lines which correspond to points of V. This set is called the *associated cone* $C(V)$ to the projective variety V and the graded coordinate ring $k[C(V)]$ of $C(V)$ the *homogeneous coordinate ring* of V. Of course $k[C(V)]$ is not made of functions on V. If one wants to retain the functional language one has to introduce a new concept of line bundles and sections of line bundles which we do not want to discuss now.

The main feature of projective space over the complex numbers is that it has a natural topology which makes it compact. In fact, without giving too many details, the reader can understand that one can normalize the homogeneous coordinates x_i so that $\sum_{i=0}^{n} |x_i|^2 = 1$. This is a compact set, the $2n+1$-dimensional sphere S^{2n+1}. Two points in S^{2n+1} give the same point in $P^n(\mathbb{C})$ if and only if they differ by multiplication by a complex number of absolute value 1. One then gives $P^n(\mathbb{C})$ the quotient topology.

Clearly now, in order to give a regular map $f : V \to W$ between two such varieties $V \subset P^n(k)$, $W \subset P^m(k)$ one has to give a map in coordinates, given by regular functions $(f_0(x_0, x_1, \dots, x_n), f_1(x_0, x_1, \dots, x_n), \dots, f_m(x_0, x_1, \dots, x_n))$ which, to respect the homogeneity of the coordinates, must necessarily be *homogeneous of the same degree.*

It is useful also to introduce two auxiliary notions:

A *quasi-affine variety* is a (Zariski) open set of an affine variety.

A *quasi-projective variety* is a (Zariski) open set of a projective variety.

In projective geometry one has to use the ideas of affine varieties in the following way. If we fix a coordinate x_i (but we could also first make a linear change of coordinates), the points in projective space $P^n(k)$ where the coordinate $x_i \neq 0$ can be described by *dehomogenizing this coordinate* and fixing it to be $x_i = 1$. Then this open set is just an n-dimensional affine space $U_i = k^n \subset P^n(k)$. So projective space is naturally *covered* by these $n+1$ *affine charts.* Given a projective variety $V \subset P^n(k)$, its intersection with U_i is an affine variety, obtained by setting $x_i = 1$ in the homogeneous equations of V.

The theory of projective varieties is developed by analyzing how the open affine sets $V \cap U_i$ glue together to produce the variety V.

What makes projective geometry essentially different from affine geometry is the fact that projective varieties are really *compact.* In characteristic 0 this really means that they are compact in the usual topology. In positive characteristic one rather uses the word *complete* since the usual topology does not exist.

1.2 Algebraic Groups

Consider $GL(n, k)$, the group of all invertible $n \times n$ matrices, and the *special linear group* $SL(n, k) := \{A \in M_n(k) \mid \det(A) = 1\}$ given by a single equation in the space of matrices. This latter group thus has the natural structure of an *affine variety.* Also the full linear group is an affine variety. We can identify it with the set of pairs A, c, $A \in M_n(k)$, $c \in k$ with $\det(A)c = 1$. Alternatively, we can embed $GL(n, k)$ as a closed subgroup of $SL(n+1, k)$ as block matrices $\begin{vmatrix} A & 0 \\ 0 & \det(A)^{-1} \end{vmatrix}$.

The *regular algebraic functions* on $GL(n, k)$ are the rational functions $f(x_{i,j})d^{-p}$ where f is a polynomial in the entries of the matrix, d is the determinant and p can be taken to be a nonnegative integer; thus its coordinate ring (over the field k) is the ring of polynomials $k[x_{ij}, d^{-1}]$ in n^2 variables with the determinant d inverted.

Definition 1. A subgroup H of $GL(n, k)$ is called a *linear group*.

A Zariski closed subgroup H of $GL(n, k)$ is a *linear algebraic group*.

The coordinate ring of such a subgroup is then of the form $k[x_{ij}, d^{-1}]/I$, with I the defining ideal of H.

Examples of Linear Algebraic Groups

(1) As $GL(n+1, k), SL(n+1, k)$ act linearly on the space k^{n+1}, they induce a group of *projective transformations* on the projective space $P^n(k)$ of lines in k^{n+1}.

In homogeneous coordinates these actions are still given by matrix multiplication. We remark that if one takes a scalar matrix, i.e., a scalar multiple of the identity zI_{n+1}, $z \neq 0$, this acts trivially on projective space, and conversely a matrix acts trivially if and only if it is a scalar. We identify the multiplicative group k^* of nonzero elements of k with invertible scalar matrices. Thus the group of projective transformations is

$$P\,GL(n+1, k) = GL(n+1, k)/k^*, \qquad \textit{projective linear group}$$

the quotient of $GL(n+1, k)$ or of $SL(n+1, k)$ by the respective centers. In the case of $GL(n+1, k)$ the center is formed by the nonzero scalars z while for $SL(n+1, k)$ we have the constraint $z^{n+1} = 1$. The fact that $P\,GL(n+1, k)$ is a linear algebraic group is not evident at this point; we will explain why it is so in Section 2.

(2) The *orthogonal and symplectic groups* are clearly algebraic since they are given by quadratic equations (cf. Chapter 5, §3):

$$O(n, k) := \{A \in GL(n, k) \mid A^t A = 1\},$$

$$Sp(2n, k) := \{A \in GL(2n, k) \mid AJA = J\},$$

where $J = \begin{vmatrix} 0 & 1_m \\ -1_m & 0 \end{vmatrix}$.

(3) The special orthogonal group is given by the further equation $\det(X) = 1$.

(4) The group T_n, called a *torus*, of invertible diagonal matrices, given by the equations $x_{i,j} = 0, \forall i \neq j$.

(5) The group $B_n \subset GL(n, k)$ of invertible upper triangular matrices, given by the equations $x_{i,j} = 0, \forall i > j$.

(6) The subgroup $U_n \subset B_n$ of strictly upper triangular matrices, given by the further equations $x_{i,i} = 1, \forall i$.

It may be useful to remark that the group

$$U_2 = \left\{ \begin{vmatrix} 1 & a \\ 0 & 1 \end{vmatrix} \right\}, \qquad a \in k, \qquad \begin{vmatrix} 1 & a \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a+b \\ 0 & 1 \end{vmatrix} \tag{1.2.1}$$

is the *additive group* of the field k.

Exercise. Show that the Clifford and spin group (Chapter 5, §4, §5) over $\mathbb{C}$ are algebraic.

Although we will not really use it seriously, let us give the more general definition:

Definition 2. An algebraic group G is an algebraic variety with a group structure, such that the two maps of multiplication $G \times G \to G$ and inverse $G \to G$ are algebraic.

If G is an affine variety it is called an affine algebraic group.

For an elementary introduction to these concepts one can see [Sp].

1.3 Rational Actions

For algebraic groups it is important to study regular algebraic actions.

Definition 1. An action $\pi : G \times V \to V$ is algebraic if V is an algebraic variety and the map π is algebraic (i.e., it is a morphism of varieties).

If both G and V are affine, it is very important to treat all these concepts using the coordinate rings and the comorphism.

The essential theorem is that, given two affine varieties V, W, we have (cf. [Ha], [Sh]):

$$k[V \times W] = k[V] \otimes k[W].$$

Thus, to an action $\pi : G \times V \to V$ is associated a comorphism $\pi^* : k[V] \to k[G] \otimes k[V]$, satisfying conditions corresponding to the definition of action.

The first class of algebraic actions of a linear algebraic group are the induced actions on tensors (and the associated ones on symmetric, skew-symmetric tensors).

Basic example. Consider the action of $GL(n, k)$ on the linear space k^n. We have $k[k^n] = k[x_1, \dots, x_n]$, $k[GL(n, k)] = k[y_{i,j}, d^{-1}]$ and the comorphism is

$$x_i \mapsto \sum_{j=1}^{n} y_{i,j} x_j.$$

Among algebraic actions there are the left and right action of G on itself:

$$(g, h) \mapsto gh, \ (g, h) \mapsto hg^{-1}.$$

Remark. The map $h \mapsto h^{-1}$ is an isomorphism between the left and right action.

For algebraic groups we will usually restrict to regular algebraic homomorphisms. A linear representation $\rho : G \to GL(n, k)$ is called *rational* if the homomorphism is algebraic.

It is useful to extend the notion to infinite-dimensional representations.

Definition 2. A linear action of an algebraic group G on a vector space V is called *rational* if V is the union of finite-dimensional subrepresentations which are algebraic.

Example. For the general linear group $GL(n, k)$ a rational representation is one in which the entries are polynomials in x_{ij} and d^{-1}.

Thus linear algebraic groups are affine. Non-affine groups belong essentially to the disjoint theory of abelian varieties.

Consider an algebraic action $\pi : G \times V \to V$ on an affine algebraic variety V. The action induces an action on functions. The regular functions $k[V]$ on V are then a representation.

Proposition. *$k[V]$ is a rational representation.*

Proof. Let $k[G]$ be the coordinate ring of G so that $k[G] \otimes k[V]$ is the coordinate ring of $G \times V$. The action π induces a map $\pi^* : k[G] \to k[G] \otimes k[V]$, where $\pi^* f(g, v) := f(gv)$.

For a given function $f(v)$ on V we have that $f(gv) = \sum_i a_i(g)b_i(v)$. Thus we see that the translated functions f^g lie in the linear span of the functions b_i.

This shows that any finite-dimensional subspace U of the space $k[V]$ is contained in a finite-dimensional G-stable subspace W. Given a basis u_i of W, we have that $u_i(g^{-1}v) = \sum_j a_{ij}(g)u_i(v)$, with the a_{ij} regular functions on G. Thus W is a rational representation. The union of these representations is clearly $k[V]$. $\square$

The previous proposition has an important consequence.

Theorem. *(i) Given an action of an affine group G on an affine variety V there exists a linear representation W of G and a G-equivariant embedding of V in W.*

(ii) An affine group is isomorphic to a linear algebraic group.

Proof. (i) Choose a finite set of generators of the algebra $k[V]$ and then a finite-dimensional G-stable subspace $W \subset k[V]$ containing this set of generators.

W defines an embedding i of V into W^* by $\langle i(v)|w\rangle := w(v)$. This embedding is clearly equivariant if on W^* we put the dual of the action on W.

(ii) Consider the right action of G on itself. If W is as before and $u_i, i = 1, \ldots, n$, is a basis of W, we have $u_i(xy) = \sum_j a_{ij}(y)u_j(x)$.

Consider the homomorphism ρ from G to matrices given by the matrix $(a_{ij}(y))$. Since $u_i(y) = \sum_j a_{ij}(y)u_j(1)$ we have that the functions a_{ij} generate the coordinate ring of G and thus ρ is an embedding of G into matrices. Thus the image of ρ is a linear algebraic group and ρ is an isomorphism from G to its image. $\square$

1.4 Tensor Representations

We start with:

Lemma. *Any finite-dimensional rational representation U of an algebraic group G can be embedded in an equivariant way in the direct sum of finitely many copies of the coordinate ring $k[G]$ under the right (or the left) action.*

Proof. Take a basis $u^i, i = 1, \dots, n$, of the dual of U. The map

$$j : U \to k[G]^n, \quad j(u) := (\langle u^1 | gu \rangle, \dots, \langle u^n | gu \rangle)$$

is clearly equivariant, with respect to the right action on $k[G]$. Computing these functions in $g = 1$ we see that this map is an embedding. □

Remark. An irreducible representation U can be embedded in a single copy, i.e., in $k[G]$.

Most of this book deals with methods of tensor algebra. Therefore it is quite useful to understand a general statement on rational representations versus tensor representations.

Theorem. *Let $G \subset GL(V)$ be a linear algebraic group. Denote by d the determinant, as a 1-dimensional representation.*

Given any finite-dimensional rational representation U of G we have that, after possibly tensoring by d^r for some r and setting $M := U \otimes d^r$, the representation M is a quotient of a subrepresentation of a direct sum of tensor powers $V^{\otimes k_i}$.

Proof. Let A, B be the coordinate rings of the space $\mathrm{End}(V)$ of all matrices, and of the group $GL(V)$, respectively. The coordinate ring $k[G]$ is a quotient of $B \supset A$.

By the previous lemma, U embeds in a direct sum $\oplus^p k[G]$. We consider the action of G by right multiplication on these spaces.

Since the algebra $B = \cup_{i=0}^{\infty} d^{-i} A$, where d is the determinant function, for some r we have that $d^r U$ is in the image of A^p.

The space of endomorphisms $\mathrm{End}(V)$ as a $G \times G$ module is isomorphic to $V \otimes V^*$, so the ring A is isomorphic to $S(V^* \otimes V) = S(V^{\oplus m})$ as a right G-module if $m = \dim V$.

As a representation $S(V^{\oplus m})$ is a quotient of the tensor algebra $\bigoplus_n (V^{\oplus m})^{\otimes n}$ which in turn is isomorphic to a direct sum of tensor powers of V. Therefore, we can construct a map from a direct sum of tensor powers $V^{\otimes m_i}$ to $k[G]^m$ so that $d^r U$ is in its image.

Since a sum of tensor powers is a rational representation we deduce that $d^r U$ is also the image of a finite-dimensional submodule of such a sum of tensor powers, as required. □

1.5 Jordan Decomposition

The previous theorem, although somewhat technical, has many corollaries. An essential tool in algebraic groups is Jordan decomposition. Given a matrix X we have seen in Chapter 4, §6.1 its *additive* Jordan decomposition, $X = X_s + X_n$ where X_s is diagonalizable, i.e., semisimple, X_n is nilpotent and $[X_s, X_n] = 0$. If X is invertible so is X_s, and it is then better to use the *multiplicative* Jordan decomposition, $X = X_s X_u$ where $X_u := 1 + X_s^{-1} X_n$ is unipotent, i.e., all of its eigenvalues are 1. We still have $[X_s, X_u] = 0$.

It is quite easy to see the following compatibility if X, Y are two invertible matrices:

$$(X \oplus Y)_s = X_s \oplus Y_s, \ (X \oplus Y)_u = X_u \oplus Y_u,$$
$$(X \otimes Y)_s = X_s \otimes Y_s, \ (X \otimes Y)_u = X_u \otimes Y_u. \tag{1.5.1}$$

Furthermore, if X acts on the space V and U is stable under X, then U is stable under X_s, X_u, and the Jordan decomposition of X restricts to the Jordan decomposition of the operator that X induces on U and on V/U.

Finally we can obviously extend this language to infinite-dimensional rational representations. This can be given a very general framework.

Theorem (Jordan–Chevalley decomposition). *Let $G \subset GL(n, k)$ be a linear algebraic group, $g \in G$ and $g = g_s g_u$ its Jordan decomposition. Then*

(i) $g_s, g_u \in G$.
(ii) For every rational representation $\rho : G \to GL(W)$ we have that $\rho(g) = \rho(g_s)\rho(g_u)$ is the Jordan decomposition of $\rho(g)$.

Proof. From Theorem 1.4, formulas 1.5.1 and the compatibility of the Jordan decomposition with direct sums, tensor products and subquotients, clearly (ii) follows from (i).

(i) is subtler. Consider the usual homomorphism $\pi : k[GL(n, k)] \to k[G]$, and the action R_g of g on functions $f(x) \mapsto f(xg)$ on $k[GL(n, k)]$ and $k[G]$, which are both rational representations.

On $k[GL(n, k)]$ we have the Jordan decomposition $R_g = R_{g_s} R_{g_u}$ and from the general properties of submodules and subquotients we deduce that the two maps R_{g_s}, R_{g_u} also induce maps on $k[G]$ which decompose the right action of g. This means in the language of algebraic varieties that the right multiplication by g_s, g_u on $GL(n, k)$ preserves the subgroup G, but this means exactly that $g_s, g_u \in G$.

(ii) From the proof of (i) it follows that $R_g = R_{g_s} R_{g_u}$ is also the Jordan decomposition on $k[G]$. We apply Lemma 1.4 and have that W embeds in $k[G]^m$ for some m. Now we apply again the fact that the Jordan decomposition is preserved when we restrict an operator to a stable subspace. □

1.6 Lie Algebras

For an algebraic group G the Lie algebra of left-invariant vector fields can be defined algebraically. For an algebraic function f and $g \in G$ we have an expression $L_g f(x) = f(gx) = \sum_i f_1^{(i)}(g) f_2^{(i)}(x)$. Applying formula 3.1.1 of Chapter 4, we have that a left-invariant vector field gives the derivation of $k[G]$ given by the formula

$$X_a f(g) := dL_g(a)(f) = a(f(gx)) = \sum_i f_1^{(i)}(g) a(f_2^{(i)}(x)).$$

The linear map a is a tangent vector at 1, i.e., a derivation of the coordinate ring of G at 1. According to basic algebraic geometry, such an a is an element of the dual of m/m^2, where m is the maximal ideal of $k[G]$ of elements vanishing at 1. Notice that this construction can be carried out over any base field.

2 Quotients

2.1 Quotients

To finish the general theory we should understand, given a linear algebraic group G and a closed subgroup H, the nature of G/H as an algebraic variety and, if H is a normal subgroup, of G/H as an algebraic group. The key results are due to Chevalley:

Theorem 1. *(a) For a linear algebraic group G and a closed subgroup H, there is a finite-dimensional rational representation V and a line $L \subset V$ such that $H = \{g \in G \mid gL = L\}$.*

(b) If H is a normal subgroup, we can assume furthermore that H acts on V by diagonal matrices (in some basis).

Proof. (a) Consider $k[G]$ as a G-module under the right action. Let I be the defining ideal of the subgroup H. Since I is a rational representation of H and finitely generated as an ideal, we can find an H-stable subspace $W \subset I$ which generates I as an ideal.

Next we can find a G-stable subspace U of $k[G]$ containing W. Thus U is a rational representation of G and we claim that $H = \{g \in G \mid gW = W\}$.

To see this let $u_1(x), \dots, u_m(x)$ be a basis of W. If $u_i(xg) \in W$ for all i we have that $u_i(xg) = \sum_{j=1}^m a_{ij}(g)u_j(x)$ is a change of basis. Compute both sides at $x = 1$ and obtain

$$u_i(g) = \sum_{j=1}^{m} a_{ij}(g)u_j(1) = 0, \quad \forall i = 1, \dots, m,$$

as the $u_i(x)$ vanish on H. As the $u_i(x)$ generate the ideal of H, we have $g \in H$, as desired.

Let $V := \bigwedge^m U$ be the exterior power and $L = \bigwedge^m W$. Given a linear transformation A of V we have that $\bigwedge^m(A)$ fixes L if and only if A fixes W,[44] and so the claim follows.

(b) Assume now that H is a normal subgroup, and let V, L be as in the previous step. Consider the sum $S \subset V$ of all the 1-dimensional H-submodules (eigenvectors). Let us show that S is a G-submodule. For this consider a vector $v \in S$ which is an eigenvector of H, $hv = \chi(h)v$.

[44] We have not proved this rather simple fact; there is a very simple proof in Chapter 13, Proposition 3.1.

We have for $g \in G$ that $hgv = gg^{-1}hgv$ and $g^{-1}hg \in H$. So $hgv = g\chi(g^{-1}hg)v = \chi(g^{-1}hg)gv \in S$. If we replace V with S we have satisfied the condition (b). □

Theorem 2. *Given a linear algebraic group G and a closed normal subgroup H, there is a linear rational representation $\rho : G \to GL(Z)$ such that H is the kernel of ρ.*

Proof. Let L, S be as in part (b) of the previous theorem. Let Z be the set of linear transformations centralizing H. If $g \in G$ and $a \in Z$ we have, for every $h \in H$,

$$h(gag^{-1})h^{-1} = g(g^{-1}hg)a(g^{-1}h^{-1}g)g^{-1}$$

$$(2.1.1) \qquad = ga(g^{-1}hg)(g^{-1}h^{-1}g)g^{-1} = gag^{-1}.$$

In other words, the conjugation action of G on $\mathrm{End}(S)$ preserves the linear space Z.

By definition of centralizer, H acts trivially by conjugation on Z. We need only prove that if $g \in G$ acts trivially by conjugation in Z, then $g \in H$.

For this, observe that since H acts as diagonal matrices in S, there is an H-invariant complementary space to L in S and so an H-invariant projection $\pi : S \to L$. By definition $\pi \in Z$. If $g\pi g^{-1} = \pi$, we must have that $gL = L$, hence that $g \in H$. □

Example. In the case of the projective linear group $PGL(V) = GL(V, k)/k^*$ there is a very canonical representation. If we act with $GL(V)$ on the space $\mathrm{End}(V)$ by conjugation we see that the scalar matrices are exactly the kernel.

The purpose of these two theorems is to show that G/H can be thought of as a *quasi-projective variety*.

When H is normal we have shown that H is the kernel of a homomorphism. In Proposition 1 we will show that the image of a homomorphism is in fact a closed subgroup which will allow us to define G/H as a linear algebraic group.

Now we should make two disclaimers. First, it is not completely clear what we mean by these two statements, nor is it clear if what we have proved up to now is enough. The second point is that in characteristic $p > 0$ one has to be more precise in the theorems in order to avoid the difficult problems coming from possible inseparability. Since this discussion would really take us away from our path we refer to [Sp] for a thorough understanding of the issues involved. Here we will limit ourselves to some simple geometric remarks which, in characteristic 0, are sufficient to justify all our claims.

The basic facts we need from algebraic geometry are the following.

1. An algebraic variety decomposes uniquely as a union of irreducible varieties.

2. An irreducible variety V has a *dimension*, which can be characterized by the following inductive property: if $W \subset V$ is a maximal proper irreducible subvariety of V, we have $\dim V = \dim W + 1$. A zero-dimensional irreducible variety is a single point.

For a non-irreducible variety W one defines its dimension as the maximum of the dimensions of its irreducible components.

3. Given a map $\pi : V \to W$ of varieties, if V is irreducible, then $\overline{\pi(V)}$ is irreducible. If $\overline{\pi(V)} = W$, we say that π is *dominant*.

4. $\pi(V)$ contains a nonempty Zariski open set U of $\overline{\pi(V)}$.

For a non-irreducible variety V it is also useful to speak of the dimension of V at a point $P \in V$. By definition it is the maximum dimension of an irreducible component of V passing through P.

Usually an open set of a closed subset is called a *locally closed set*. Then what one has is that $\pi(V)$ is a finite union of locally closed sets; these types of sets are usually called *constructible*.

The second class of geometric ideas needed is related to the concept of smoothness.

Algebraic varieties, contrary to manifolds, may have *singularities*. Intuitively, a point P of an irreducible variety of dimension n is *smooth* if the variety can be described locally by n-parameters. This means that given the maximal ideal m_P of functions vanishing in P, the vector space $T_P^*(V) := m_P/m_P^2$ (which should be thought of as the space of *infinitesimals of first order*) has dimension exactly n (over the base field k).

One then defines the *tangent space* of V in P as the dual space :

$$T_P(V) := \hom(m_P/m_P^2, k).$$

This definition can be given in general. Then we say that a variety V is *smooth at a point* $P \in V$ if the dimension of $T_P(V)$ equals the dimension of P in V.

5. If V is not smooth at P, the dimension of $T_P(V)$ is strictly bigger than the dimension of P in V.

6. The set of smooth points of V is open and dense in V.

Given a map $\pi : V \to W$, $\pi(P) = Q$ and the comorphism $\pi^* : k[W] \to k[V]$ we have $\pi^*(m_Q) \subset m_P$. Thus we obtain a map $m_Q/m_Q^2 \to m_P/m_P^2$ and dually the *differential*:

$$d\pi_P : T_P(V) \to T_{\pi(P)}(W). \tag{2.1.2}$$

In characteristic 0, one has the following:

Theorem 3. *Given a dominant map $\pi : V \to W$ between irreducible varieties, there is a nonempty open set $U \subset W$ such that U, $\pi^{-1}(U)$ are smooth. If $P \in \pi^{-1}(U)$, then $d\pi_P : T_P(V) \to T_{\pi(P)}(W)$ is surjective.*

This basic theorem fails in positive characteristic due to the phenomenon of *inseparability*. This is best explained by the simplest example. We assume the characteristic $p > 0$ and take as varieties the affine line $V = W = k$. Consider the map $x \mapsto x^p$. By simple field theory it is a bijective map. Its differential can be computed with the usual rules of calculus as $dx^p = px^{p-1}dx \equiv 0$. If the differential is not identically 0, we will say that the map is *separable*, otherwise *inseparable*. Thus, in positive characteristic one usually has to take care also of these facts. Theorem 3 remains valid if we assume π to be separable. Thus we also need to add the separability condition to the properties of the orbit maps. This is discussed, for instance, in [Sp].

Apart from this delicate issue, let us see the consequences of this analysis.

Proposition 1. *(a) An orbit variety G/H is a smooth quasiprojective variety.*

(b) The image G/H under a group homomorphism $\rho : G \to GL(V)$ with kernel H is a closed subgroup.

Proof. (a) From the first lemma of this section there is a linear representation V of G and a line L such that H is the stabilizer of L. Hence if we consider the projective space $P(V)$ of lines in V, the line L becomes a point of $P(V)$. H is the stabilizer of this point and G/H its orbit. According to the previous property 4, G/H contains a nonempty set U, open in the closure $\overline{G/H}$. From property 6 we may assume that U is made of smooth points. Since G acts algebraically on G/H, it also acts on its closure and $\cup_{g\in G} gU$ is open in $\overline{G/H}$ and made of smooth points. Clearly $\bigcup_{g\in G} gU = G/H$.

(b) By (a) G/H is a group, open in $\overline{G/H} \subset GL(V)$. If we had an element $x \in \overline{G/H} - G/H$ we would have also the coset $(G/H)x \subset \overline{G/H} - G/H$. This is absurd by a dimension argument, since as varieties $(G/H)x$ and G/H are isomorphic, and thus they have the same dimension, so $(G/H)x$ cannot be contained properly in the closure of G/H. □

The reader should wonder at this point if our analysis is really satisfactory. In fact it is not, the reason being that we have never explained if G/H really has an intrinsic structure of an algebraic variety. A priori this may depend on the embeddings that we have constructed. In fact what one would like to have is a canonical structure of an algebraic variety on G/H such that:

Universal property (of orbits). If G acts on any algebraic variety X, $p \in X$ is a point fixed by H, then the map, which is defined set-theoretically by $gH \mapsto gp$, is a regular map of algebraic varieties from G/H to the orbit of p.

Similarly, when H is a normal subgroup we would like to know that:

Universal property (of quotient groups). If $\rho : G \to K$ is a homomorphism of algebraic groups and H is in the kernel of ρ, then the induced homomorphism $G/H \to K$ is algebraic.

To see what we really need in order to resolve this question we should go back to the general properties of algebraic varieties.

One major difficulty that one has to face is the following: if a map $\pi : V \to W$ is bijective it is not necessarily an isomorphism of varieties!

We have already seen the example of $x \to x^p$, but this may happen also in characteristic 0. The simplest example is the bijective *parameterization* of the cubic $C := \{x^3 - y^2 = 0\}$ given by $x = t^2$, $y = t^3$ (a cusp). In the associated comorphism the image of the coordinate ring $k[C]$, in the coordinate ring $k[t]$ of the line, is the proper subring $k[t^2, t^3] \subsetneq k[t]$, so the map cannot be an isomorphism.

The solution to this puzzle is in the concept of *normality*: For an affine variety V this means that its coordinate ring $k[V]$ is integrally closed in its field of fractions.

For a projective variety it means that its affine open sets are indeed normal. Normality is a weaker condition than smoothness and one has the basic fact that (cf. [Ra]):

Theorem (ZMT, Zariski's main theorem).[45] *A bijective separable morphism $V \to W$, where W is normal, is an isomorphism.*

Assume that we have found an action of G on some projective space $\mathbb{P}$, and a point $p \in \mathbb{P}$ such that H is the stabilizer. We then have the orbit map $\pi : G \to \mathbb{P}, \pi(g) = gp$ which identifies, set-theoretically, the orbit of p with G/H. The key observation is:

Proposition 2. *If π is separable, then the orbit map satisfies the universal property.*

Proof. Then let G act on some other algebraic variety X and $q \in X$ be fixed by H. Consider the product $X \times P$. Inside it the point (q, p) is clearly stabilized by exactly H. Let $A = G(x, p)$ be its orbit which set-theoretically is G/H and $\phi : g \mapsto g(q, p)$ the orbit map. The two projections p_1, p_2 on the two factors X, P give rise to maps $G \xrightarrow{\phi} A \xrightarrow{p_1} Gq, \quad G \xrightarrow{\phi} A \xrightarrow{p_2} Gp$. The second map $A \to Gp$ is bijective. Since the map from G to Gp is separable also the map from A to Gp must be separable. Hence by ZMT it is an isomorphism. Then its inverse composed with the first projection is the required map. □

Summing up we see that in characteristic 0 we do not need any further requirements for the constructions of the lemma and theorem in order to obtain the required universal properties. In positive characteristic one needs to prove (and we send the reader to [Sp]) that in fact the separability can also be granted by the construction.

One final remark, given a group homomorphism $\rho : G \to K$ with kernel H, we would like to say that ρ induces an isomorphism between G/H and the image $\rho(G)$. In fact this is *not always true* as the homomorphism $x \to x^p$ of the additive group shows. This phenomenon of course can occur only in positive characteristic and only if the morphism is not separable. Notice that in this case the notion of *kernel* has to be refined. In our basic example the kernel is defined by the equation $x^p = 0$. This equation defines the point 0 with some multiplicity, as a *scheme*. In general this can be made into the rather solid but complicated theory of *group schemes*, for which the reader can consult [DG].

3 Linearly Reductive Groups

3.1 Linearly Reductive Groups

We come to the main class of algebraic groups of our interest.

Proposition 1. *For an affine group G the following are equivalent:*

(i) Every finite-dimensional rational representation is semisimple.

[45] This is actually just a version of ZMT.

(ii) Every rational representation is semisimple.
(iii) The coordinate ring $k[G]$ is semisimple under the right (or the left) action.
(iv) If G is a closed subgroup of $GL(V)$ then all the tensor powers V^n are semisimple.

A group satisfying the previous properties is called a ***linearly reductive group.***

Proof. (i) implies (ii) by abstract representation theory (Chapter 6, Theorem 2.1). Clearly (ii) implies (iii) and (iv), (iii) implies (i) by Lemma 1.4 and the fact that direct sums and submodules of semisimple modules are semisimple. Assume (iv); we want to deduce (i).

Let d be a multiplicative character of G, as for instance the determinant in a linear representation. A finite-dimensional representation U is semisimple if and only if $U \otimes d^r$ is semisimple. Now we apply Theorem 1.4: since tensor powers are semisimple, so is any subrepresentation of a direct sum. Finally the quotient of a semisimple representation is also semisimple, so $U \otimes d^r$ is semisimple. □

Then let G be a linearly reductive group. For every irreducible representation U, the argument of Chapter 6, Theorem 2.6 proves that:

Lemma. *$U^* \otimes U$ appears in $k[G]$ as a $G \times G$ submodule: namely, the isotypic component of type U.*

It follows that

Theorem. *If G is a linearly reductive group we have only countably many non-isomorphic irreducible representations and*

$$k[G] = \bigoplus_i U_i^* \otimes U_i \ (as \quad G \times G \quad modules), \tag{3.1.1}$$

where U_i runs over the set of all non-isomorphic irreducible representations of G.

Proof. Since G is linearly reductive and $k[G]$ is a rational representation we must have that $k[G]$ is the direct sum of its isotypic components $U_i^* \otimes U_i$. □

Remark. Observe that this formula is the exact analogue of the decomposition formula for the group algebra of a finite group, Chapter 6, §2.6.3 (see also Chapter 8, §3.2).

Corollary. *If G, H are linearly reductive, so is $G \times H$. The irreducible representations of $G \times H$ are $U \otimes V$ where U (respectively, V) is an irreducible representation of G (respectively, H).*

Proof. We have $k[G] = \bigoplus_i U_i^* \otimes U_i$, $k[H] = \bigoplus_j V_j^* \otimes V_j$; so

$$k[G \times H] = \bigoplus_{i,j} U_i^* \otimes U_i \otimes V_j^* \otimes V_j = \bigoplus_{i,j} (U_i \otimes V_j)^* \otimes U_i \otimes V_j.$$

The theorem follows from Chapter 6, §2.6. □

Lemma. *An algebraic group G is linearly reductive if and only if, given any finite dimensional module U there is a G equivariant projection π_U to the invariants U^G, such that if $f : U_1 \to U_2$ is a G-equivariant map of modules we have a (functorial) commutative diagram:*

$$\begin{tikzcd} U_1 \arrow[r, "f"] \arrow[d, "\pi_{U_1}"'] & U_2 \arrow[d, "\pi_{U_2}"'] \\ U_1^G \arrow[r, "f"] & U_2^G \end{tikzcd}$$

Proof. If G is linearly reductive we have a canonical decomposition $U = U^G \oplus U_G$ where U_G is the sum of the non trivial irreducible submodules which induces a functorial projection. Conversely assume such a functorial projection exists, for every module U.

It is enough to prove that, given a module V and a submodule W we have a G-invariant complement P in V to W. In other words it is enough to prove that there is a G-equivariant projection of V to W.

For this let $\rho : V \to W$ be any projection, think of $\rho \in \hom(V, W)$ and $\hom(V, W)$ is a G-module. Then by hypothesis there is a G-equivariant projection π of $\hom(V, W)$ to the invariant elements which are $\hom_G(V, W)$. Let us show that $\pi(\rho)$ is the required projection. It is equivariant by construction so it is enough to show that, restricted to W, it is the identity.

Since ρ ia a projection to W under the restriction map $\hom(V, W) \to \hom(W, W)$ we have that ρ maps to the identity 1_W. Thus the functoriality of the commutative diagram implies that also $\pi(\rho)$ maps to 1_W that is $\pi(\rho)$ is a projection. □

Proposition 2. *An algebraic group G is linearly reductive if and only if its algebra of regular functions $k[G]$ has an* ***integral****, that is a $G \times G$ equivariant projection $f \mapsto \int f$ to the constant functions.*

Proof. We want to show that G satisfies the conditions of the previous lemma. Let V be any representation, $v \in V$, $\phi \in V^*$ consider the function $c_{\phi,v}(g) := \langle \phi \mid gv \rangle \in k[G]$, the map $\phi \mapsto c_{\phi,v}(g)$ is linear. Its integral $\int \langle \phi \mid gv \rangle$ is thus a linear function of ϕ i.e. it can be uniquely represented in the form:

$$\int \langle \phi \mid gv \rangle = \langle \phi \mid \pi(v) \rangle, \ \pi(v) \in V.$$

By right invariance we deduce that $\pi(hv) = \pi(v)$ and by left invariance that

$$\langle \phi \mid \pi(v) \rangle = \int \langle \phi \mid gv \rangle = \int \langle \phi \mid h^{-1}gv \rangle = \langle h\phi \mid \pi(v) \rangle,$$

hence $\pi(v) \in V^G$. By linearity we also have that π is linear and finally, if v is invariant $\langle \phi \mid gv \rangle$ is constant, hence $\pi(v) = v$. We have thus found an equivariant linear projection from V to V^G.

We have thus only to verify that the projection to the invariants is functorial (in the sense of the commutative diagram).

If $f : U_1 \to U_2$ is a map by construction we have

$$\langle \phi \mid \pi_{U_2}(f(v)) = \int \langle \phi \mid gf(v) \rangle = \int \langle f^*(\phi) \mid gv \rangle = \langle f^*(\phi) \mid \pi_{U_1}(v) = \langle \phi \mid f\pi_{U_1}(v)$$

Hence $\pi_{U_2}(f(v)) = f\pi_{U_1}(v)$ we can apply the previous lemma and finish the proof. □

3.2 Self-adjoint Groups

Given a linearly reductive group, as in the case of finite groups, an explicit description of the decomposition 3.1.1 implies a knowledge of its representation theory.

We need some condition to recognize that an algebraic group is linearly reductive. There is a very simple sufficient condition which is easy to apply. This has been proved in Chapter 6, Proposition 1.2. We recall the statement.

Theorem. *Given a subgroup $G \subset GL(V) = GL(n, \mathbb{C})$, let $G^* := \{g^* = \overline{g}^t \mid g \in G\}$. If $G = G^*$, all tensor powers $V^{\otimes m}$ are completely reducible under G. In particular, if G is an algebraic subgroup, then it is linearly reductive.*

As a consequence one easily verifies:

Corollary. *The groups $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $O(n, \mathbb{C})$, $SO(n, \mathbb{C})$, $Sp(n, \mathbb{C})$, D are linearly reductive (D denotes the group of invertible diagonal matrices).*

Exercise. Prove that the spin group is self-adjoint under a suitable Hilbert structure.

We should finally remark from the theory developed:

Proposition. *If $G \subset GL(V)$ is linearly reductive, all of the irreducible representations of G, up to tensor product with powers of the determinant, can be found as subrepresentations of $V^{\otimes n}$ for some n.*

If $G \subset SL(V)$, all of the irreducible representations of G can be found as subrepresentations of $V^{\otimes n}$ for some n.

Proof. In Theorem 1.4 we have seen that a representation tensored by the determinant appears in the quotient of a direct sum of tensor powers. If it is irreducible, it must appear in one on these tensor powers. □

We will apply this idea to classify irreducible representations of classical groups.

Remark. For a connected linear group G to be self-adjoint it is necessary and sufficient that its Lie algebra be self-adjoint.

Proof. G is generated by the exponentials $\exp(a)$, $a \in L$, and $\exp(a)^* = \exp(a^*)$. □

3.3 Tori

The simplest example of a linearly reductive group is the torus T_n, isomorphic to the product of n copies of the multiplicative group, which can be viewed as the group D of invertible diagonal matrices. Its coordinate ring is the ring of Laurent polynomials $k[T] = k[x_i, x_i^{-1}]$ in n variables. A basis of $k[T]$ is given by the monomials:

$$x^{\underline{m}} = x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}, \tag{3.3.1}$$

as $\underline{m} = (m_1, m_2, \dots, m_n)$ varies in the free abelian group $\mathbb{Z}^n$.

The 1-dimensional subspace $x^{\underline{m}}$ is a subrepresentation. Under right action, if $t = (t_1, t_2, \dots, t_n)$, we have

$$(x^{\underline{m}})^t = (x_1 t_1)^{m_1} (x_2 t_2)^{m_2} \dots (x_n t_n)^{m_n} = t^{\underline{m}} x^{\underline{m}}. \tag{3.3.2}$$

Theorem 1. *The irreducible representations of the torus T_n are the irreducible characters $t \mapsto t^{\underline{m}}$. They form a free abelian group of rank n called the* ***character group****.*

Proof. Apply 3.1.1. □

Proposition. *Every rational representation V of T has a basis in which the action is diagonal.*

Proof. This is the consequence of the fact that every rational representation is semisimple and that the irreducible representations are the 1-dimensional characters. □

Definition. (1) A vector generating a T-stable subspace is called a *weight vector* and the corresponding character χ or eigenvalue is called the *weight.*

(2) The set $V_\chi := \{v \in V \mid tv = \chi(t)v, \forall t \in T\}$ is called the *weight space* of V of weight χ.

The weights of the representation can of course appear with any multiplicity, and the corresponding character $\mathrm{tr}(t)$ can be identified with a Laurent polynomial $\mathrm{tr}(t) = \sum_{\underline{m}} c_{\underline{m}} t^{\underline{m}}$ with the $c_{\underline{m}}$ positive integers. One should remark that weights are a generalization of degrees of homogeneity. Let us illustrate this in the simple case of a vector space $V = U_1 \oplus U_2$.

To such a decomposition of a space corresponds a (2-dimensional) torus T, with coordinates x, y, formed by the linear transformations $(u_1, u_2) \to (xu_1, yu_2)$. The decompositions of the various spaces one constructs from V associated to the given direct sum decomposition are just weight space decompositions. For instance

$$S^n(V) = \bigoplus_{i=0}^n S^i(U_1) \otimes S^{n-i}(U_2), \quad \bigwedge^n(V) = \bigoplus_{i=0}^n \bigwedge^i(U_1) \otimes \bigwedge^{n-i}(U_2).$$

Both $S^i(U_1) \otimes S^{n-i}(U_2)$, $\bigwedge^i(U_1) \otimes \bigwedge^{n-i}(U_2)$ are weight spaces of weight $x^i y^{n-i}$.

We complete this section discussing the structure of subgroups of tori.

Let T be an n-dimensional torus, H a closed subgroup. Since T is abelian, H is normal, and hence there is a linear representation of T such that H is the kernel of the representation.

We know that such a linear representation is a direct sum of 1-dimensional characters. Thus we deduce that H is the subgroup where a set of characters χ take value 1.

The character group is $\hat{T} = \mathbb{Z}^n$ and setting $H^\perp := \{\chi \in \hat{T} \mid \chi(h) = 1, \forall h \in H\}$ we have that $H^\perp$ is a subgroup of $\hat{T} = \mathbb{Z}^n$. By the elementary theory of abelian groups we can change the character basis of $\hat{T} = \mathbb{Z}^n$ to a basis e_i so that $H^\perp = \sum_{i=1}^h \mathbb{Z} n_i e_i$ for some positive integers n_i and a suitable h. It follows that in these coordinates, H is the set

$$(3.3.3) \qquad H = \{(t_1, \ldots, t_n) \mid t_i^{n_i} = 1, \ i = 1, \ldots, h\} = \prod_{i=1}^h \mathbb{Z}/(n_i) \times T_{n-h}.$$

Here T_k denotes a k-dimensional torus. In particular we have:

Theorem 2. *A closed connected subgroup of a torus is a torus. A quotient of a torus is a torus.*

Proof. We have seen in 3.3.3 the description of any closed subgroup. If it is connected we have $n_i = 1$ and $H = T_{n-h}$.

As for the second part, T/H is the image of the homomorphism $(t_1, \ldots, t_n) \mapsto (t_1^{n_1}, \ldots, t_h^{n_h})$ with image the torus T_h. □

Remark. We have again a problem in characteristic $p > 0$. In this case the n_i should be prime with p (or else we have to deal with group schemes).

3.4 Additive and Unipotent Groups

Although we will not be using them as much as tori, one should take a look at the other algebraic abelian groups. For tori, the prototype is the multiplicative group. For the other groups as the prototype we have in mind the *additive group.* We have seen in 1.2.1 that this is made of unipotent matrices, so according to Theorem 1.4 in all representations it will be unipotent. In particular let us start from the action on its coordinate ring $k[x]$. If $a \in k$ is an element of the additive group, the right action on functions is given by $f(x + a)$, so we see that:

Proposition 1. *For every positive integer m, the subspace P_m of $k[x]$ formed by the polynomials of degree $\leq m$ is a submodule. In characteristic 0 these are the only finite-dimensional submodules of P_m.*

Proof. That these spaces P_m of polynomials are stable under the substitutions $x \mapsto x + a$ is clear. Conversely, let M be a finite-dimensional submodule. Suppose that m is the maximum degree of a polynomial contained in M, so $M \subset P_k$. Let $f(x) = x^m + u(x) \in M$ where $u(x) = bx^{m-1} + \cdots$ has degree strictly less than m. We have that for $a \in k$

$$f_a(x) := f(x+a) - f(x) = (ma+b)x^{m-1} + \cdots \in M.$$

If $ma + b \neq 0$, this is a polynomial of degree exactly $m - 1$. In characteristic 0, $m \neq 0$, and so we can find an a with $ma + b \neq 0$. By induction all the monomials $x^i, i \leq m - 1$ are in M, hence also $x^m \in M$ and $P_m \subset M$. □

Remark. In characteristic $p > 0$, we have that $k[x^p]$ is a submodule.

Unipotent elements tend to behave in a more complicated way in positive characteristic. One reason is this. Let us work in characteristic 0, i.e., $k = \mathbb{C}$. If A is a nilpotent matrix, the exponential series $\exp(A) = \sum_{i=0}^{\infty} \frac{A^i}{i!}$ terminates after finitely many steps, so it is indeed a polynomial, and $\exp(A)$ is a unipotent matrix.

Similarly, let $1 + A$ be unipotent, so that A is nilpotent. The logarithmic series terminates after finitely many steps: it is a polynomial, and $\log(A)$ is a nilpotent matrix. We have:

Proposition 2. *The variety of complex nilpotent matrices is isomorphic to the variety of unipotent matrices by the map* exp, *and its inverse* log.

Notice that both maps are equivariant with respect to the conjugation action.

In positive characteristic neither of these two series makes sense (unless the characteristic is large with respect to the size of the matrices). The previous proposition is not true in general.

We complete this discussion by studying the 1-dimensional connected algebraic groups. We assume $k = \mathbb{C}$.

Lemma. *Let $g = \exp(A)$, with $A \neq 0$ a nilpotent matrix. Let $\{g\}$ be the Zariski closure of the cyclic subgroup generated by g. Then $\{g\} = \exp(t\,A)$, $t \in \mathbb{C}$.*

Proof. Since $\{g\}$ is made of unipotent elements, it is equivalent to prove that $\log(\{g\}) = \mathbb{C}A$. Since $\exp(mA) = \exp(A)^m$ we have that $mA \in \log(\{g\})$, $\forall m \in \mathbb{Z}$. By the previous proposition it follows that $\log(\{g\})$ is a closed subvariety of the nilpotent matrices and the closure of the elements $\log(g^m) = mA$. In characteristic 0 we easily see that the closure of the integral multiples of A (in the Zariski topology) is $\mathbb{C}A$, as desired. □

Theorem. *A 1-dimensional connected algebraic group is isomorphic to the additive or the multiplicative group.*

Proof. Let G be such a group. The proof in positive characteristic is somewhat elaborate and we send the reader to ([Sp], [Bor]). Let us look in characteristic 0. First, if $g \in G$, we know that $g_s, g_u \in G$. Let us study the case in which there is a nontrivial unipotent element. By the previous lemma G contains the 1-dimensional group $\exp(\mathbb{C}\,A)$. Hence, being connected and 1-dimensional, it must coincide with $\exp(\mathbb{C}\,A)$ and the map $t \mapsto \exp(t\,A)$ is an isomorphism between $(\mathbb{C}, +)$ and G.

Consider next the case in which G does not contain unipotent elements; hence it is made of commuting semisimple elements and thus it can be put into diagonal form.

In this case it is no longer true that the closure of a cyclic group is 1-dimensional; in fact, it can be of any given dimension. But from Theorem 2 of 3.3 we know that a closed connected subgroup of a torus is a torus. Thus G is a torus, but since it is 1-dimensional it is isomorphic to the multiplicative group. $\square$

We have seen in Chapter 4, §7 that a connected Lie group is solvable if and only if its Lie algebra is solvable. Thus Lie's theorem, Chapter 4, §6.3, implies that in characteristic 0, a connected solvable algebraic linear group is conjugate to a subgroup of the group of upper triangular matrices. In other words, in a suitable basis, it is made of upper triangular matrices. In fact this is true in any characteristic ([Sp], [Bor], [Hu2]), as we will discuss in §4. Let us assume this basic fact for the moment.

Definition. A linear group is called *unipotent* if and only if its elements are all unipotent.

Thus, from the previous discussion, we have seen in characteristic 0 that a unipotent group is conjugate to a subgroup of the group U_n of strictly upper triangular matrices. In characteristic 0 in fact we can go further. The argument of Proposition 2 shows that the two maps exp, log are also bijective algebraic isomorphisms between the variety N_n of upper triangular matrices with 0 on the diagonal, a Lie algebra, and the variety U_n of upper triangular matrices with 1 on the diagonal, an algebraic unipotent group.

Theorem. *Under the map* $\exp : N_n \to U_n$ *the image of a Lie subalgebra is an algebraic group.*

Under the map $\log : U_n \to N_n$ *the image of an algebraic group is a Lie algebra.*

Proof. Let $A \subset N_n$ be a Lie algebra. We know by the isomorphism statement that $\exp(A)$ is a closed subvariety of U_n. On the other hand, exp maps A into the analytic Lie group of Lie algebra A and it is even a local isomorphism. It follows that this analytic group is closed and coincides with $\exp(A)$. As for the converse, it is enough, by the previous statement, to prove that any algebraic subgroup H of U_n is connected, since then it will follow that it is the exponential of its Lie algebra. Now if $a = \exp(b) \neq 1$, $a \in H$ is a unipotent matrix, we have that $\exp(\mathbb{C}b) \subset H$ by the previous lemma, thus H is connected. $\square$

3.5 Basic Structure Theory

When studying the structure of algebraic groups it is important to recognize which constructions produce algebraic groups. Let G be an algebraic group with Lie algebra L. The first important remark is that L is in a natural way a complex Lie algebra: in fact, for $GL(n, \mathbb{C})$ it is the complex matrices and for an algebraic subgroup it is clearly a complex subalgebra of matrices. Given a Lie subgroup $H \subset G$, a necessary condition for H to also be algebraic is that its Lie algebra M should be also complex. This is *not sufficient* as the following trivial example shows. Let A be a matrix which is neither semisimple nor nilpotent. The complex Lie algebra generated by A is not the Lie algebra of an algebraic group (from 1.5). On the other hand, we want to prove that the derived and lower central series as well as the solvable and nilpotent radicals of an algebraic group (considered as a Lie group) are algebraic. We thus need a criterion to ensure that a subgroup is algebraic. We use the following:

Proposition 1. *Let G be an algebraic group and $V \subset G$ an irreducible subvariety, $1 \in V$. Then the closed (in the usual complex topology) subgroup H generated by V is algebraic and connected.*

Proof. Let V^{-1} be the set of inverses of V, which is still an irreducible variety containing 1. Let $U = \overline{VV^{-1}}$. Since U is the closure of the image of an irreducible variety $V \times V^{-1}$ under an algebraic map $\pi : V \times V^{-1} \to G, \pi(a, b) = ab$, it is also an irreducible variety. Since $1 \in V \cap V^{-1}$ we have that $V, V^{-1} \subset U$ and $U = U^{-1}$. Consider for each positive integer m the closure $\overline{U^m}$ of the product $UU \dots U = U^m$. For the same reasons as before $\overline{U^m}$ is an irreducible variety, and $\overline{U^m} \subset \overline{U^{m+1}} \subset H$. An increasing sequence of irreducible varieties must at some point stop increasing.[46] Assume that $\overline{U^n} = \overline{U^k}, \forall k \geq n$. By continuity $\overline{U^n}\,\overline{U^n} \subset \overline{U^{2n}} = \overline{U^n}$. A similar argument shows $\overline{U^n} = \overline{U^n}^{-1}$, so $\overline{U^n}$ is a group and a subvariety; hence $\overline{U^n} = H$. □

Theorem. *Let G be a connected algebraic group. The terms of the derived and lower central series are all algebraic and connected.*

Proof. The proof is by induction. We do one case; the other is similar. Assume we know that G^i is algebraic and irreducible. G^{i+1} is the algebraic subgroup generated by the set X_i of elements $\{x, y\}$, $x \in G$, $y \in G^i$. The map $(x, y) \mapsto \{x, y\}$ is algebraic, so X_i is dense in an irreducible subvariety, and we can apply the previous proposition. □

Proposition 2. *The center of an algebraic group is algebraic (but in general not connected).*

The solvable radical of an algebraic group, as a Lie group, is algebraic.

Proof. The center is the kernel of the adjoint representation which is algebraic.

For the second part look at the image R in the adjoint representation of the solvable radical. Since Lie(R) is solvable, R can be be put into some basis in the form of upper triangular matrices. Then the Zariski closure in G of R is still made of upper triangular matrices and hence solvable, and clearly a normal connected subgroup. Since R is maximal closed connected normal solvable, it must be closed in the Zariski topology. □

In the same way one has, for subgroups of an algebraic group G, the following.

Proposition 3. *The Zariski closure of a connected solvable Lie subgroup of G is solvable.*

A maximal connected solvable Lie subgroup of G is algebraic.

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[46] Here is where we use the fact that we started from an irreducible set; otherwise the number of components could go to infinity.

3.6 Reductive Groups

Definition 1. A linear algebraic group is called *reductive* if it does not contain any closed unipotent normal subgroup.

A linear algebraic group is called *semisimple* if it is connected and its solvable radical is trivial.

Notice that semisimple implies reductive (but not conversely).

Proposition 1. *(1) If G is a connected abelian linearly reductive group, it is a torus.*

(2) If G is solvable and connected, $\{G, G\}$ is unipotent.

(3) A unipotent linearly reductive group reduces to $\{1\}$.

Proof. (1) Let V be a faithful representation of G. Decompose it into irreducibles. Since G is abelian each irreducible is 1-dimensional, hence G is a subgroup of the diagonal matrices. From Theorem 2 of 3.3 it is a torus.

(2) In a linear representation G can be put into triangular form; in characteristic 0 this follows from Chapter 4. §7.1 and the analogous theorem of Lie for solvable Lie algebras. In general it is the Lie–Kolchin theorem which we discuss in Section 4.1. Then all the commutators lie in the strictly upper triangular matrices, a unipotent group.

(3) A unipotent group U, in a linear representation, in a suitable basis is made of upper triangular matrices, which are unipotent by Theorem 1.5. Hence U has a fixed vector. If U is linearly reductive, this fixed vector has an invariant complement. By induction U acts trivially on this complement. So U acts trivially on any representation, hence $U = 1$. □

Theorem 1. *Let G be an algebraic group and N a normal subgroup. Then G is linearly reductive if and only if G/N and N are linearly reductive.*

Proof. Assume G is linearly reductive. Since every representation of G/N is also a representation of G, clearly G/N is linearly reductive. We have to prove that $k[N]$ is completely reducible as an N-module. Since $k[N]$ is a quotient of $k[G]$, it is enough to prove that $k[G]$ is completely reducible as an N-module. Let M be the sum of all irreducible N-submodules of $k[G]$, we need to show that $M = k[G]$. In any case, since N is normal, it is clear that M is G-stable. Since G is linearly reductive, we have a G-stable complement P, and $k[G] = M \oplus P$. P must be 0, otherwise it contains an irreducible N-module, which by definition is in M.

Conversely, using the fact that N is linearly reductive we have an $N \times N$-equivariant projection $\pi : k[G] \to k[G]^{N\times N} = k[G/N]$. This projection is canonical since it is the only projection to the isoptypic component of invariants. Therefore it commutes with the $G \times G$ action. On $k[G]^{N\times N}$ we have an action of $G/N \times G/N$ so, since this is also linearly reductive, we have an integral projecting to the $G/N \times G/N$ invariants. Composing, we have the required projection $k[G] \to k$. We apply now Proposition 2 of §3.1. □

Proposition 2. *Let G be an algebraic group and G_0 the connected component of* 1. *Then G is linearly reductive if and only if G_0 is linearly reductive and the order of the finite group G/G_0 is not divisible by the characteristic of the base field.*

Proof. Since G_0 is normal, one direction comes from the previous theorem. Assume G_0 linearly reductive, M a rational representation of G, and N a submodule. We need to prove that N has a G-stable complement. N has a G_0-stable complement: in other words, there is a projection $\pi : M \to N$ which commutes with G_0. It follows that given $a \in G$, the element $a\pi a^{-1}$ depends only on the class of a modulo G_0. Then we can average and define $\rho = \frac{1}{|G/G_0|}\sum_{a\in G/G_0} a\pi a^{-1}$ and obtain a G-equivariant projection to N. □

The main structure theorem is (cf. [Sp]):

Theorem 2. *In characteristic* 0 *a linear algebraic group is linearly reductive if and only if it is reductive.*[47]

For a reductive group the solvable radical is a torus contained in the center.

Proof. Let G be linearly reductive. If it is not reductive it has a normal unipotent subgroup U, which by Theorem 1 would be linearly reductive, hence trivial by Proposition 1. Conversely, let G be reductive. Since the radical is a torus it is enough to prove that a semisimple algebraic group is linearly reductive. This is not easy but is a consequence of the theory of Chapter 10. □

The second statement of Theorem 2 follows from the more precise:

Lemma. *Let G be a connected algebraic group and $T \subset G$ a torus which is also a normal subgroup; then T is in the center.*

Proof. The idea is fairly simple. By assumption G induces by conjugation a group of automorphisms of T. Since the group of automorphisms of T is discrete and G is connected it must act trivially. To make this proof more formal, let M be a faithful representation of G and decompose M into eigenspaces for T for the different eigenvalues present. Clearly G permutes these spaces, and (now it should be clear) since G is connected the permutations which it induces must be the identity. Hence for any weight λ of T appearing in M and $g \in G, t \in T$ we have $\lambda(gtg^{-1}) = \lambda(t)$. Since M is faithful, the weights λ generate the character group. Hence $gtg^{-1} = t$, $\forall g \in G, \forall t \in T$. □

With these facts we can explain the program of classification of linearly reductive groups over $\mathbb{C}$.

Linearly reductive groups and their representations can be fully classified.

The steps are the following. First, decompose the adjoint representation L of a linearly reductive group G into irreducibles, $L = \bigoplus_i L_i$. Each L_i is then a simple Lie

[47] Linear reductiveness in characteristic $p > 0$ is a rare event and one has to generalize most of the theory in a very nontrivial way.

algebra. We separate the sum of the trivial representations, which is the Lie algebra of the center Z. By Proposition 1, the connected component Z_0 of Z is a torus.

In Chapter 10 we classify simple Lie algebras, and prove that the simply connected group associated to such a Lie algebra is a linearly reductive group. It follows that if G_i is the simply connected group associated to the nontrivial factors L_i, we have a map $Z_0 \times \prod_i G_i \to G$ which is an isomorphism of Lie algebras (and algebraic).

Then G is isomorphic to $Z_0 \times \prod_i G_i/A$ where A is a finite group contained in the center $Z_0 \times \prod_i Z_i$ of $Z_0 \times \prod_i G_i$. The center of a simply connected linearly reductive group is described explicitly in the classification and is a cyclic group or $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$; hence the possible subgroups A can be made explicit. This is then a classification.

4 Borel Subgroups

4.1 Borel Subgroups

The notions of *maximal torus* and *Borel subgroup* play a special role in the theory of linear algebraic groups.

Definition 1. A subgroup of an algebraic group is called a *maximal torus* if it is a closed subgroup, a torus as an algebraic group, and maximal with respect to this property.

A subgroup of an algebraic group is called a *Borel subgroup* if it is closed, connected and solvable, and maximal with respect to this property.

The main structure theorem is:

Theorem 1. *All maximal tori are conjugate. All Borel subgroups are conjugate.*

We illustrate this theorem for classical groups giving an elementary proof of the first part (see Chapter 10, §5 for more details on this topic and a full proof).

Example 1. $GL(V)$. In the general linear group of a vector space V a maximal torus is given by the subgroup of all matrices which are diagonal for some fixed basis of V.

A Borel subgroup is the subgroup of matrices which fix a *maximal flag*, i.e., a sequence $V_1 \subset V_2 \subset \cdots V_{n-1} \subset V_n = V$ of subspaces of V with $\dim V_i = i$ (assuming $n = \dim V$).

Example 2. $SO(V)$. In the special orthogonal group of a vector space V, equipped with a nondegenerate symmetric form, a maximal torus is given as follows.

If $\dim V = 2n$ is even we take a hyperbolic basis $e_1, f_1, e_2, f_2, \dots, e_n, f_n$. That is, the 2-dimensional subspaces V_i spanned by e_i, f_i are mutually orthogonal and the matrix of the form on the vectors e_i, f_i is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

For such a basis we get a maximal torus of matrices which stabilize each V_i and, restricted to V_i in the basis e_i, f_i, has matrix $\begin{pmatrix} \alpha_i & 0 \\ 0 & \alpha_i^{-1} \end{pmatrix}$.

The set of maximal tori is in 1-1 correspondence with the set of all decompositions of V as the direct sum of 1-dimensional subspaces spanned by hyperbolic bases. The set of Borel subgroups is in 1-1 correspondence with the set of *maximal isotropic flags*, i.e., the set of sequences $V_1 \subset V_2 \subset \cdots V_{n-1} \subset V_n$ of subspaces of V with $\dim V_i = i$ and such that the subspace V_n is totally isotropic for the form. To such a flag one associates the maximal flag $V_1 \subset V_2 \subset \cdots V_{n-1} \subset V_n = V_n^\perp \subset V_{n-1}^\perp \cdots \subset V_2^\perp \subset V_1^\perp \subset V$ which is clearly stable under the subgroup fixing the given isotropic flag. If $\dim V = 2n + 1$ is odd, we take bases of the form $e_1, f_1, e_2, f_2, \dots, e_n, f_n, u$ with $e_1, f_1, e_2, f_2, \dots, e_n, f_n$ hyperbolic and u orthogonal to $e_1, f_1, e_2, f_2, \dots, e_n, f_n$. As a maximal torus we can take the same type of subgroup which now fixes u.

The analogue statement holds for Borel subgroups except that now a maximal flag is $V_1 \subset V_2 \subset \dots V_{n-1} \subset V_n \subset V_n^\perp \subset V_{n-1}^\perp \cdots \subset V_2^\perp \subset V_1^\perp \subset V$.

Example 3. $Sp(V)$. In a symplectic group of a vector space V equipped with a nondegenerate skew-symmetric form a maximal torus is given as follows. Let $\dim V = 2n$. We say that a basis $e_1, f_1, e_2, f_2, \dots, e_n, f_n$ is symplectic if the 2-dimensional subspaces V_i spanned by e_i, f_i are mutually orthogonal and the matrix of the form on the vectors e_i, f_i is: $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. For such a basis we get a maximal torus of matrices that stabilize each V_i and, restricted to V_i in the basis e_i, f_i, has matrix $\left(\begin{smallmatrix} \alpha_i & 0 \\ 0 & \alpha_i^{-1} \end{smallmatrix}\right)$.

The set of maximal tori is in 1-1 correspondence with the set of all decompositions of V as the direct sum of 1-dimensional subspaces spanned by symplectic bases. The set of Borel subgroups is again in 1-1 correspondence with the set of *maximal isotropic flags*, i.e., the set of sequences $V_1 \subset V_2 \subset \cdots V_{n-1} \subset V_n$ of subspaces of V with $\dim V_i = i$ and such that the subspace V_n is totally isotropic for the form. To such a flag one associates the maximal flag $V_1 \subset V_2 \subset \cdots V_{n-1} \subset V_n = V_n^\perp \subset V_{n-1}^\perp \cdots \subset V_2^\perp \subset V_1^\perp \subset V$ which is clearly stable under the subgroup fixing the given isotropic flag.

Proof of previous statements. We use the fact that a torus action on a vector space decomposes as a direct sum of 1-dimensional irreducible representations (cf. §3.3). This implies immediately that any torus in the general linear group has a basis in which it is diagonal, and hence the maximal tori are the ones described.

For the other two cases we take advantage of the fact that, given two eigenspaces relative to two characters χ_1, χ_2, these subspaces are orthogonal under the given invariant form unless $\chi_1\chi_2 = 1$. For instance, assume we are in the symmetric case (the other is identical). Given two eigenvectors u_1, u_2 and an element t of the maximal torus $(u_1, u_2) = (tu_1, tu_2) = (\chi_1\chi_2)(t)(u_1, u_2)$. It follows that if $\chi_1\chi_2 \neq 1$, the two eigenvectors are orthogonal.

By the nondegenerate nature of the form when $\chi_1\chi_2 = 1$, the two eigenspaces relative to the two characters must be in perfect duality since they are orthogonal to the remaining weight spaces. We thus choose, for each pair of characters χ, χ^{-1}, a basis in the eigenspace V_χ of χ and the basis in $V_{\chi^{-1}}$ dual to the chosen basis. We complete these bases with an hyperbolic basis of the eigenspace of 1. In this way we have constructed a hyperbolic basis $\mathcal{B}$ for which the given torus is contained in the torus associated to $\mathcal{B}$.

All hyperbolic bases are conjugate under the orthogonal group. If the orthogonal transformation which conjugates two hyperbolic bases is improper, we may compose it with the exchange of e_i, f_i in order to get a special (proper) orthogonal transformation. So, up to exchanging e_i, f_i, an operation leaving the corresponding torus unchanged, two hyperbolic bases are also conjugate under the special orthogonal group. So the statement for maximal tori is complete in all cases. □

The discussion of Borel subgroups is a little subtler; here one has to use the basic fact:

Theorem (Lie–Kolchin). *A connected solvable group G of matrices is conjugate to a subgroup of upper triangular matrices.*

There are various proofs of this statement which can be found in the literature at various levels of generality. In characteristic 0 it is an immediate consequence of Lie's theorem and the fact that a connected Lie group is solvable if and only if its Lie algebra is solvable. The main step is to prove the existence of a common eigenvector for G from which the statement follows immediately by induction.

A particularly slick proof follows immediately from a stronger theorem.

Borel fixed-point theorem. *Given an action of a connected solvable group G on a projective variety X there exists a fixed point.*

Proof. Work by induction on the dimension of G. If G is trivial there is nothing to prove; otherwise G contains a proper maximal connected normal subgroup H. Since G is solvable $H \supset \{G, G\}$ and thus G/H is abelian (in fact it is easy to prove that it is 1-dimensional). Let X^H be the set of fixed points of H. It is clearly a projective subvariety. Since H is normal in G we have that X^H is G-stable and G/H acts on X^H. Take an orbit Dp of minimal dimension for D on X^H, and let E be the stabilizer of p. We claim that $E = D$, and hence p is the required fixed point. In any event D/E is closed and hence, since X is projective, it is compact. Now D/E is also a connected affine algebraic group. We thus have to use a basic fact from algebraic geometry: *an irreducible affine variety is complete if and only if it reduces to a point.* So D/E is a point, or $D = E$. □

The projective variety to which this theorem has to be applied to obtain the Lie–Kolchin Theorem is the flag variety whose points are the complete flags of linear subspaces $F := V_1 \subset V_2 \subset \cdots \subset V_n = V$ with $\dim V_i = i$. The flag variety is easily seen to be projective (cf. Chapter 10, §5.2).

Clearly a linear map fixes the flag F if and only if it is an upper triangular matrix with respect to a basis $e_1, \dots, e_n$ with the property that V_i is spanned by $e_1, \dots, e_i$ for each $i \leq n$. This shows that Borel's fixed point theorem implies the Lie–Kolchin theorem.

In fact, it is clear that these statements are more or less equivalent. When we have a linear group G acting on a vector space V, finding a vector which is an eigenvector of all the elements of G is the same as finding a line in V which is fixed by G, i.e., a fixed point for the action of G on the projective space of lines of V.

The geometric argument of Borel thus breaks down into two steps. The first step is a rather simple statement of algebraic geometry: when one has an algebraic group G acting on a variety V one can always find at least one closed orbit, for instance, choosing an orbit of minimal dimension. The next point is that for a solvable group the only possible projective closed orbits are points.

Given the theorem of Lie–Kolchin, the study of Borel subgroups is immediate. For the linear group it is clearly a restatement of this theorem. For the other groups, let G be a connected solvable group of linear transformations fixing the form. We do the symmetric case since the other is similar but simpler. We work by induction on the dimension of V. If $\dim V = 1$, then $G = 1$, and a maximal isotropic flag is empty and there is nothing to prove. Let u be an eigenvector of G and $u^\perp$ its orthogonal subspace which is necessarily G-stable. If u is isotropic, $u \in u^\perp$ and the space $u^\perp/\mathbb{C}u$ is equipped with the induced symmetric form (which is nondegenerate) for which G acts again as a group of orthogonal transformations, and we can apply induction.

In the case where u is not isotropic we have a direct sum orthogonal decomposition $V = u^\perp \oplus \mathbb{C}u$. If $g \in G$, we have $gu = \pm u$ since g is orthogonal. The induced map $G \to \pm 1$ is a homomorphism and, since G is connected, it must be identically 1. If $\dim u^\perp > 1$, by induction we can find an isotropic vector stabilized by G in $u^\perp$ and go back to the previous case. If $\dim u^\perp = 1$, the same argument as before shows that $G = 1$. In this case of course G fixes any isotropic flag.

Furthermore, for a connected algebraic group G we have the following important facts.

Theorem 4. *Every element of G is contained in a Borel subgroup.*

If G is a reductive group, then the union of all maximal tori is dense in G.

We leave as exercise to verify these statements directly for classical groups. They will be proved in general in Chapter 10.