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THE WAGNER CURVATURE TENSOR IN NONHOLONOMIC MECHANICS

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We present the classical Wagner construction from 1935 of the curvature tensor for the completely nonholonomic manifolds in both invariant and coordinate way. The starting point is the Shouten curvature tensor for the nonholonomic connection introduced by Vranceanu and Shouten. We illustrate the construction by two mechanical examples: the case of a homogeneous disc rolling without sliding on a horizontal plane and the case of a homogeneous ball rolling without sliding on a fixed sphere. In the second case we study the conditions imposed on the ratio of diameters of the ball and the sphere to obtain a flat space — with the Wagner curvature tensor equal to zero.

1. Introduction

1.1. Historical overview

It is well known that the full difference between the nonholonomic variational problems and nonholonomic mechanics was understood after Hertz [5]. The geometrization of nonholonomic mechanics started in the late 20'th of the XX century with the works of Vranceanu, Synge and Shouten. Vranceanu defined the notion of the nonholonomic structure on a manifold (see [11]). Synge and Shouten made the first steps toward the definition of the curvature in the nonholonomic case (see [9, 8]). It was Shouten who introduced the notion of partial, or nonholonomic connection. However, the highlights of that pioneers period of development of mechanically motivated nonholonomic geometry was the work of V. V. Wagner, published in several papers from 1935 till 1941 (see [13, 14, 15]). Wagner constructed the curvature tensor as an extension of the Shouten tensor. This construction is performed in several steps, following the flag of the distribution. In that sence, the structure of nonholonomicity of a given distribution is reflected in the Wagner construction. For those achievements, Wagner was awarded in 1937 by Kazan University (see [16]).

The main aim of this paper is to present Wagner's construction, both in invariant and coordinate way. The existence of Gorbatenko's recent, review [17] is very helpful in understanding THE original Wagner's works. Since we want to follow the original Wagner ideas, there are some differences from the Gorbatenko's presentation.

We also give two mechanical examples. The first one is the problem of a homogeneous disc rolling without sliding on a horizontal plane and the second is the problem of a homogeneous ball rolling without sliding on a fixed sphere. In both cases we produced the complete computations of the construction of the Wagner curvature tensor. Although the first problem is of degree 2 of

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nonholonomicity, and the second one is of degree 1, the computations in the second case are much more complicated.

The problem of a homogeneous ball rolling without sliding on a fixed sphere is interesting because it gives a family of $(3,5)$-problems depending on a parameter k, which is the ratio between the diameters of the ball and the sphere. We investigate the Wagner flatness in these cases, in terms of this parameter k.

Geometry of the nonholonomic variational problems is deing intensively developed nowadays, (see [6, 7, 1]) motivated by the Control Theory. As an important example, we mention the Agrachev curvature tensor and related invariants of Sub-Riemannian Geometry (see [1]). These natural geometric constructions were developed further in [2, 3], and Agrachev and Zelenko implied their theory to the situation of a homogeneous ball rolling without sliding on a fixed sphere. It appears that there exist k for which their invariants are zero, exactly in the same cases when the Cartan tensor is zero (see [4, 7]).

So, summarizing, we can make a conclusion that the Wagner construction of the curvature tensor is natural, and essentially different from other natural constructions, such as the Cartan and the Agrachev curvatures.

1.2. Basic notions from the nonholonomic geometry

Let us fix some basic notions from the theory of distributions [16].

Definition 1. Let $TM = \bigcup\limits_{x\in M} T_xM$, be the tangent bundle of a smooth n-dimensional manifold M. The sub-bundle $V = \bigcup\limits_{x\in M} V_x$, where V_x is the vector subspace of T_xM, smoothly dependent on points $x \in M$, is the *distribution*. If the manifold M is connected with $\dim V_x$ it is called *the dimension* of the distribution.

A vector field X on M *belongs* to the distribution V if $X(x) \subset V_x$. A curve γ is *admissible relatively to* V, if the vector field $\dot\gamma$ belongs to V.

A differential system is the linear space of vector fields having a structure of $C^\infty(M)$-module. Vector fields which belong to the distribution V form a differential system $N(V)$.

The k-dimensional distribution V is *integrable* if the manifold M is foliated into k-dimensional sub-manifolds, having V_x as the tangent space at the point x. According to the Frobenius theorem, V is integrable if and only if the corresponding differential system $N(V)$ is *involutive*, i.e. if it is a Lie sub-algebra of Lie algebra of the vector fields on M.

Definition 2. *The flag* of a differential system N is a sequence of differential systems: $N_0 = N,\ N_1 = [N, N],\ \ldots,\ N_l = [N_{l-1}, N],\ \ldots$.

The differential systems N_i are not always differential systems of some distributions V_i, but if for every i, there exists V_i, such that $N_i = N(V_i)$, then there exists *a flag* of the distribution V: $V = V_0 \subset V_1 \ldots$. Such distributions, which have flags, will be called *regular*. It is clear that the sequence $N(V_i)$ is going to stabilize, and there exists a number r such that $N(V_{r-1}) \subset N(V_r) = N(V_{r+1})$.

Definition 3. If there exists a number r such that $V_r = TM$, the distribution V is called *completely nonholonomic*, and the minimal such r is *the degree of nonholonomicity* of the distribution V.

We are going to consider only regular and completely nonholonomic distributions.

1.3. The equations of motion of mechanical nonholonomic systems

One of the basic references on nonholonomic mechanics is [18], see also [12]. Let us consider a nonholonomic mechanical system corresponding to a Riemannian manifold (M, g), where g is a metric defined by the kinetic energy. It is well-known that to every Riemannian metric g on M corresponds the connection ∇ with the properties:

$$
\begin{aligned}
&i) \quad \nabla_X g(Y,Z) = X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0,\\
&ii) \quad T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0,
\end{aligned}
$$

where X, Y, Z are the smooth vector fields on M. This *symmetric, metric* connection is usually called *the Levi-Chivita* connection.

We assume that the distribution V is defined by $(n-m)$ 1-forms ω_α; in the local coordinates $q = (q^1, \ldots, q^n)$ on M

$$\omega_\rho(q)(\dot q) = a_{\rho i}(q)\dot q^i = 0, \quad \rho = m+1, \ldots, n \quad ; \quad i = 1, \ldots, n. \tag{1}$$

Definition 4. *The virtual displacement* is the vector field X on M, such that $\omega_\rho(X) = 0$, i.e. X belongs to the differential system $N(V)$.

The differential equations of motion of a given mechanical system follow from the D'Alambert-Lagrange principle: *the trajectory γ of a given system is a solution of the equation*

$$\langle \nabla_{\dot\gamma}\dot\gamma - Q, X\rangle = 0, \tag{2}$$

where X is an arbitrary virtual displacement, Q is the vector field of the internal forces, and ∇ is the metric connection for the metric g.

The vector field $R(x)$ on M, such that $R(x) \in V_x^\perp$, $V_x^\perp \oplus V_x = T_x M$, is called *a reaction of the ideal nonholonomic connections.* Equation (2) can be written in the form:

$$
\begin{gathered}
\nabla_{\dot\gamma}\dot\gamma - Q = R,\\
\omega_\alpha(\dot\gamma) = 0.
\end{gathered}
\tag{3}
$$

If the system is potential, by introducing $L = T - U$, where U is the potential energy of the system ($Q = -\operatorname{grad} U$), then in the local coordinates q on M, equations (3) become:

$$
\begin{gathered}
\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = \widetilde R,\\
\omega_\alpha(\dot q) = 0.
\end{gathered}
\tag{4}
$$

Now $\widetilde R$ is a 1-form in $(V^\perp)$, and it can be represented as a linear combination of 1-forms $\omega^{m+1}, \ldots, \omega^n$ which define the distribution: $\widetilde R = \sum\limits_{\alpha=m+1}^{n} \lambda_\alpha \omega_\alpha$.

Suppose $e_1, \ldots, e_n$ are the vector fields on M, such that $e_1(x), \ldots, e_n(x)$ form a base of the vector space $T_x M$ at every point $x \in M$, and $e_1, \ldots, e_m$ generate the differential system $N(V)$. Express them through the coordinate vector fields:

$$e_i = A_i^j(q)\frac{\partial}{\partial q^j}, \quad i, j = 1, \ldots, n.$$

Denote by p a projection $p : TM \to V$ orthogonal to the metric g. The corresponding homomorphism of C^∞-modules of the sections of TM and V will also be denoted by p:

$$p\left(\frac{\partial}{\partial q^i}\right) = p_i^a e_a, \quad a = 1, \ldots, m, \quad i = 1, \ldots, n.$$

Projecting by p the equations (3), we get $p(R) = 0$, from $R(x) \in V^\perp(x)$, and denoting $p(Q) = \widetilde Q$ we get

$$\widetilde\nabla_{\dot\gamma}\dot\gamma = \widetilde Q, \tag{5}$$

where $\widetilde{\nabla}$ is the projected connection. A relationship between the coefficients $\widetilde{\Gamma}^c_{ab}$ of the connection $\widetilde{\nabla}$, defined by the formula

$$\widetilde{\nabla}_{e_a} e_b = \widetilde{\Gamma}^c_{ab} e_c$$

and the Christoffel symbols Γ^k_{ij} of the connection ∇ follows from

$$\begin{aligned}\widetilde{\nabla}_{e_a} e_b = \widetilde{\Gamma}^c_{ab} e_c = p\,(\nabla_{e_a} e_b) = \\ &= p\left(\nabla_{A^i_a \frac{\partial}{\partial q^i}} A^j_b \frac{\partial}{\partial q^j}\right) = \\ &= p\left(A^i_a \frac{\partial A^j_b}{\partial q^i}\frac{\partial}{\partial q^j} + A^i_a A^j_b \nabla_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j}\right) = \\ &= A^i_a \frac{\partial A^j_b}{\partial q^i} p^c_j e_c + A^i_a A^j_b \Gamma^k_{ij} p^c_k e_c.\end{aligned}$$

Thus we get

$$\widetilde{\Gamma}^c_{ab} = \Gamma^k_{ij} A^i_a A^j_b p^c_k + A^i_a \frac{\partial A^j_b}{\partial q^i} p^c_j. \tag{6}$$

If the motion is taking place under the inertia ($Q = \widetilde{Q} = 0$), the trajectories of the nonholonomic mechanical problem are going to be geodesics for the projected connection $\widetilde{\nabla}$. Equations (5) were derived by Vrancheanu and Shouten.

Note. The projected connection $\widetilde{\nabla}$ is not a connection on the vector bundle V over M, because the parallel transport is defined only along the admissible curves. So, it is called *partial* or *nonholonomic* connection. (The exact definition is in Section 2.2.)

2. The Shouten tensor

Let V be the distribution on M. Denote $C^\infty(M)$-module of sections on V by $\Gamma(V)$.

Definition 1. **Definition 1.** *A nonholonomic connection* on the sub-bundle V of TM is a map $\nabla : \Gamma(V) \times \Gamma(V) \to \Gamma(V)$ with the properties:

$$\begin{aligned} i)\quad & \nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z, \\ ii)\quad & \nabla_X(f \cdot Y) = X(f)Y + f\nabla_X Y, \\ iii)\quad & \nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z, \\ & X, Y, Z \in \Gamma(V); \quad f, g \in C^\infty(M).\end{aligned}$$

Having a morphism of vector bundles $p_0 : TM \to V$, formed by the projection on V, denote by $q_0 = 1_{TM} - p_0$ the projection on W, $V \oplus W = TM$.

Definition 2. The tensor field $T_\nabla : \Gamma(V) \times \Gamma(V) \to \Gamma(V)$ defined in the following way:

$$T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - p_0[X, Y]; \quad X, Y \in \Gamma(V)$$

is called *the tensor of torsion* for the connection ∇.

Suppose there is a positively defined metric tensor g on V:

$$g : \Gamma(V) \times \Gamma(V) \to C^\infty(M), \quad g(X, Y) = g(Y, X).$$

Theorem 1. *Given the distribution V, with p_0 and g, there exists a unique nonholonomic connection ∇ with the properties:*

$$\begin{aligned} &i) \quad \nabla_X g(Y,Z) = X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0, \\ &ii) \quad T_\nabla = 0. \end{aligned} \tag{1}$$

The Theorem 1 is the generalization of a well-known theorem from differential geometry. The proof can be found in [17].

The conditions (1) can be rewritten in the form:

$$\begin{aligned} &i) \quad \nabla_X Y = \nabla_Y X + p_0[X,Y], \\ &ii) \quad Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \end{aligned} \tag{2}$$

By cyclic permutation of X, Y, Z in (2 *ii*)) and by summation we get:

$$\begin{aligned} g(\nabla_X Y, Z) = \frac{1}{2}\{&X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) + \\ &+ g(Z, p_0[X,Y]) + g(Y, p_0[Z,X]) - g(X, p_0[Y,Z]\}. \end{aligned} \tag{3}$$

Let q^i, $(i = 1, \ldots, n)$ be the local coordinates on M, such that the first m coordinate vector fields $\dfrac{\partial}{\partial q^i}$ are projected by projection p_0 into the vector fields e_a, $(a = 1, \ldots, m)$, generating the distribution V: $p_0(\dfrac{\partial}{\partial q^i}) = p^a_i(q) e_a$. Vector fields e_a can be expressed in the basis $\dfrac{\partial}{\partial q^i}$ as $e_a = B^i_a \dfrac{\partial}{\partial q^i}$, with $B^i_a p^b_i = \delta^b_a$. Now we give the coordinate expressions for the coefficients of the connection Γ^c_{ab}, defined as $\nabla_{e_a} e_b = \Gamma^c_{ab} e_c$. From (3) we get:

$$\Gamma^c_{ab} = \{{}^{\,c}_{ab}\} + g_{ae} g^{cd} \Omega^e_{bd} + g_{be} g^{cd} \Omega^e_{ad} - \Omega^c_{ab}, \tag{4}$$

where Ω is obtained from $p_0[e_a, e_b] = -2\Omega^c_{ab} e_c$ as:

$$2\Omega^c_{ab} = p^c_i e_a(B^i_b) - p^c_i e_b(B^i_a),$$

and $\{{}^{\,c}_{ab}\} = \frac{1}{2} g^{ce}(e_a(g_{be}) + e_b(g_{ae}) - e_e(g_{ab}))$.

It was shown in Section 1.3. that the equations of the nonholonomic mechanical problem, without external forces, are the geodesic equations for the connection $\widetilde{\nabla}$. The connection $\widetilde{\nabla}$ is obtained by projection on the sub-bundle V of the Levi-Civita connection ∇ for the metric g. The question is: what is the relationship between the connection $\widetilde{\nabla}$ and the metric $\widetilde{g}$, induced from g on V.

Proposition 1. *The connection $\widetilde{\nabla}$, obtained by projecting the metric torsion-less connection ∇ for the metric g, is the metric torsion-less connection for the induced metric $\widetilde{g}$ if the projector p_0 is orthogonal.*

Proof.

Let $p_0 : TM \to V$ be the orthogonal projector.

a) We need to prove $\widetilde{\nabla}\widetilde{g} = 0$. For the arbitrary $X, Y, Z \in \Gamma(V)$ we have:

$$\widetilde{\nabla}_X \widetilde{g}(Y,Z) = X(\widetilde{g}(Y,Z)) - \widetilde{g}(\widetilde{\nabla}_X Y, Z) - \widetilde{g}(Y, \widetilde{\nabla}_X Z). \tag{5}$$

Since $\widetilde{g}$ is induced by g, it follows that $\widetilde{g}(Y,Z) = g(Y,Z)$. In the same way, $\widetilde{\nabla}_X Y = p_0 \nabla_X Y = \nabla_X Y - U$, where $U \in \Gamma(V^\perp)$ is a vector field projected with p_0 into 0. From the orthogonality condition, U is orthogonal on X, Y and Z relatively to the metric g, so we get: $\widetilde{g}(\widetilde{\nabla}_X Y, Z) = g(\widetilde{\nabla}_X Y, Z) = g(\nabla_X Y - U, Z) = g(\nabla_X Y, Z)$. Similarly, $\widetilde{g}(Y, \widetilde{\nabla}_X Z) = g(Y, \nabla_X Z)$. Plugging into (5), we get:

$$\widetilde{\nabla}_X \widetilde{g}(Y,Z) = \nabla_X g(Y,Z) \quad X, Y, Z \in \Gamma(V),$$

and from the assumption $\nabla g = 0$ we get $\widetilde{\nabla}\widetilde{g} = 0$.

b) We need to show that the connection $\widetilde{\nabla}$ is torsion-less.

$$\begin{aligned} T_{\widetilde{\nabla}}(X,Y) &= \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - p_0[X,Y] \\ &= p_0\nabla_X Y - p_0\nabla_Y X - p_0[X,Y] = p_0(\nabla_X Y - \nabla_Y X - [X,Y]), \end{aligned}$$

and since ∇ is free of torsion, the same is valid for $\widetilde{\nabla}$. ■

Note. Both the Wagner and the Shouten tensor, as we will see later, depend on the choice of the projector. Wagner defined curvature tensor for a metric which is defined on the distribution V. If we start from some mechanical problem, then there is a metric on the whole TM, which is afterwards induced on V. According to the last Proposition, in order to get the projected connection which is metric for the induced metric, one must choose the orthogonal projector. That means, that for the mechanical systems there is a unique choice of the projector.

The problem of definition of the curvature tensor for the nonholonomic connections was considered for the first time by Shouten. He defined the curvature tensor in the following way:

Definition 3. *The Shouten tensor* is a mapping $K : \Gamma(V) \times \Gamma(V) \times \Gamma(V) \to \Gamma(V)$ defined by:

$$K(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{p_0[X,Y]} Z - p_0[q_0[X,Y], Z], \tag{6}$$

where $X, Y, Z \in \Gamma(V)$.

To check that the Definition 3 is correct, one has to verify that K is of tensor nature, i.e. that it is linear on X, Y, Z relatively to multiplication by the smooth functions on M. Really, by direct check [17] we get:

$$\begin{aligned} K(fX,Y)Z &= fK(X,Y)Z, \\ K(X,Y)(fZ) &= fK(X,Y)Z, \\ K(X,Y)Z &= -K(Y,X)Z. \end{aligned}$$

In comparison to the curvature tensor for the connections on M, we see that the Shouten tensor (6) has one term more, the last one in (6), and that in the third term p_0 appears. The last term gives a correction in order that K be a tensor. Note that without that last term linearity for Z relatively to multiplication by the smooth functions would not be satisfied.

A mapping $K(X,Y) : Z \to K(X,Y)Z$ is a morphism of $C^\infty(M)$- module $\Gamma(V)$. Since K is anti-symmetric relatively to X, Y, a $C^\infty(M)$-linear mapping $\Gamma(K) : \Gamma(\wedge^2 V) \to \Gamma(End(V,V))$ can be associated with the Shouten tensor by the condition:

$$\Gamma(K)(X \wedge Y)Z = K(X,Y)Z, \quad X, Y, Z \in \Gamma(V),$$

where $\wedge^2 V$ is the space of bivectors.

3. The Wagner tensor

3.1. The Wagner construction

Wagner constructed a curvature tensor starting from the integrability condition for the tensor equation $\nabla X = U$ where $U \in End(V,V)$, $X \in \Gamma(V)$. If the curvature tensor is zero, then absolute parallelism should take place, i.e. a covariantly constant vector field in any direction should exist, which is equivalent to the integrability of the equations $\nabla X = 0$. Wagner noticed that if the degree of nonholonomicity is greater then 1, then the Shouten tensor does not satisfy the condition of the absolute parallelism, and he suggested a correction. The idea is the following. One starts with a some metric g on V. The metric g is going to be extended to each sub-bundle V_i of the flag $V = V_0 \subset V_1 \subset \ldots \subset V_N = TM$. The next step, the connection on V_i and the curvature tensor analogous to the Shouten tensor are going to be defined. In this way, in the N-th step, the curvature tensor which satisfies the absolute parallelism condition is constructed. The basic Wagner's paper where this was performed is [13].

Let the metric g be defined on the k-dimensional vector space W. Then the metric $g^\wedge$ on $\wedge^2 W$ is defined by the expression:

$$g^\wedge(x_1 \wedge y_1, x_2 \wedge y_2) = \begin{vmatrix} g(x_1,x_2) & g(x_1,y_2) \\ g(y_1,x_2) & g(y_1,y_2) \end{vmatrix}. \tag{1}$$

(The isomorphism $\varphi : \wedge^2 W^* \to (\wedge^2 W)^*$

$$\varphi(f \wedge g)(x \wedge y) = \omega(x,y) = f(x)g(y) - f(y)g(x)$$

is used here.)

Lemma 1. *If g is a positively definite form on W, then $g^\wedge$ is also a positively defined form on $\wedge^2 W$.*

Consider a mapping

$$\Delta : \wedge^2\Gamma(V) \to \Gamma(TM)/\Gamma(V),$$

defined by

$$\Delta(X \wedge Y) = [X,Y] \bmod \Gamma(V)\ , \quad X, Y \in \Gamma(V).$$

The mapping Δ is $C^\infty(M)$ - linear:

$$\begin{aligned}\Delta(fX \wedge Y) = [fX,Y] \bmod \Gamma(V) = \{-Y(f)X + f[X,Y]\} \bmod \Gamma(V) = \\ = f[X,Y] \bmod \Gamma(V) = f\Delta(X \wedge Y).\end{aligned}$$

Observe that $\mathrm{Im}(\Delta)$ is not always equal to $\Gamma(TM)/\Gamma(V)$, but it is its $C^\infty(M)$-submodule, and denote

$$\Gamma(V_1) = \{X \in \Gamma(TM) | X \bmod \Gamma(V) \in Im(\Delta)\}\,.$$

So, we get a sequence of the C^∞ submodules $\Gamma(V_0) \subset \ldots \subset \Gamma(V_N) = \Gamma(TM)$, defined by:

$$\begin{aligned}&\Gamma(V_i) = \{X \in \Gamma(TM) | X \bmod \Gamma(V_{i-1}) \in \mathrm{Im}(\Delta_{i-1})\}\,, \\ &\Delta_i(X \wedge Y) = [X,Y] \bmod \Gamma(V_i), \quad i = 1, \ldots, N,\end{aligned} \tag{2}$$

where $V = V_0$, $\Delta = \Delta_0$. Note that the sequence of sub-bundles $V_0 \subset V_1 \subset \ldots \subset V_N = TM$ is a flag of the distribution V, and N is the degree of nonholonomicity, since we reduced our attention to the case of regular distributions. The mapping $\Delta_i : \wedge^2 V_i \to TM/V_i$ is called *the i-th tensor of nonholonomicity* of the distribution V.

For every point $x \in M$, there is the factor space $V_{i+1,x}/V_{i,x}$ with the projection $\pi_i : V_{i+1,x} \to V_{i+1,x}/V_{i,x}$. Suppose the mappings $\theta_{i,x} : V_{i+1,x}/V_{i,x} \to R_{i,x}$ are defined, where $R_{i,x}$ are some sub-spaces, chosen transversely to $V_{x,i}$, so that $V_{i,x} \oplus R_{i,x} = V_{i+1,x}$. Mappings $q_i = \theta_i \cdot \pi_i$ and $p_i = 1_{V_{i+1}} - q_i$ are the projectors onto R_i and V_i respectively. Now we are going to extend the metric from V to the whole TM.

Theorem 1. *Let the distribution V with the metric g and the mappings $\theta_0, \ldots \theta_{N-1}$ are given. Then there exists the unique metric tensor G on TM, which satisfies the conditions:*

1. $G|_V = g$.
2. *In the direct sum $TM = V_0 \oplus R_0 \oplus \ldots \oplus R_{N-1}$ the components are mutually orthogonal.*
3. $(G|_{R_i})^{-1} = \theta_i \cdot \Delta_i \cdot ((G|_{V_i})^\wedge)^{-1} \cdot (\theta_i \cdot \Delta_i)^*$.

Proof.

For an arbitrary point x on M we have $T_xM = V_{0,x} \oplus R_{0,x} \oplus \ldots \oplus R_{N-1,x}$. Define $G|_{R_i,x} = g_{i+1,x}$ by the condition 3 of this Theorem. By the previous Lemma, $g^{\wedge}_{0,x}$ is a positively defined form on $\wedge^2 V_0$, so we have $(g^{\wedge}_{0,x})^{-1}$ on $(\wedge^2 V_0)^*$. The operation of conjugation preserves positive definitness, so $g_{1,x}$ is also a positively definite form. By iterations we get that $g_{i+1,x}$ are positively definite. ■

The coordinate expressions for the metric enlarged from V_{i-1} to $V_i = V_{i-1} \oplus R_{i-1}$ are obtained in the following way. Let the vectors $e_{a_{i-1}}$ span V_{i-1}. Corresponding dual base we denote by $e^{a_{i-1}}$. If $X_{a_i}e^{a_i}$ is a given 1-form on R_{i-1}, then:

$$
\begin{aligned}
\overset{i}{\rightarrow} g(X_{a_i}e^{a_i}) =& \overset{i}{\rightarrow} g^{a_ib_i}X_{a_i}e_{b_i} = \\
&= (\theta_{i-1}\cdot\Delta_{i-1})(G^{\wedge}_{V_{i-1}})^{-1}(\theta_{i-1}\cdot\Delta_{i-1})^*(X_{a_i}e^{a_i}) = \\
&= (\theta_{i-1}\cdot\Delta_{i-1})(G^{\wedge}_{V_{i-1}})^{-1}(X_{a_i} \overset{i}{-} 1 \rightarrow M^{a_i}_{a_{i-1}b_{i-1}} e^{a_{i-1}} \wedge e^{b_{i-1}}) = \\
&= (\theta_{i-1}\cdot\Delta_{i-1})(g^{\wedge})^{a_{i-1}b_{i-1}c_{i-1}d_{i-1}}(\overset{i}{-}1 \rightarrow M^{a_i}_{a_{i-1}b_{i-1}} X_{a_i} e_{c_{i-1}} \wedge e_{d_{i-1}}) = \\
&= (g^{\wedge})^{a_{i-1}b_{i-1}c_{i-1}d_{i-1}}(X_{a_i} \overset{i}{-} 1 \rightarrow M^{a_i}_{a_{i-1}b_{i-1}} \overset{i}{-} 1 \rightarrow M^{b_i}_{c_{i-1}d_{i-1}} e_{b_i}),
\end{aligned}
$$

where $g^{\wedge a_{i-1}b_{i-1}c_{i-1}d_{i-1}}$ is the inverse metric tensor for $g^{\wedge}$ defined by (1), and $\overset{i}{-}1 \rightarrow M^{b_i}_{c_{i-1}d_{i-1}}$ are the coordinate expressions for the $(i-1)$-th tensor of nonholonomicity Δ_{i-1}. It is obvious that

$$
g^{\wedge a_{i-1}b_{i-1}c_{i-1}d_{i-1}} = \frac{1}{2}(\overset{i}{-}1 \rightarrow g^{a_{i-1}c_{i-1}} \ \overset{i}{-} 1 \rightarrow g^{b_{i-1}d_{i-1}} - \overset{i}{-}1 \rightarrow g^{a_{i-1}d_{i-1}} \ \overset{i}{-} 1 \rightarrow g^{b_{i-1}c_{i-1}}),
$$

so, finally we get

$$
\overset{i}{\rightarrow} g^{a_ib_i} = \overset{i}{-}1 \rightarrow M^{a_i}_{a_{i-1}b_{i-1}} \overset{i}{-} 1 \rightarrow M^{b_i}_{c_{i-1}d_{i-1}} \overset{i}{-} 1 \rightarrow g^{a_{i-1}c_{i-1}} \ \overset{i}{-} 1 \rightarrow g^{b_{i-1}d_{i-1}}.
$$

Let us define morphism of the vector bundles $\mu_i : V_{i+1} \rightarrow \wedge^2 V_i$, by:

$$
\mu_i = \left(B^{\wedge}_i\right)^{-1} \cdot (\theta_i \cdot \Delta_i)^* \cdot G_{i+1}|_{R_i} \cdot \theta_i \cdot \pi_i. \tag{3}
$$

So, if $X \in \Gamma(V_i)$, then $\mu_i(X) = 0$.

Now we get the coordinate expressions for μ_i:

$$
\begin{aligned}
\mu_{i-1}(e_{a_i}) =& \overset{*}{\rightarrow} M^{a_{i-1}b_{i-1}}_{a_i} e_{a_{i-1}} \wedge e_{b_{i-1}} \\
&= \overset{i}{-}1 \rightarrow M^{b_i}_{c_{i-1}d_{i-1}} \overset{i}{\rightarrow} g_{a_ib_i} \ \overset{i}{-} 1 \rightarrow g^{c_{i-1}a_{i-1}} \ \overset{i}{-} 1 \rightarrow g^{d_{i-1}b_{i-1}} e_{a_{i-1}} \wedge e_{b_{i-1}}.
\end{aligned}
$$

The coordinate expressions for μ_i and those for the metrics are in the agreement with the original Wagner's paper [13].

We are ready to expose the Wagner's construction for the curvature tensor for the nonholonomic systems.

Denote by $\overset{0}{\rightarrow} \nabla$ the connection for the metric g_0 on V_0, and by $\overset{0}{\rightarrow} K_{\square}$ the Shouten tensor. Define $\overset{1}{\rightarrow} \square : \Gamma(V_1) \times \Gamma(V_0) \rightarrow \Gamma(V_0)$ by:

$$
\overset{1}{\rightarrow} \square_X U = \overset{0}{\rightarrow} \nabla_{p_0X}U + \overset{0}{\rightarrow} K_{\square}(\mu_0(X))(U) + p_0[q_0X, U],
$$

and $\overset{1}{\rightarrow} K_{\square} : \wedge^2 V_1 \rightarrow End(V_0)$ by the condition:

$$
\Gamma(\overset{1}{\rightarrow} K_{\square})(X \wedge Y)(U) = \overset{1}{\rightarrow} \square_X \overset{1}{\rightarrow} \square_Y U - \overset{1}{\rightarrow} \square_Y \overset{1}{\rightarrow} \square_X U - \overset{1}{\rightarrow} \square_{p_1[X,Y]} U - p_0[q_1[X,Y], U],
$$

where $X, Y \in \Gamma(V_1),\ U \in \Gamma(V_0)$.

Similarly, by induction: $\overset{i}{\to}\Box : \Gamma(V_i)\times\Gamma(V_0)\to\Gamma(V_0)$

$$\overset{i}{\to}\Box_X U = \overset{i-1}{\to}\Box_{p_{i-1}X}U + \overset{i-1}{\to} K_{\Box}(\mu_{i-1}(X))(U) + p_0[q_{i-1}X, U],$$
$$\overset{i}{\to} K_{\Box} : \wedge^2 V_i \to End(V_0) \quad X, Y \in \Gamma(V_i),\ U \in \Gamma(V_0),$$
$$\Gamma(\overset{i}{\to} K_{\Box})(X\wedge Y)U = \overset{i}{\to}\Box_X \overset{i}{\to}\Box_Y U - \overset{i}{\to}\Box_Y \overset{i}{\to}\Box_X U - \overset{i}{\to}\Box_{p_i[X,Y]}U - p_0[q_i[X,Y],U].$$

Finally for $i = N$ we get:

$$\begin{aligned}&\overset{N}{\to}\Box : \Gamma(V_N)\times\Gamma(V_0)\to\Gamma(V_0),\\ &\overset{N}{\to}\Box_X U = \overset{N-1}{\to}\nabla_{p_{N-1}X}U + \overset{N-1}{\to} K_{\Box}(\mu_{N-1}(X))(U) + p_0[q_{N-1}X, U],\end{aligned} \tag{4}$$

$$\begin{aligned}&\overset{N}{\to} K_{\Box} : \wedge^2 V_N \to End(V_0), \quad X, Y \in \Gamma(V_N),\ U \in \Gamma(V_0),\\ &\Gamma(\overset{N}{\to} K_{\Box})(X\wedge Y)U = \overset{N}{\to}\Box_X \overset{N}{\to}\Box_Y U - \overset{N}{\to}\Box_Y \overset{N}{\to}\Box_X U - \overset{N}{\to}\Box_{[X,Y]}U,\end{aligned} \tag{5}$$

because $p_N = id$, and $q_N = 0$.

Theorem 2. *The mappings $\overset{i}{\to}\Box$, satisfy the following conditions:*

1. $\overset{i}{\to}\Box_{fX+gY}U = f\overset{i}{\to}\Box_X U + g\overset{i}{\to}\Box_Y U, \quad f, g\in C^\infty(M)$
2. $\overset{i}{\to}\Box_X(fU) = X(f)U + f\overset{i}{\to}\Box_X U, \quad X, Y\in\Gamma(V_i)$
3. $\overset{N}{\to}\Box$ *is a linear connection on the vector bundle* V.

The proof follows from the direct calculations.

Since $\overset{N}{\to}\Box$ is the connection on the vector bundle, according to the Theorem 2, we get that $\overset{N}{\to} K_{\Box}$ is the curvature tensor of the vector bundle V over M, relative to the connection $\overset{N}{\to}\Box$, and it is called *the Wagner tensor* of a nonholonomic manifold.

Note. In [17], the Wagner tensor is defined in a slightly different manner, as the mapping $K_{\Box} : \wedge^2\Gamma(V_N)\to\Gamma(End(V_{N-1}))$. The way presented here is in agreement with the original Wagner paper [13], as it is going to be clear from the coordinate expressions given below.

3.2. Coordinate expressions for the Wagner tensor

Now we are going to derive the coordinate expressions for the Shouten tensor and the Wagner tensor. The Latin indices a_i run in the intervals $1,\ldots,n_i$, where $n_i = \dim V_i$, and Greek indices α in the interval $1,\ldots,n$. Let e_a be the vector fields spanning the distribution V, and p_0 and q_0 the projectors to V and $V^\perp$ respectively. The components of the Shouten tensor K^d_{abc} are derived from:

$$K(e_a, e_b)(e_c) = K^d_{abc}e_d.$$

Plugging into (2.6) and using the properties of the connection ∇ we get:

$$K^d_{abc} = e_a(\Gamma^d_{bc}) - e_b(\Gamma^d_{ac}) + \Gamma^d_{ae}\Gamma^e_{bc} - \Gamma^d_{be}\Gamma^e_{ac} + 2\Omega^e_{ab}\Gamma^d_{ec} - M^p_{ab}\Lambda^d_{pc}. \tag{6}$$

The coefficients Λ^d_{pc} are defined by $p_0[e_p, e_c] = \Lambda^d_{pc}e_d$, $p = m+1,\ldots,n$ and M^p_{ab} are the components of the tensor of nonholonomicity Δ defined by $M^p_{ab}e_p = q_0[e_a, e_b]$. Expressing e_a in the basis of the coordinate vector fields $\dfrac{\partial}{\partial q^i}$ as $e_a = B^i_a\dfrac{\partial}{\partial q^i}$ and plugging into (6), we get the coordinate expressions for the Shouten tensor, which coincide with those obtained in [13].

Denote by $\overset{i}{\to}\Pi^c_{a_i b}$ the components of the connection for $\overset{i}{\to}\Box$ defined by $\overset{i}{\to}\Box_{e_{a_i}} e_b = \overset{i}{\to}\Pi^c_{a_i b} e_c$, where the vector fields e_{a_i} span the distribution V_i. So, we get:

$$\overset{i}{\to}\Pi^c_{a_i b} = \overset{i-1}{\to} p^{a_{i-1}}_{a_i} \overset{i-1}{\to}\Pi^c_{a_{i-1} b} + \overset{*}{\to} M^{a_{i-1} b_{i-1}}_{a_i} \overset{i-1}{\to} K^c_{a_{i-1} b_{i-1} b} + \overset{i-1}{\to} q^p_{a_i} \Lambda^c_{pb} \tag{7}$$

In the same way we get the coordinate expressions for $\overset{i}{\to}K_{\Box}$:

$$\overset{i}{\to}K^d_{a_i b_i c} = e_{a_i}(\overset{i}{\to}\Pi^d_{b_i c}) - e_{b_i}(\overset{i}{\to}\Pi^d_{a_i c}) + \overset{i}{\to}\Pi^d_{a_i e}\overset{i}{\to}\Pi^e_{b_i c} - \overset{i}{\to}\Pi^d_{b_i e}\overset{i}{\to}\Pi^e_{a_i c} + 2\overset{i}{\to}\Omega^{c_i}_{a_i b_i}\overset{i}{\to}\Pi^d_{c_i c} - \overset{i}{\to}M^p_{a_i b_i}\Lambda^d_{pc}. \tag{8}$$

$\overset{i}{\to}p$ and $\overset{i}{\to}q$ are the corresponding projectors to V_i and $V_i^{\perp}$ and $\overset{i}{\to}\Omega^{c_i}_{a_i, b_i}$ is defined by $2\overset{i}{\to}\Omega^{c_i}_{a_i, b_i} e_{c_i} = -\overset{i}{\to}p[e_{a_i}, e_{b_i}]$, while $\overset{i}{\to}M^p_{a_i b_i}$ are the components of the i-th tensor of nonholonomicity, defined by (2).

Finally, for $i = N$, we get the coordinate expressions for the Wagner tensor

$$\overset{N}{\to}K^d_{a_N b_N c} = e_{a_N}(\overset{N}{\to}\Pi^d_{b_N c}) - e_{b_N}(\overset{N}{\to}\Pi^d_{a_N c}) + \overset{N}{\to}\Pi^d_{a_N e}\overset{N}{\to}\Pi^e_{b_N c} - \overset{N}{\to}\Pi^d_{b_N e}\overset{N}{\to}\Pi^e_{a_N c} + 2\overset{N}{\to}\Omega^{c_N}_{a_N b_N}\overset{N}{\to}\Pi^d_{c_N c}. \tag{9}$$

The vector fields e_{a_N} are now spanning the whole TM.

3.3. Absolute parallelism and the Wagner tensor

We start from the equation

$$\nabla W = U\ , \quad U \in \Gamma(End(V)),\ W \in \Gamma(V). \tag{10}$$

The question is if for a given endomorphism U and for every $X \in \Gamma(V)$, the equation:

$$\nabla_X W = U_X$$

has a solution. From (10) we get:

$$\nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{p_0[X,Y]} W - p_0[q_0[X,Y], W] =$$
$$= \nabla_X U_Y - \nabla_Y U_X - U_{p_0[X,Y]} - p_0[q_0[X,Y], W].$$

So, there exists $X \in \Gamma(V_1)$ such that:

$$\overset{0}{\to}K(\mu_0(X))(W) + p_0[q_0 X, W] = U^{\nabla}(\mu_0(X)),$$

where $U^{\nabla}(\mu_0(X)) = \nabla_X U_Y - \nabla_Y U_X - U_{p_0[X,Y]}$. Then:

$$\nabla_{p_0 X} W + \overset{0}{\to}K(\mu_0(X))(W) + p_0[q_0 X, W] = \overset{1}{\to}U_X = U^{\nabla}(\mu_0(X)) + U_{p_0 X}.$$

The integrability conditions for the equation (10) are reduced to:

$$\overset{1}{\to}\Box W = \overset{1}{\to}U. \tag{11}$$

In the same way, iteratively, we reduce the integrability condition for the equation (10) to the condition:

$$\overset{i}{\to}\Box W = \overset{i}{\to}U.$$

Finally, for $i = N$ we get:

$$\overset{N}{\to}\Box W = \overset{N}{\to}U.$$

So:

$$\begin{aligned}\overset{N}{\to}K(X\wedge Y)(W) &= \overset{N}{\to}\Box_X \overset{N}{\to}\Box_Y W - \overset{N}{\to}\Box_Y \overset{N}{\to}\Box_X W - \overset{N}{\to}\Box_{[X,Y]} W = \\ &= \overset{N}{\to}\Box_X \overset{N}{\to}U_Y - \overset{N}{\to}\Box_Y \overset{N}{\to}U_X - \overset{N}{\to}U_{[X,Y]}.\end{aligned} \tag{12}$$

This equation is the integrability condition for the equation (10). Therefore, in the case $U = 0$, the necessary and sufficient condition for the existence of the vector fields parallel along any direction is that the Wagner tensor is equal to zero.

4. The rolling disc

Now, we are going to illustrate the theory exposed before by calculating the Wagner tensors in two mechanical problems. In this section, we deal with a homogeneous disc of the unit mass and the radius R rolling without sliding on a horizontal plane.

Note that we are going to present only basic steps of the calculations. As it is well known, the configuration space is $M = R^2 \times SO(3)$. For the local coordinates we chose x and y as coordinates of the mass center of the disc, and the Euler angles φ, ψ, θ. The nonholonomic constraints follow from the condition that the velocity of the contact point of the disc and the plane should be equal to zero. The two nonholonomic constraints are:

$$\dot{x} + R\cos\varphi\dot{\psi} + R\cos\theta\cos\varphi\dot{\varphi} - R\sin\theta\sin\varphi\dot{\theta} = 0,$$
$$\dot{y} + R\sin\varphi\dot{\psi} + R\cos\theta\sin\varphi\dot{\varphi} + R\sin\theta\cos\varphi\dot{\theta} = 0.$$

Corresponding 1-forms which define the three-dimensional distribution V are:

$$\omega_1 = dx + R\cos\varphi d\psi + R\cos\theta\cos\varphi d\varphi - R\sin\theta\sin\varphi d\theta,$$
$$\omega_2 = dy + R\sin\varphi d\psi + R\cos\theta\sin\varphi d\varphi + R\sin\theta\cos\varphi d\theta.$$

The vector fields which span the differential system $N(V)$ are:

$$\begin{aligned}
e_1 &= R\cos\varphi\frac{\partial}{\partial x} + R\sin\varphi\frac{\partial}{\partial y} - \frac{\partial}{\partial\psi},\\
e_2 &= \cos\theta\frac{\partial}{\partial\psi} - \frac{\partial}{\partial\varphi},\\
e_3 &= R\sin\theta\sin\varphi\frac{\partial}{\partial x} - R\sin\theta\cos\varphi\frac{\partial}{\partial y} + \frac{\partial}{\partial\theta}.
\end{aligned}$$

First, let us calculate the degree of nonholonomicity of this mechanical system:

$$\begin{aligned}
[e_1, e_2] &= -R\sin\varphi\frac{\partial}{\partial x} + R\cos\varphi\frac{\partial}{\partial y} = T,\\
[e_1, e_3] &= 0,\\
[e_2, e_3] &= -\sin\theta e_1.
\end{aligned}$$

So, the distribution V is nonintegrable, and the whole TM is not generated in the first step. From:

$$[e_1, e_2] = T, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = -\sin\theta e_1,$$
$$[e_1, T] = 0, \quad [e_2, T] = R\cos\varphi\frac{\partial}{\partial x} + R\sin\varphi\frac{\partial}{\partial y} = U,$$

since e_1, e_2, e_3, T, U span the tangent space in the every point of M, the degree of nonholonomicity is 2.

It is well known that the kinetic energy of the system is:

$$2T = \dot{x}^2 + \dot{y}^2 + (A\sin^2\theta + C\cos^2\theta)\dot{\varphi}^2 + 2C\cos\theta\dot{\varphi}\dot{\psi} + C\dot{\psi}^2 + (A + R^2\cos^2\theta)\dot{\theta}^2$$

where A and C are the principle central moments of inertia of the disc in the moving frame. This gives the metric on M:

$$(g_{ij}) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0\\
0 & 0 & A\sin^2\theta + C\cos^2\theta & C\cos\theta & 0\\
0 & 0 & C\cos\theta & C & 0\\
0 & 0 & 0 & 0 & A + R^2\cos^2\theta
\end{pmatrix}.$$

As it was pointed out below the Proposition 2.1, in mechanical problems we chose the orthogonal projector p_0 from TM onto V. The vector fields annulated by p_0 are:

$$
\begin{aligned}
e_4 &= -\sin\varphi(A + R^2\cos^2\theta)\frac{\partial}{\partial x} + \cos\varphi(A + R^2\cos^2\theta)\frac{\partial}{\partial y} + R\sin\theta\frac{\partial}{\partial\theta},\\
e_5 &= C\cos\varphi\frac{\partial}{\partial x} + C\sin\varphi\frac{\partial}{\partial y} + R\frac{\partial}{\partial\psi}.
\end{aligned}
$$

The vector fields e_a are expressed in the basis $\frac{\partial}{\partial x^i}$ by $e_a = B_a^i\frac{\partial}{\partial x_i}$. So we get:

$$
(B_a^i) = \begin{pmatrix} R\cos\varphi & R\sin\varphi & 0 & -1 & 0\\ 0 & 0 & -1 & \cos\theta & 0\\ R\sin\theta\sin\varphi & -R\sin\theta\cos\varphi & 0 & 0 & 1 \end{pmatrix}.
$$

From $p_0\Big(\frac{\partial}{\partial x^i}\Big) = p_i^a e_a$, we get the coordinates of the projector:

$$
(p_i^a) = \begin{pmatrix} \dfrac{R\cos\varphi}{C+R^2} & 0 & \dfrac{R\sin\theta\sin\varphi}{A+R^2}\\[2ex] \dfrac{R\sin\varphi}{C+R^2} & 0 & \dfrac{-R\sin\theta\cos\varphi}{A+R^2}\\[2ex] \dfrac{-C\cos\theta}{C+R^2} & -1 & 0\\[2ex] \dfrac{-C}{C+R^2} & 0 & 0\\[2ex] 0 & 0 & \dfrac{A+R^2\cos^2\theta}{A+R^2} \end{pmatrix}.
$$

Similarly, for q_0 we get:

$$
(q_i^p) = \begin{pmatrix} \dfrac{-\sin\varphi}{A+R^2} & \dfrac{\cos\varphi}{C+R^2}\\[2ex] \dfrac{\cos\varphi}{A+R^2} & \dfrac{\sin\varphi}{C+R^2}\\[2ex] 0 & \dfrac{R\cos\theta}{C+R^2}\\[2ex] 0 & \dfrac{R}{C+R^2}\\[2ex] \dfrac{R\sin\theta}{A+R^2} & 0 \end{pmatrix}.
$$

The induced metric g_{ab} on V, is derived from g_{ij}:

$$
(g_{ab}) = \begin{pmatrix} R^2 + C & 0 & 0\\ 0 & A\sin^2\theta & 0\\ 0 & 0 & A + R^2 \end{pmatrix}.
$$

Now we calculate the components of the connection Γ^c_{ab} for the metric connection using the coordinate expressions (2.4). We start with determining $\left\{ {c \atop ab} \right\}$. The only nonzero coefficients are:

$$\left\{ {2 \atop 23} \right\} = \left\{ {2 \atop 32} \right\} = \frac{\cos\theta}{\sin\theta}, \quad \left\{ {3 \atop 22} \right\} = \frac{-A\sin\theta\cos\theta}{A+R^2}.$$

The coefficients Ω we derive from $-2\Omega^c_{ab} = p_0[e_a, e_b]$. Having the expressions for the commutators of e_a, it can easily be seen that the nonzero elements are:

$$\Omega^3_{12} = -\Omega^3_{21} = \frac{R^2\sin\theta}{2(A+R^2)}, \quad \Omega^1_{23} = -\Omega^1_{32} = \frac{\sin\theta}{2}.$$

From (2.4) we get the following nonzero components of the connection:

$$\Gamma^1_{23} = \frac{-(2R^2+C)\sin\theta}{2(C+R^2)}, \quad \Gamma^1_{32} = \frac{C\sin\theta}{2(C+R^2)}, \quad \Gamma^2_{23} = \Gamma^2_{32} = \frac{\cos\theta}{\sin\theta},$$

$$\Gamma^2_{13} = \Gamma^2_{31} = \frac{-C}{2A\sin\theta}, \quad \Gamma^3_{12} = \frac{C\sin\theta}{2(A+R^2)},$$

$$\Gamma^3_{21} = \frac{(2R^2+C)\sin\theta}{2(A+R^2)}, \quad \Gamma^3_{22} = \frac{-A\sin\theta\cos\theta}{A+R^2}.$$

In order to get the components of the Shouten tensor (see (3.6)), we are calculating the coefficients Λ. From:

$$[e_4, e_1] = 0, \quad [e_4, e_2] = -\cos\varphi(A + R^2\cos^2\theta)\frac{\partial}{\partial x} - \sin\varphi(A + R^2\cos^2\theta)\frac{\partial}{\partial y} - R\sin^2\theta\frac{\partial}{\partial\psi},$$

$$[e_4, e_3] = -R^2\sin\varphi\cos\theta\sin\theta\frac{\partial}{\partial x} + R^2\cos\varphi\cos\theta\sin\theta\frac{\partial}{\partial y} - R\cos\theta\frac{\partial}{\partial\theta}, \quad [e_5, e_1] = 0,$$

$$[e_5, e_2] = -C\sin\varphi\frac{\partial}{\partial x} + C\cos\varphi\frac{\partial}{\partial y}, \quad [e_5, e_3] = 0,$$

we get:

$$\Lambda^1_{42} = \frac{-R(A + R^2\cos^2\theta - C\sin^2\theta)}{C+R^2}, \quad \Lambda^3_{43} = -R\cos\theta, \quad \Lambda^3_{52} = \frac{-RC\sin\theta}{A+R^2}.$$

Similarly, for the components of the tensor of nonholonomicity we get:

$$\overset{0}{\to} M^4_{12} = \frac{R}{A+R^2}, \quad \overset{1}{\to} M^5_{24} = \frac{A+R^2}{C+R^2},$$

where the projectors p_1 and q_1 to V_1 and $V_1^\perp$ are used. Here V_1 is generated by the vector fields e_1, e_2, e_3, e_4:

$$(p^a_i) = \begin{pmatrix} \frac{R\cos\varphi}{C+R^2} & 0 & \frac{R\sin\theta\sin\varphi}{A+R^2} & \frac{-\sin\varphi}{A+R^2} \\ \frac{R\sin\varphi}{C+R^2} & 0 & \frac{-R\sin\theta\cos\varphi}{A+R^2} & \frac{\cos\varphi}{A+R^2} \\ \frac{-C\cos\theta}{C+R^2} & -1 & 0 & 0 \\ \frac{-C}{C+R^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{A+R^2\cos^2\theta}{A+R^2} & \frac{R\sin\theta}{A+R^2} \end{pmatrix}, \quad (q^p_i) = \begin{pmatrix} \frac{\cos\varphi}{C+R^2} \\ \frac{\sin\varphi}{C+R^2} \\ \frac{R\cos\theta}{C+R^2} \\ \frac{R}{C+R^2} \\ 0 \end{pmatrix}.$$

Expansion of the metric from V_0 to V_1 is obtained from the coordinate expression: $\overset{i}{\rightarrow} g^{a_1 b_1} = M^{a_1}_{ab} M^{b_1}_{cd} g^{ac}\ g^{bd}$ as:

$$g^{44} = \frac{2R^2}{(A+R^2)^2(C+R^2)A\sin^2\theta},$$

$$g_{44} = \frac{1}{g^{44}}.$$

Similarly, we get the coordinate expressions for the metric expanded on $V_2 = TM$ by:

$$g^{55} = \frac{4R^2}{A^2(C+R^2)^3\sin^4\theta},$$

$$g_{55} = \frac{1}{g^{55}}.$$

From the expanded metric, as it was mentioned before, we get the components for the morphisms μ_0 and μ_1:

$$\overset{*}{\rightarrow} M^{12}_4 = (\overset{0}{\rightarrow} M^4_{12})^2 g^{11}\ g^{22} = \frac{A+R^2}{2R}, \quad \overset{*}{\rightarrow} M^{24}_5 = \frac{C+R^2}{2(A+R^2)}.$$

Everything is prepared for the calculation of the Wagner tensor. In the coordinate expressions for the Wagner tensor, the first two indices take values from 1 to 5, and the second two from 1 to 3. From the antisymmetry for the first two indexes, there are 90 independent components of the Wagner tensor. We are going to calculate three components. All calculations are performed in three steps: the first step is the Shouten tensor, then the tensor $\overset{1}{\rightarrow} K$ on V_1, and finally the Wagner tensor. We are calculating only the necessary components.

We calculate the component K^2_{451} of the Wagner tensor.

$$K^2_{451} = e_4(\overset{2}{\rightarrow} \Pi^2_{51}) - e_5(\overset{2}{\rightarrow} \Pi^2_{41}) + \overset{2}{\rightarrow} \Pi^2_{4c} \overset{2}{\rightarrow} \Pi^c_{51} - \overset{2}{\rightarrow} \Pi^2_{5c} \overset{2}{\rightarrow} \Pi^c_{41},$$

$$\overset{2}{\rightarrow} \Pi^c_{51} = \overset{*}{\rightarrow} M^{24}_5 \overset{1}{\rightarrow} K^c_{241}, \quad \overset{2}{\rightarrow} \Pi^2_{5c} = \overset{*}{\rightarrow} M^{24}_5 \overset{1}{\rightarrow} K^2_{24c},$$

$$\overset{2}{\rightarrow} \Pi^c_{41} = \overset{1}{\rightarrow} \Pi^c_{41} = \overset{*}{\rightarrow} M^{12}_4 \overset{0}{\rightarrow} K^c_{121},$$

$$\overset{2}{\rightarrow} \Pi^2_{4c} = \overset{1}{\rightarrow} \Pi^2_{4c} = \overset{*}{\rightarrow} M^{12}_4 \overset{0}{\rightarrow} K^2_{12c},$$

$$\overset{1}{\rightarrow} K^c_{241} = e_2(\overset{1}{\rightarrow} \Pi^c_{41}) - e_4(\Gamma^c_{21}) + \Gamma^c_{2d} \overset{1}{\rightarrow} \Pi^d_{41} - \overset{1}{\rightarrow} \Pi^c_{43}\Gamma^3_{21},$$

$$\overset{1}{\rightarrow} K^2_{24c} = e_2(\overset{1}{\rightarrow} \Pi^2_{4c}) - e_4(\Gamma^2_{1c}) + \Gamma^2_{23} \overset{1}{\rightarrow} \Pi^3_{4c} - \overset{1}{\rightarrow} \Pi^2_{4d}\Gamma^d_{2c} + 2 \overset{1}{\rightarrow} \Omega^1_{24}\Gamma^2_{1c},$$

$$\overset{1}{\rightarrow} \Pi^c_{43} = \overset{*}{\rightarrow} M^{12}_4 \overset{0}{\rightarrow} K^c_{123} + \Lambda^c_{43}, \quad \overset{1}{\rightarrow} \Pi^3_{4c} = \overset{*}{\rightarrow} M^{12}_4 \overset{0}{\rightarrow} K^3_{12c}.$$

So, for the component K^2_{451}, we need first the coordinate expressions for the components $\overset{0}{\rightarrow} K^d_{12c}$ of the Shouten tensor. From (3.6) we get:

$$\overset{0}{\rightarrow} K^1_{121} = 0, \quad \overset{0}{\rightarrow} K^2_{121} = \frac{-C(4R^2+C)}{4A(A+R^2)},$$

$$\overset{0}{\rightarrow} K^3_{121} = 0, \quad \overset{0}{\rightarrow} K^1_{122} = \frac{4R^2A + 4R^4\cos^2\theta + C^2\sin^2\theta}{4(A+R^2)(C+R^2)},$$

$$\overset{0}{\rightarrow} K^2_{122} = \frac{R^2\cos\theta}{A+R^2}, \quad \overset{0}{\rightarrow} K^3_{122} = 0,$$

$$\overset{0}{\rightarrow} K^1_{123} = 0, \quad \overset{0}{\rightarrow} K^2_{123} = 0, \quad \overset{0}{\rightarrow} K^3_{123} = \frac{R^2\cos\theta}{A+R^2}.$$

Similarly, we get:

$$\overset{1}{\to}\Pi^1_{41}=0,\quad \overset{1}{\to}\Pi^3_{41}=0,\quad \overset{1}{\to}\Pi^2_{41}=\frac{-C(4R^2+C)}{4AR},$$

$$\overset{1}{\to}\Pi^2_{42}=R\cos\theta,\quad \overset{1}{\to}\Pi^3_{42}=0,$$

$$\overset{1}{\to}\Pi^1_{43}=0,\quad \overset{1}{\to}\Pi^2_{43}=0,\quad \overset{1}{\to}\Pi^3_{43}=0.$$

Therefore:

$$\overset{1}{\to}K^2_{241}=0,\quad \overset{1}{\to}K^2_{242}=0,\quad \overset{1}{\to}K^1_{241}=0,$$

$$\overset{1}{\to}K^2_{243}=\frac{8R^4A\sin^2\theta-10R^2C^2\sin^2\theta-C^3\sin^2\theta+8R^2AC\sin^2\theta+4R^2AC-8R^4C\sin^2\theta+4R^4C\cos^2\theta}{8AR\sin\theta(C+R^2)}.$$

So

$$\overset{2}{\to}\Pi^1_{51}=\overset{*}{=\to}M^{24}_5\overset{1}{\to}K^1_{241}=0,\quad \overset{2}{\to}\Pi^2_{51}=\overset{*}{=\to}M^{24}_5\overset{1}{\to}K^2_{241}=0,\quad \overset{2}{\to}\Pi^2_{52}=\overset{*}{=\to}M^{24}_5\overset{1}{\to}K^2_{242}=0.$$

Finally, we get

$$K^2_{451}=0.$$

In the same way, we can calculate the other components of the Wagner tensor. For example, we are calculating also K^2_{121} and K^3_{121}.

From

$$K^2_{121}=e_1(\overset{2}{\to}\Pi^2_{21})-e_2(\overset{2}{\to}\Pi^2_{11})+\overset{2}{\to}\Pi^2_{1c}\overset{2}{\to}\Pi^c_{21}-\overset{2}{\to}\Pi^2_{2c}\overset{2}{\to}\Pi^c_{11}+2\overset{2}{\to}\Omega^{a_2}_{12}\overset{2}{\to}\Pi^2_{a_21},$$

we get:

$$K^2_{121}=\Gamma^2_{1c}\Gamma^c_{21}+2\overset{2}{\to}\Omega^{a_2}_{12}\overset{2}{\to}\Pi^2_{a_21},$$

and finally:

$$K^2_{121}=0.$$

Similarly $K^1_{133}=\dfrac{C^2}{4A(R^2+C)}$.

5. Ball rolling on a fixed sphere

Now we will give a construction of the Wagner tensor for the system of a homogeneous ball of unit mass rolling on a fixed sphere S^2. Denote the diameters of the ball and the sphere by r_2, r_1 respectively. This system has five degrees of freedom. Let us introduce the following coordinates: the spherical coordinates α,β on S^2 and the Euler angles ψ,φ,θ which determine the position of the ball. The nonholonomic constraints are derived from the condition that the velocity of the contact point is equal to zero. There are two independent nonholonomic constraints:

$$(1+k)\dot\beta+\sin(\psi-\alpha)\dot\theta-\sin\theta\cos(\psi-\alpha)\dot\varphi=0,$$
$$(1+k)\dot\alpha+\tan\beta\cos(\psi-\alpha)\dot\theta+[\tan\beta\sin\theta\sin(\psi-\alpha)-\cos\theta]\dot\varphi-\dot\psi=0,$$

where $k=r_1/r_2$. So, we assume $r_2=1$. Corresponding 1-forms that define the three-dimensional distribution V are:

$$\omega_1=(1+k)d\beta+\sin(\psi-\alpha)d\theta-\sin\theta\cos(\psi-\alpha)d\varphi,$$
$$\omega_2=(1+k)d\alpha+\tan\beta\cos(\psi-\alpha)d\theta+[\tan\beta\sin\theta\sin(\psi-\alpha)-\cos\theta]d\varphi-d\psi=0.$$

The vector fields:

$$X_1 = \frac{\partial}{\partial\alpha} + (1+k)\frac{\partial}{\partial\psi},$$
$$X_2 = \tan\beta\sin\theta\frac{\partial}{\partial\alpha} - (1+k)\sin\theta\cos(\psi-\alpha)\frac{\partial}{\partial\theta} - (1+k)\sin(\psi-\alpha)\frac{\partial}{\partial\varphi} + $$
$$+ (1+k)\cos\theta\sin(\psi-\alpha)\frac{\partial}{\partial\psi},$$
$$X_3 = \sin\theta\frac{\partial}{\partial\beta} - (1+k)\sin\theta\sin(\psi-\alpha)\frac{\partial}{\partial\theta} - (1+k)\cos(\psi-\alpha)\frac{\partial}{\partial\varphi} - $$
$$- (1+k)\cos\theta\cos(\psi-\alpha)\frac{\partial}{\partial\psi}$$

span the differential system $N(V)$. Since

$$[X_1, X_2] = (0, 0, k\cos\theta\cos(\psi-\alpha)\,(1+k), -k\cos(\psi-\alpha)\,(1+k), k\sin\theta\sin(\psi-\alpha)\,(1+k)),$$
$$[X_1, X_3] = (0, 0, k\cos\theta\sin(\psi-\alpha)\,(1+k), -k\sin(\psi-\alpha)\,(1+k), -k\sin\theta\cos(\psi-\alpha)\,(1+k)),$$
$$[X_2, X_3] = \Big(\frac{-\sin^2\theta + (1+k)\sin\theta\sin(\psi-\alpha)\cos\theta\sin\beta\cos\beta}{\cos^2\beta}, -sin\theta\cos(\psi-\alpha)\cos\theta(1+k),$$
$$\frac{(1+k)^2(2\cos^2\theta\cos\beta - \cos\beta) - (1+k)\sin\theta\sin(\psi-\alpha)\sin\beta\cos\theta)}{\cos\beta},$$
$$-\frac{(1+k)^2\cos\theta\,\cos\beta\, - (1+k)\sin\beta\sin\theta\sin(\psi-\alpha)}{\cos\beta},$$
$$\frac{\sin\beta\cos(\psi-\alpha)\,\sin^2\theta\,(1+k)}{cos\beta}\Big)$$

the degree of nonholonomicity is equal to one.

From the kinetic energy of the system:

$$2T = (1+k)^2(\dot\beta^2 + \cos^2\beta\dot\alpha^2) + A(\dot\psi^2 + \dot\varphi^2 + \dot\theta^2 + 2\cos\theta\dot\varphi\dot\psi),$$

where A is the inertia momentum of the ball, the formula for the metric is derived

$$(g_{ij}) = \begin{pmatrix} (1+k)^2\cos^2\beta & 0 & 0 & 0 & 0 \\ 0 & (1+k)^2 & 0 & 0 & 0 \\ 0 & 0 & A & A\cos\theta & 0 \\ 0 & 0 & A\cos\theta & A & 0 \\ 0 & 0 & 0 & 0 & A \end{pmatrix}.$$

We choose the orthogonal projector p_0. The vector fields orthogonal to the distribution V are:

$$X_4 = A\cos(\psi-\alpha)\frac{\partial}{\partial\alpha} + A\tan\beta\cos^2\beta\sin(\psi-\alpha)\frac{\partial}{\partial\beta} - $$
$$- (1+k)\cos^2\beta\cos(\psi-\alpha)\frac{\partial}{\partial\psi} + (1+k)\tan\beta\cos^2\beta\frac{\partial}{\partial\theta}$$
$$X_5 = A\sin\theta\frac{\partial}{\partial\beta} + (1+k)\cos\theta\cos(\psi-\alpha)\frac{\partial}{\partial\psi} - $$
$$- (1+k)\cos(\psi-\alpha)\frac{\partial}{\partial\varphi} + (1+k)\sin\theta\sin(\psi-\alpha)\frac{\partial}{\partial\theta}.$$

So the induced metric on the distribution V is

$$(g_{ab}) = \begin{pmatrix} A+\cos^2\beta & \sin\beta\cos\beta\sin\theta & 0 \\ \sin\beta\cos\beta\sin\theta & \sin^2\theta(A+\sin^2\beta) & 0 \\ 0 & 0 & \sin^2\theta(1+A) \end{pmatrix}.$$

Using formula (2.4) we get:

$$\Gamma^3_{11} = \frac{\sin\beta\cos\beta}{\sin\theta\,(1+A)},\Gamma^3_{12} = -\frac{1}{2}\,\frac{A\,k - A - 2 + 2\,\cos^2\beta}{1+A},$$

$$\Gamma^1_{13} = -\frac{(1+k)\sin\theta\sin\beta\cos\beta}{(1+A)},\quad \Gamma^2_{13} = \frac{1}{2}\,\frac{A\,k - A + \cos^2\beta\,k - 2 + \cos^2\beta}{1+A},$$

$$\Gamma^3_{21} = \frac{1}{2}\,\frac{A + A\,k + 2 - 2\cos^2\beta}{1+A},\quad \Gamma^2_{22} = -(1+k)\cos\theta\cos(\psi-\alpha),$$

$$\Gamma^3_{22} = \frac{(A+\sin^2\beta)\sin\theta\sin\beta}{\cos\beta\,(1+A)},$$

$$\Gamma^1_{23} = \frac{k+1}{2}\,\frac{-A\sin^2\theta + \cos^2\beta - 1 + \cos^2\beta\cos^2\theta + \cos^2\theta}{1+A},$$

$$\Gamma^2_{23} = -\frac{(2\,A - (1+k)\cos^2\beta + 2)\sin\theta\sin\beta}{2\cos\beta\,(1+A)},$$

$$\Gamma^3_{23} = -(1+k)\cos\theta\cos(\psi-\alpha),\quad \Gamma^1_{31} = \frac{1}{2}\,\frac{(-1+k)\cos\beta\sin\beta\sin\theta}{1+A},$$

$$\Gamma^2_{31} = -\frac{1}{2}\,\frac{A + A\,k + \cos^2\beta\,k - \cos^2\beta + 2}{1+A},$$

$$\Gamma^1_{32} = \frac{1}{2}\,\frac{(1+k)(-A\sin^2\theta - 1) + (1-k)(\cos^2\beta\cos^2\theta + \cos^2\theta - \cos^2\beta)}{1+A},$$

$$\Gamma^2_{32} = \frac{1}{2}\,\frac{-2(1+k)(1+A)\sin(\psi-\alpha)\cos\theta + (1-k)\sin\theta\sin\beta\cos\theta}{1+A},$$

$$\Gamma^3_{33} = -(1+k)\sin(\psi-\alpha)\cos\theta.$$

Other Γ are equal to zero. Some components of the Shouten tensor different from zero are:

$$\overset{0}{\to} K^1_{121} = -\overset{0}{\to} K^2_{122} = \frac{((k-1)^2A + 4\,k^2)\,\sin\beta\,\cos\beta\,\sin\theta}{4(1+A)^2},$$

$$\overset{0}{\to} K^2_{121} = -\frac{(1+k^2)(A^2 + A\,\cos^2\beta) + 4Ak(1+k) + 2k(A^2 - A\,\cos^2\beta + 2\,k\,\cos^2\beta)}{(1+A)^2},$$

$$\overset{0}{\to} K^2_{132} = \overset{0}{\to} K^3_{231} = \frac{(-5\,A + 2\,A\,k + 3\,A\,k^2 - 4)\,\cos\beta\,\sin\beta\,\sin\theta}{4(1+A)^2},$$

$$\overset{0}{\to} K^2_{133} = -\frac{(-1+k^2)\,\sin\theta\,\sin\beta\,\cos\beta}{1+A}.$$

The following components of the Shouten tensor are zero:

$$\overset{0}{\to} K^3_{121} = \overset{0}{\to} K^3_{122} = \overset{0}{\to} K^1_{123} = \overset{0}{\to} K^2_{123} = \overset{0}{\to} K^3_{123} = \overset{0}{\to} K^1_{131} = \overset{0}{\to} K^2_{131} = \overset{0}{\to} K^1_{132} =$$
$$\overset{0}{\to} K^2_{132} = \overset{0}{\to} K^3_{133} = \overset{0}{\to} K^1_{231} = \overset{0}{\to} K^2_{231} = \overset{0}{\to} K^1_{232} = \overset{0}{\to} K^2_{232} = \overset{0}{\to} K^3_{233} = 0.$$

Expansion of the metric is given by the following formulae:

$$g^{44} = \frac{2k^2}{A(A+1)^3 \cos^2\beta \cos^2(\psi-\alpha)},$$
$$g^{45} = \frac{-\,2k^2 \sin\beta \sin(\psi-\alpha)}{A(A+1)^3 \sin\theta \cos\beta \cos(\psi-\alpha)},$$
$$g^{55} = \frac{k^2(1-\cos^2\beta \sin^2(\psi-\alpha))}{A(1+A)^3 \sin^2\theta cos^2(\psi-\alpha)}.$$

One of the components of the Wagner tensor is:

$$K^1_{133} = \frac{\sin^2\theta \cos^2\beta(k^2(A+4\sin^2\beta)+2Ak+A+4\cos^2\beta)}{4(1+A)}.$$

From the last formula we get

Theorem 1. *For any k the Wagner curvature tensor is different from zero.*

6. Conclusion

From the Theorem 5.1, it follows that the Wagner tensor is essentially different from the tensors constructed by Cartan [4] and Agrachev's school [1, 2, 3], since it doesn't recognize the nilpotent case. A natural question is to find the theory of Jacobi fields which corresponds to the Wagner curvature.

At the end let us note that the paper [10] the appeared very recently, dealing with geometrization of nonholonomic mechanics, based on some later Cartan's work. The connections studied in [10] are generally not torsionless.

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