I. Introduction to theories without the independence property

Anand Pillay

University of Leeds

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1 Preliminaries

Notation

- $T$ will denote a complete theory in a language $L$, possibly many-sorted. There is no harm for now in assuming $T = T^{eq}$. Although subsequently we may want to work in a one-sorted theory for example and distinguish between 1-types and $n$-types.

- $M, N, ..$ denote models of $T$, $A, B, ..$ subsets of models of $T$, $a, b, c, ..$ (usually finite) tuples from models of $T$, and $x, y, ..$ (finite tuples of) variables.

- It is convenient to fix a “large” saturated model $\bar{M}$ of $T$, and now $M, N, ..$ will denote “small” elementary submodels, $A, B, ..$ small subsets of $\bar{M}$ etc.

- So for example for a sentence $\sigma$ of $L_{\bar{M}}$ we may just write $\models \sigma$ for $\bar{M} \models \sigma$.

- Here “large” may mean of cardinality $\bar{\kappa}$ with $\bar{\kappa}$ inaccessible.

- By a definable set $X$ we usually mean a subset of some sort of $\bar{M}$, definable with parameters in $\bar{M}$. (Remember the set of $n$-tuples from a given sort is also a sort.) We may say $A$-definable to exhibit the parameters over which the set is definable.

- By a type-definable set we mean the solution set in $\bar{M}$ of a partial type over a small set of parameters, equivalently the intersection of a small collection of definable sets (in a given sort). We also have the notion “type-definable over $A$”.

- By a “global complete type” we mean a complete type (usually in a finite tuple of variables) over $\bar{M}$.

- We sometimes may want to realize a global complete type in which case we do so in an elementary extension of $\bar{M}$.
Indiscernibles

Definition 1.1.  
• (i) If $\alpha$ is an ordinal and $b_i$ for $i < \alpha$ are tuples from $\bar{M}$ (of same sort), we say that $(b_i : i < \alpha)$ is an indiscernible sequence over $A$ if whenever $i_1 < ... < i_n < \alpha$ and $j_1 < ... < j_n < \alpha$, then $tp((b_{i_1}, b_{i_2}, ..., b_{i_n})/A) = tp((b_{j_1}, ..., b_{j_n})/A)$. If $A = \emptyset$ we just say “indiscernible sequence”.

• (ii) We have the same notion for an an arbitrary ordered set in place of $(\alpha, <)$.

• (iii) Given an infinite indiscernible sequence $(b_i)_{i}$ over $A$, by the $EM$-type of this sequence we mean the collection of $tp((b_{i_1}, .. b_{i_n})/A)$ for $n = 1, 2, ...$

Lemma 1.2. If $(b_i)_{i}$ is an infinite indiscernible sequence over $A$, then for any infinite totally ordered set $(J, <)$ there is an indiscernible sequence $(c_j : j \in J)$ over $A$ with $EM$ type the same as that of $(b_i)_{i}$.

Proof. Compactness (exercise).

The most powerful tool for producing indiscernible sequences uses the Erdos-Rado theorem, and is:

Lemma 1.3. Given $A$, there is some $\lambda$ such that whenever $(b_i : i < \lambda)$ is a set of tuples of the same sort then there is an indiscernible sequence $(c_i : i < \omega)$ over $A$ such that for each $n$ there are $j_1 < ... < j_n < \lambda$ such that $tp(c_1, ..., c_n/A) = tp(b_{j_1}, ..., b_{j_n}/A)$.

The following special case can be proved using Ramsey’s theorem

Lemma 1.4. Suppose that for each $n$, $\Sigma_n(x_1, ..., x_n)$ is a partial type over $A$, and that $(b_i : i < \omega)$ is a sequence (of suitable tuples) such that for each $n$, and $i_1 < ... < in < \omega$, $\models \Sigma_n(b_{i_1}, ..., b_{i_n})$. Then we can find an indiscernible sequence $(c_i : i < \omega)$ over $A$ with the same feature.

Lascar strong types

The material here is relevant to later talks.

Definition 1.5.  
• (i) $a$ and $b$ are said to have the same strong type over $A$, if $E(a, b)$ for each finite (finitely many classes) equivalence relation $E$ definable over $A$.

• (ii) $a$ and $b$ have the same compact (or $KP$) strong type over $A$ if $E(a, b)$ whenever $E$ is an equivalence relation, type-definable over $A$, and with a bounded ($< \bar{\kappa}$) number of classes, equivalently with $\leq 2^{[E] + \omega + |A|}$ classes.

• (iii) $a$ and $b$ have the same Lascar strong type over $A$, if $E(a, b)$ whenever $E$ is a bounded equivalence relation which is $Aut(M/A)$-invariant.
Some remarks.

- We write \( \text{stp}(a/A) \), \( \text{KPstp}(a/A) \), \( \text{Lstp}(a/A) \), for strong type of \( a \) over \( A \) etc.

- To be honest we have only really defined when e.g. \( \text{Lstp}(a/A) = \text{Lstp}(b/A) \), but we can identify \( \text{Lstp}(a/A) \) with the class of \( a \) modulo the smallest bounded \( \text{Aut}(\bar{M}/A) \)-invariant equivalence relation, etc.

- \( \text{stp}(b/A) \) \( \equiv \) \( \text{tp}(b/\text{acl}^e(A)) \) and \( \text{KPstp}(b/A) \) \( \equiv \) \( \text{tp}(b/\text{bdd}^{beq}(A)) \).

- Fleshing out the details of the above is left as an exercise.

- Of course \( \text{Lstp}(b/A) \) implies (or refines) \( \text{KPstp}(b/A) \) implies \( \text{stp}(b/A) \) implies \( \text{tp}(b/A) \), and these all coincide when \( A \) is a model \( M \).

**Lemma 1.6.** \( \text{Lstp}(a/A) = \text{Lstp}(b/A) \) if and only if there are \( a = a_0, a_1, \ldots, a_n = b \) such that for each \( i = 1, \ldots, n - 1 \), \( a_i, a_{i+1} \) begin (are the first two members of) some infinite \( A \)-indiscernible sequence.

We leave the full proof as an exercise (??). But note the easy part: suppose that \( (a_i : i < \omega) \) is an indiscernible sequence over \( A \). Then all members have the same Lascar strong type over \( A \). For if not, then by “stretching” the sequence using Lemma 1.2 we could find arbitrarily many Lascar strong types over \( A \), contradiction.

- A Lascar strong automorphism over \( A \) is an automorphism of \( \bar{M} \) which fixes all Lascar strong types over \( A \). We let \( \text{Autf}(\bar{M}/A) \) denote the group of such automorphisms.

- Fact. \( \text{Lstp}(a/A) = \text{Lstp}(b/A) \) iff there is a Lascar strong automorphism over \( A \) taking \( a \) to \( b \).

- For a given theory \( T \) (or class of theories) it is important to known when various notions of strong type coincide.

- For example in a stable theory they are all the same.

- In a simple theory, \( \text{Lstp} = \text{KPstp} \) but it is open whether this coincides with \( \text{stp} \).

## 2 NIP

### Definitions and equivalences

**Definition 2.1.**

- (i) The \( L \)-formula \( \phi(x, y) \) is said to be unstable if there are \( a_i, b_i \) for \( i < \omega \) such that for \( i, j < \omega, \models \phi(a_i, b_j) \) iff \( i < j \). \( T \) is said to be unstable if some formula \( \phi(x, y) \) is unstable.
• (ii) $\phi(x,y) \in L$ has the independence property if there are $a_i$ for $i < \omega$ and $b_s$ for $s \subseteq \omega$ such that for all $i, s, \models \phi(a_i, b_s)$ iff $i \in s$. $T$ is said to have the independence property if some formula $\phi(x,y)$ has it.

• (iii) We say that $\phi(x,y)$ is stable if it is not unstable, and has $\text{NIP}$ (or is dependent) if does not have the independence property. Likewise for theories.

**Lemma 2.2.**

• (i) If $\phi(x,y)$ has the independence property then it is unstable.

• (ii) $\phi(x,y)$ is stable iff there is $n_\phi < \omega$ such that for any indiscernible sequence $(a_i : i < \omega)$ and any $b$ (of appropriate sorts), $|\{i < \omega : \models \phi(a_i, b)\}| \leq n_\phi$ or $|\{i < \omega : \models \neg\phi(a_i, b)\}| \leq n_\phi$.

• (iii) $\phi(x,y)$ has $\text{NIP}$ iff there is $n_\phi$ such that for any indiscernible sequence $(a_i : i < \omega)$ and $b$ there do not exist $i_1 < i_2 < \ldots < i_{n_\phi}$ such that for each $j = 1, \ldots, n - 1$, $\models \phi(a_{i_j}, b) \leftrightarrow \neg\phi(a_{i_j}, b)$. (i.e. the truth value of $\phi(a_i, b)$ cannot change its mind at least $n_\phi$ times.)

**Proof.** Exercise.

**Corollary 2.3.** Suppose $T$ has $\text{NIP}$ and $(b_i : i < \omega)$ is an indiscernible sequence, and $\phi(x,y) \in L$. Then $\{\phi(x,b_i) \Delta \phi(x,b_{i+1}) : i = 0, 2, 4, \ldots\}$ is inconsistent.

(Where $\phi(x,y) \Delta \phi(x,z)$ denotes $(\phi(x,y) \land \neg\phi(x,z)) \lor (\neg\phi(x,y) \land \phi(x,z))$.)

**Proof.** If not let $a$ realize this set of formulas. Either there are infinitely many even $i$ such that $\models \phi(a, b_i)$ or infinitely many even $i$ such that $\models \neg\phi(a, b_i)$. In either case we contradict Lemma 2.2 (iii).

**Average types and eventual types**

• In this mini-section we will assume that $T$ has $\text{NIP}$.

• Suppose $I = (a_i : i < \omega)$ is an indiscernible sequence (or more generally an indiscernible sequence where the index set has no greatest element).

• By Lemma 2.2 for any formula $\phi(x,b)$, either for eventually all $i < \omega$, $\models \phi(a_i, b)$, or for eventually all $i < \omega$, $\models \neg\phi(a_i, b)$.

• So for any set $B$ of parameters (or even for $B = \bar{M}$) we can define $Av(I/B)$, the average type of $I$ over $B$ to be those formulas $\phi(x)$ over $B$ which are true of eventually all $a_i$.

• So $Av(I/B) \in S_x(B)$.
We now prove (i) using the above characterization. Let

We want to show that for any

(ii) Suppose $I$ is $A$-special, and $B \supseteq A$. We define $Ev(I/B)$ (eventual type of $I$ over $B$) to be the set of formulas $\phi(x)$ over $B$ such that for any $I'$ realizing $tp(I/A)$ there is $J$ realizing $tp(I/A)$ such that $I'J$ is indiscernible over $A$ and $\phi(x) \in Av(I'/J/B)$ (equivalently $Av(J/B)$).

Lemma 2.5. Let $I$ be $A$-special. Then

(i) For any $B \supseteq A$, $Ev(I/A) \subseteq S(B)$.

(ii) For $\phi(x)$ over $B$, $\phi(x) \in Ev(I/B)$ iff there is $I'$ realizing $tp(I/A)$ and witnesses a greatest possible alternation of truth values of $\phi(x)$, such that $\phi(x) \in Av(I'/B)$, i.e. $\phi(x)$ is true for eventually all elements of $I'$.

(iii) If $A \subseteq B \subseteq C$ then $Ev(I/B) \subseteq Ev(I/C)$.

(iv) $Ev(I/B)$ depends only on $tp(I/A)$ (in fact on its $EM$-type over $A$).

Proof.

We start by proving (ii).

Suppose $\phi(x) \in Ev(I/B)$. Let $I'$ realize $tp(I/A)$ and witness a greatest possible alternation of truth values of $\phi(x)$. By definition of $Ev(I/A)$ there is $J$ such that $I'J$ is indiscernible over $A$ and $\phi(x) \in Av(I'/J/B)$. But by choice of $I'$ this implies that $\phi(x) \in Av(I'/B)$.

On the other hand suppose the RHS holds, and let $I'$ witness greatest alternation of truth values of $\phi(x)$. Let $I_1$ realize $tp(I/A)$ and let $J$ be such that both $I'J$ and $I_1J$ are $A$-indiscernible. So $\phi(x) \in Av(I'/J/B) = Av(I_1J/B)$. Good.

We now prove (i) using the above characterization.

We want to show that for any $\phi(x)$ over $B$ exactly one of $\phi(x)$, $\neg \phi(x)$ is in $Ev(I/B)$.

By (ii) we have at least one.

Suppose that $I', I''$ each witness the maximum alternation of truth values of $\phi(x)$. Let $J$ be such that both $I'J$, $I''J$ are indiscernible over $A$. Then $Av(I'/B) = Av(I'J/B) = Av(J/B) = Av(I''J/B) = Av(I''/B)$. So $\phi(x) \in Av(I'/B)$ iff $\phi(x) \in Av(I''/B)$. Good.

The rest is left as an exercise.

Exercise: Give an example of an indiscernible sequence in an $\omega$-minimal theory which is independent (in the $\omega$-minimal sense) but not special.