Lecture 004 (September 30, 2007)

9 Simplicial sets

A *simplicial set* is a functor $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$, ie. contravariant set-valued functor on the ordinal number category $\mathbf{\Delta}$. Such things are usually written $\mathbf{n} \mapsto X_n$, and X_n is called the set of n-*simplices* of X.

A *simplicial map* $f : X \to Y$ is a natural transformation of such functors. The simplicial sets and simplicial maps form the category of simplicial sets, which will be denoted by $s\mathbf{Set}$ in these notes — one also sees the notation $\mathbf{S}$ for this category.

A simplicial set is a simplicial object in the set category. Generally, if $\mathcal{A}$ is some category, then a simplicial object in $\mathcal{A}$ is a functor $A : \mathbf{\Delta}^{op} \to \mathcal{A}$, maps between such things are natural transformations, and the simplicial objects in $\mathcal{A}$ and their morphisms assemble to form a category $s\mathcal{A}$. Thus, for example, one has the categories $s\mathbf{Gr}$ of simplicial groups, $s(R-\mathbf{Mod})$ of simplicial R-modules, $s(s\mathbf{Set}) = s^2\mathbf{Set}$ of bisimplicial sets, and so on. There are simplicial objects everywhere you look.

Examples:

1) We've already met a simplicial set, namely the singular set $S(X)$ associated to a topological space X. Recall that the singular set is defined by the cosimplicial space (covariant functor) $\mathbf{n} \mapsto |\Delta^n|$ by

$$S(X)_n = \hom(|\Delta^n|, X),$$

and any ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ defines a function

$$S(X)_n = \hom(|\Delta^n|, X) \xrightarrow{\theta^*} \hom(|\Delta^m|, X) = S(X)_m$$

by precomposition with the map $\theta_* : |\Delta^m| \to |\Delta^m|$ that was defined in Section 4 (Lecture 002).

The assignment $X \mapsto S(X)$ defines a covariant functor

$$S : \mathbf{CGHaus} \to s\mathbf{Set}$$

in the obvious way, and this functor is called the *singular functor*.

2) The ordinal number $\mathbf{n}$ represents a contravariant functor

$$\Delta^n = \hom_{\mathbf{\Delta}}(\ ,\mathbf{n}),$$

which is called the *standard n-simplex.* Write

$$\iota_n = 1_{\mathbf{n}} \in \hom_{\mathbf{\Delta}}(\mathbf{n}, \mathbf{n}).$$

The n-simplex ι_n is often called the *classifying n-simplex*, because it's a consequence of the Yoneda Lemma that there is a natural bijection

$$\hom_{s\mathbf{Set}}(\Delta^n, Y) \cong Y_n$$

defined by sending the map $\sigma : \Delta^n \to Y$ to the element $\sigma(\iota_n) \in Y_n$. I usually say that a map $\Delta^n \to Y$ is an n-simplex of Y.

Note that every ordinal number morphism $\theta : \mathbf{m} \to \mathbf{n}$ induces a simplicial set map

$$\theta : \Delta^m \to \Delta^n$$

by composition. In this way one obtains a covariant functor $\Delta : \mathbf{\Delta} \to s\mathbf{Set}$ with $\mathbf{n} \mapsto \Delta^n$. This is a cosimplicial object in $s\mathbf{Set}$, and is a fundamental object of study.

In general, if $\sigma : \Delta^n \to X$ is a simplex of X, then the i^{th} *face* $d_i(\sigma)$ is the composite

$$\Delta^{n-1} \xrightarrow{d^i} \Delta^n \xrightarrow{\sigma} X,$$

while the j^{th} *degeneracy* $s_j(\sigma)$ is the composite

$$\Delta^{n+1} \xrightarrow{s^j} \Delta^n \xrightarrow{\sigma} X.$$

3) Write $\partial\Delta^n$ for the subobject of Δ^n which is generated by the $(n-1)$-simplices d^i, $0 \leq i \leq$

n, and let Λ^n_k be the subobject of $\partial\Delta^n$ which is generated by the simplices d^i, $i \neq k$. $\partial\Delta^n$ is called the *boundary* of Δ^n, and Λ^n_k is called the k^{th} *horn*.

The faces $d^i : \Delta^{n-1} \to \Delta^n$ determine a covering

$$\bigsqcup_{i=0}^{n} \Delta^{n-1} \to \partial\Delta^n,$$

and for each $i < j$ there are pullback diagrams

$$\begin{array}{ccc} \Delta^{n-2} & \xrightarrow{d^{j-1}} & \Delta^{n-1} \\ {\scriptstyle d^i}\downarrow & & \downarrow{\scriptstyle d^i} \\ \Delta^{n-1} & \xrightarrow[d^j]{} & \Delta^n \end{array}$$

(excercise!). It follows that there is a coequalizer

$$\sqcup_{i<j,0\leq i,j\leq n} \Delta^{n-2} \rightrightarrows \sqcup_{0\leq i\leq n} \Delta^{n-1} \longrightarrow \partial\Delta^n$$

in $s\mathbf{Set}$. Similarly, there is a coequalizer

$$\sqcup_{i<j,i,j\neq k} \Delta^{n-2} \rightrightarrows \sqcup_{0\leq i\leq n,i\neq k} \Delta^{n-1} \longrightarrow \Lambda^n_k.$$

4) Suppose that a category C is *small* in the sense that the morphisms $\mathrm{Mor}(C)$ and objects $\mathrm{Ob}(C)$ are sets. Examples of such things include all finite ordinal numbers $\mathbf{n}$ (because they are posets), all monoids (small categories having one object), and all groups.

If C is a small category there is a simplicial set BC with

$$BC_n = \hom(\mathbf{n}, C),$$

meaning the functors $\mathbf{n} \to C$. The simplicial structure on BC is defined by precomposition with ordinal number maps. In other words, if $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map (aka. functor) and $\sigma : \mathbf{n} \to C$ is an n-simplex, then $\theta^*(\sigma)$ is the composite functor

$$\mathbf{m} \xrightarrow{\theta} \mathbf{n} \xrightarrow{\sigma} C.$$

The object BC is called, variously, the *classifying space* or *nerve* of C (some people use the notation NC). The "classifying space" name reflects the fact that, if G is a group, then BG can be used to construct the standard classifying space for G in **CGHaus**.

Note that $B\mathbf{n} = \Delta^n$ in this notation.

5) Suppose that I is a small category, and that $X : I \to \mathbf{Set}$ is a set-valued functor. The *category of elements* $E_I(X)$ associated to X has as objects all pairs (i, x) with $x \in X(i)$. A morphism $\alpha : (i, x) \to (j, y)$ is a morphism $\alpha : i \to j$ of I such that $\alpha_*(x) = y$. The simplicial set $B(E_I X)$ has various names, but it's often called the *homotopy*

colimit for the functor X. One often write

$$\underrightarrow{\text{holim}}_I \, X = B(E_I X).$$

Here's a different point of view on the nerve BI:

$$BI = \underrightarrow{\text{holim}}_I \, *,$$

meaning that BI is the homotopy colimit of the functor $I \to \mathbf{Set}$ which associates the one-point set $*$ to every object of I.

There is a canonical functor $E_I X \to I$ which is defined by the assignment $(i, x) \mapsto i$. This functor induces a canonical simplicial set map

$$\pi : B(E_I X) = \underrightarrow{\text{holim}}_I \, X \to BI.$$

The question of when this map looks like a fibration will become a real issue for us later.

A functor $\mathbf{n} \to C$ is uniquely specified by a string of arrows

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} a_n$$

in C, for then all composites of these arrows are uniquely determined. The functors $\mathbf{n} \to E_I X$ can be identified with strings

$$(i_0, x_0) \xrightarrow{\alpha_1} (i_1, x_1) \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} (i_n, x_n).$$

Note that such a string is uniquely specified by the underlying string $i_0 \to \cdots \to i_n$ in the index category Y and $x_0 \in X(i_0)$. It follows that there is an identification

$$(\underrightarrow{\text{holim}}_I\, X)_n = B(E_I X)_n = \bigsqcup_{i_0 \to \dots \to i_n} X(i_0).$$

The construction is natural with respect to natural transformations in X. Thus a diagram $X : I \to s\mathbf{Set}$ taking values in simplicial sets, or rather a simplicial object in set-valued functors determines a simplicial category $m \mapsto E_I(X_m)$ and a corresponding bisimplicial set with (n,m) simplices

$$B(E_I X)_m = \bigsqcup_{i_0 \to \dots \to i_n} X(i_0)_m.$$

The diagonal $d(Y)$ of a bisimplicial set Y is the simplicial set with n-simplices $Y_{n,n}$. Equivalently, $d(Y)$ is the composite functor

$$\mathbf{\Delta}^{op} \xrightarrow{\Delta} \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \xrightarrow{Y} \mathbf{Set}$$

where $\mathbf{\Delta}$ is the diagonal functor.

The diagonal $dB(E_I X)$ of the bisimplicial set $B(E_I X)$ is the *homotopy colimit* $\underrightarrow{\text{holim}}_I\, X$ of the functor $X : I \to s\mathbf{Set}$, and there is a natural simplicial set map

$$\pi : \underrightarrow{\text{holim}}_I\, X \to BI.$$

6) Suppose that X and Y are simplicial sets. There is a simplicial set $\mathbf{hom}(X,Y)$ with n-simplices

$$\mathbf{hom}(X,Y)_n = \hom(X \times \Delta^n, Y).$$

If $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map and $f : X \times \Delta^n \to Y$ is an n-simplex of $\mathbf{hom}(X,Y)$, then $\theta^*(f)$ is the composite

$$X \times \Delta^m \xrightarrow{1\times\theta} X \times \Delta^m \xrightarrow{f} Y.$$

The object $\mathbf{hom}(X,Y)$ is called the *function complex.*

There is a natural simplicial set map

$$ev : X \times \mathbf{hom}(X,Y) \to Y$$

which sends the pair $(x, f : X \times \Delta^n \to Y)$ to the simplex $f(x,\iota_n)$.

Suppose that K is another simplicial set. There is a function

$$ev_* : \hom(K, \mathbf{hom}(X,Y)) \to \hom(X \times K, Y)$$

defined by sending the map $g : K \to \mathbf{hom}(X,Y)$ to the composite

$$X \times K \xrightarrow{1\times g} X \times \mathbf{hom}(X,Y) \xrightarrow{ev} Y.$$

This function is a bijection, with an inverse which takes a morphism $f : X \times K \to Y$ to the morphism $f_* : K \to \mathbf{hom}(X,Y)$ such that $f_*(y)$ is the composite

$$X \times \Delta^n \xrightarrow{1 \times y} X \times \Delta^n \xrightarrow{f} Y.$$

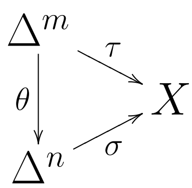
The natural bijection

$$\hom(X \times K, Y) \cong \hom(K, \mathbf{hom}(X, Y))$$

is often called the *exponential law*, and it gives $s\mathbf{Set}$ the structure of a cartesian closed category. The function complexes also give $s\mathbf{Set}$ the structure of a category enriched in simplicial sets.

10 The simplex category and realization

Suppose that X is a simplicial set. The *simplex category* for X has for objects all simplices $\Delta^n \to X$; its morphisms are the incidence relations between the simplices, meaning all commutative diagrams



(1)

There are various notations for this category: you will see it denoted by $\mathbf{\Delta} \downarrow X$ in [2] — this is based on Mac Lane's old notation for comma categories. People in the category theory community would now be inclined to write $\mathbf{\Delta}/X$, and say that this is a type of slice category. In the broader context of homotopy theories associated to a test category (long story — see [4]) it is typical to say that the simplex category is a *cell category*, and, following Cisinski, write $i_{\mathbf{\Delta}}(X)$ to denote the thing. I will stick with $\mathbf{\Delta}/X$.

The assignment $X \mapsto \mathbf{\Delta}/X$ defines a functor

$$\mathbf{\Delta}/? : s\mathbf{Set} \to \mathbf{cat}$$

taking values in small categories and all functors between them.

Exercise 10.1. Show that a simplicial set X is a colimit of its simplices, in that the simplices $\Delta^n \to X$ define a simplicial set map

$$\varinjlim_{\Delta^n \to X} \Delta^n \to X$$

which is an isomorphism.

There is a space $|X|$, called the *realization* of the simplicial set X, which is defined by

$$|X| = \varinjlim_{\Delta^n \to X} |\Delta^n|.$$

Here $|\Delta^n|$ is the topological standard n-simplex, as described in Section 4 (Lecture 002). In words, $|X|$

is the colimit of the functor $\mathbf{\Delta}/X \to \mathbf{CGHaus}$ which takes the morphism displayed in (1) to the continuous map

$$|\Delta^m| \xrightarrow{\theta} |\Delta^n|.$$

The assignment $X \mapsto |X|$ defines a functor

$$|\ | : s\mathbf{Set} \to \mathbf{CGHaus}.$$

Lemma 10.2. *The realization functor is left adjoint to the singular functor $S : \mathbf{CGHaus} \to s\mathbf{Set}$.*

Proof. A simplicial set X is a colimit of its simplices. Thus, for a simplicial set X and a space Y, there are natural isomorphisms

$$\begin{aligned} \hom(X, S(Y)) &\cong \hom(\varinjlim_{\Delta^n \to X} \Delta^n, S(Y)) \\ &\cong \varprojlim_{\Delta^n \to X} \hom(\Delta^n, S(Y)) \\ &\cong \varprojlim_{\Delta^n \to X} \hom(|\Delta^n|, Y) \\ &\cong \hom(\varinjlim_{\Delta^n \to X} |\Delta^n|, Y) \\ &= \hom(|X|, Y). \end{aligned}$$

□

Remark: This was the first example of an adjoint pair of functors. Kan introduced the adjoint functors concept to describe the relation between the realization and singular functors.

Examples:

1) $|\Delta^n| = |\Delta^n|$, since the simplex category $\mathbf{\Delta}/\Delta^n$ has a terminal object, namely $1 : \Delta^n \to \Delta^n$.
2) $|\partial\Delta^n| = |\partial\Delta^n|$ and $|\Lambda^n_k| = |\Lambda^n_k|$, since the realization functor is a left adjoint and therefore preserves coequalizers and coproducts.

The n^{th} *skeleton* $\operatorname{sk}_n X$ of a simplicial set X is the subobject generated by the simplices X_i, $0 \leq i \leq n$. The ascending sequence of subcomplexes

$$\operatorname{sk}_0 X \subset \operatorname{sk}_1 X \subset \operatorname{sk}_2 X \subset \dots$$

defines a filtration of X, and there are pushout diagrams

$$\begin{array}{ccc} \sqcup_{x \in NX_n} \partial\Delta^n & \longrightarrow & \operatorname{sk}_{n-1} X \\ \downarrow & & \downarrow \\ \sqcup_{x \in NX_n} \Delta^n & \longrightarrow & \operatorname{sk}_n X \end{array} \tag{2}$$

Here, NX_n denotes the set of non-degenerate n-simplices of X.

Exercise 10.3. Show that the diagram (2) is indeed a pushout. For this, it's helpful to know that the functor $X \mapsto \operatorname{sk}_n X$ is left adjoint to truncation up to level n. For *that*, you should know that every simplex x of a simplicial set X has a unique representation $x = s^*(y)$ where s is an iterated (or empty) codegeneracy and y is non-degenerate.

Corollary 10.4. *The realization of a simplicial set is a CW-complex.*

With a little more work (and the same ideas) you can show that every monomorphism $A \to B$ of simplicial sets induces a cofibration $|A| \to |B|$ of spaces. In fact, $|B|$ is constructed from $|A|$ by attaching cells.

Lemma 10.5. *The realization functor preserves finite limits. Equivalently, it preserves finite products and equalizers.*

Proof. There are isomorphisms

$$\begin{aligned}
|X \times Y| &\cong |\varinjlim_{\Delta^n \to X, \Delta^m \to Y} \Delta^n \times \Delta^m| \\
&\cong \varinjlim_{\Delta^n \to X, \Delta^m \to Y} |\Delta^n \times \Delta^m| \\
&\cong \varinjlim_{\Delta^n \to X, \Delta^m \to Y} |\Delta^n| \times |\Delta^m| \\
&\cong |X| \times |Y|
\end{aligned}$$

If $\sigma, \tau : \Delta^n \to Y$ are simplices such that

$$|\sigma| = |\tau| : |\Delta^n| \to |Y|,$$

then $\sigma = \tau$ (exercise).

Suppose that $f, g : X \to Y$ are simplicial set maps, and $x \in |X|$ is an element such that $f_*(x) = g_*(x)$. If σ is the "carrier" of x (ie. non-degenerate simplex of X such that x is interior to the cell defined by σ), then $f_*(y) = g_*(y)$ for all y in the interior of $|\sigma|$ (by transforming by a suitable automorphism of the cosimplicial space $|\Delta|$ — see [1]), p. 51). But then

$$|f\sigma| = |g\sigma| : |\Delta^n| \to |Y|,$$

so that $f\sigma = g\sigma$ and $x \in |E|$, where E is the equalizer of f and g in $s\mathbf{Set}$. $\square$

11 Model structure for simplicial sets

Say that a map $f : X \to Y$ of simplicial sets is a *weak equivalence* if the induced map $f_* : |X| \to |Y|$ is a weak equivalence of **CGHaus**.

A map $i : A \to B$ of simplicial sets is a *cofibration* if and only if it is a monomorphism, meaning that all functions $i : A_n \to B_n$ are injective.

A simplicial set map $p : X \to Y$ is a *fibration* if and only if it has the right lifting property with respect to all trivial cofibrations.

Remark 11.1. There is a natural commutative diagram

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{\nabla} & X \\ {\scriptstyle (i_0,i_1)} \downarrow & \nearrow {\scriptstyle pr} & \\ X \times \Delta^1 & & \end{array} \tag{3}$$

for simplicial sets X. Here, (i_0, i_1) can be identified up to isomorphism with the cofibration

$$1_X \times i : X \times \partial\Delta^1 \to X \times \Delta^1$$

which is induced by the inclusion $i : \partial\Delta^1 \subset \Delta^1$. The two inclusions i_ϵ of the end points of the cylinder are weak equivalences, as is the projection $pr : X \times \Delta^1 \to X$. The diagram (3) becomes a natural cylinder object for the model structure on simplicial sets which is developed below (see Theorem 11.7) and left homotopy with respect to this cylinder object is classical *simplicial homotopy.*

Lemma 11.2. *A map $p : X \to Y$ is a trivial fibration if and only if it has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.*

Proof. Suppose that p has the lifting property. Then p has the right lifting property with respect to all cofibrations, so that the lifting s exists in the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & \overset{s}{\nearrow} & \downarrow p \\ Y & \underset{1_Y}{\longrightarrow} & Y \end{array}$$

since all simplicial sets are cofibrant. The lifting h exists in the diagram

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{(sp,1)} & X \\ i \downarrow & \overset{h}{\nearrow} & \downarrow p \\ X \times \Delta^1 & \underset{p \cdot pr}{\longrightarrow} & Y \end{array}$$

and it follows that the map $p_* : |X| \to |Y|$ is a homotopy equivalence, hence a weak equivalence.

Suppose that p is a trivial fibration and choose a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & U \\ & {\scriptstyle p}\searrow & \downarrow q \\ & & Y \end{array}$$

such that j is a cofibration and q has the right lifting property with respect to all maps $\partial\Delta^n \subset \Delta^n$ (such things exist by a small object argument). Then q is a weak equivalence by the last paragraph, so that j is a trivial cofibration and the lifting r

exists in the diagram

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle j}\downarrow & \overset{r}{\nearrow} & \downarrow{\scriptstyle p} \\ U & \xrightarrow[q]{} & Y \end{array}$$

It follows that p is a retract of q, and therefore has the desired right lifting property. □

Here's a lemma that can be proved with simplicial approximation techniques [3]:

Lemma 11.3. *Suppose that a simplicial set X has at most countably many non-degenerate simplices. Then the set of path components $\pi_0|X|$ and all homotopy groups $\pi_n(|X|, x)$ are countable.*

Here's a consequence:

Lemma 11.4 (Bounded cofibration lemma)**.** *Suppose that $i : X \to Y$ is a trivial cofibration and that $A \subset Y$ is a countable subcomplex. Then there is a countable subcomplex $B \subset Y$ with $A \subset B$ such that the map $B \cap X \to B$ is a trivial cofibration.*

Proof. Write $B_0 = A$ and consider the map $B_0 \cap X \to B_0$. Then the homotopy groups of the spaces $|B_0|$ and $|B_0 \cap X|$ are countable by Lemma 11.3.

Note that Y is a union of its countable subcomplexes.

It two maps

$$\alpha, \beta : (|\Delta^n|, |\partial\Delta^n|) \to (|B_0 \cap X|, x)$$

become homotopic in $|B_0|$ hence in $|X|$, then the map defining the homotopy is compact, so there is a countable subcomplex $B' \subset Y$ such that $B_0 \subset B'$ such that the homotopy lives in $|B' \cap X|$. Similarly, the image in $|Y|$ of any morphism

$$\gamma : (|\Delta^n|, |\partial\Delta^n|) \to (|B_0|, x)$$

lifts to $|X|$ up to homotopy, and that homotopy lives in $|B''|$ for some countable subcomplex $B'' \subset Y$ with $B_0 \subset B''$.

It follows that there is a countable subcomplex $B_1 \subset Y$ with $B_0 \subset B_1$ such that any two elements

$$[\alpha], [\beta] \in \pi_n(|B_0 \cap X|, x)$$

which map to the same element in $\pi_n(|B_0|, x)$ must also map to the same element of $\pi_n(|B_1 \cap X|, x)$, and every element

$$[\gamma] \in \pi_n(|B_0|, x)$$

lifts to an element of $\pi_n(|B_1 \cap X|, x)$, and this for all $n \geq 0$ and all (countably many) vertices x.

Repeat the construction inductively, to form a countable collection

$$A = B_0 \subset B_1 \subset B_2 \subset \dots$$

of Y. Then $B = \cup B_i$ is a countable subcomplex of Y, and the map $B \cap X \to B$ is a weak equivalence. $\square$

Lemma 11.5. *Every simplicial set map $f : X \to Y$ has a factorization*

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle q} \\ & & Y \end{array}$$

such that q has the right lifting property with respect to all countable trivial cofibrations and i is constructed from countable trivial cofibrations by pushout and composition.

The proof of Lemma 11.5 is an example of a transfinite small object argument.

Proof. Choose an uncountable cardinal number κ, and interpret κ as the poset of ordinal numbers $s < \kappa$. Construct a system of factorizations

$$\begin{array}{ccc} X & \xrightarrow{i_s} & Z_s \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle q_s} \\ & & Y \end{array} \qquad (4)$$

of f with j_s a trivial cofibration as follows:

- given factorization of the form (4) consider all diagrams

$$D: \quad \begin{array}{ccc} A_D & \longrightarrow & Z_s \\ {\scriptstyle i_D}\downarrow & & \downarrow{\scriptstyle q_s} \\ B_D & \longrightarrow & Y \end{array}$$

  such that i_D is a countable trivial cofibration, and form the pushout

$$\begin{array}{ccc} \sqcup_D A_D & \longrightarrow & Z_s \\ \downarrow & & \downarrow{\scriptstyle j_s} \\ \sqcup_D B_D & \longrightarrow & Z_{s+1} \end{array}$$

  Then the map j_s is a trivial cofibration, and the diagrams together induce a map $q_{s+1} : Z_{s+1} \rightarrow Y$. Let $i_{s+1} = j_s i_s$.

- if s is a limit ordinal, let $Z_s = \varinjlim_{t<s} Z_t$.

Now let $Z = \varinjlim_{s<\kappa} Z_s$ with induced factorization

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

Suppose given a lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & Z \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle q} \\ B & \longrightarrow & Y \end{array}$$

with $j : A \rightarrow B$ a countable trivial cofibration. Then $\alpha(A)$ is a countable subcomplex of X, so that

$\alpha(A) \subset X_s$ for some $s < \kappa$ for otherwise $\alpha(A)$ has too many elements. Thus α factors through some subobject X_s, and then the argument finishes as usual. $\square$

Remark: Normally, we say that the map i in the saturation of the set of countable trivial cofibrations. The *saturation* of a set of cofibrations I is the smallest class of cofibrations containing I which is closed under pushout, coproducts, (long) compositions and retraction. If a map p has the right lifting property with respect to all maps of I then it has the right lifting property with respect to all maps in the saturation of I.

Classes of cofibrations which are defined by a left lifting property with respect to some family of maps are saturated in this sense.

Lemma 11.6. *A simplicial set map $p : X \to Y$ is a fibration if and only if it has the right lifting property with respect to (the set of) all trivial cofibrations $i : C \to D$ with D countable.*

We'll use a trick for the proof of this result, which trick will recur. Roughly speaking, it amounts to verifying a solution set condition.

Proof. Suppose given a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

where j is a cofibration, B is countable and f is a weak equivalence. Then by Lemma 11.5 f has a factorization $f = q \cdot i$, where i is a trivial cofibration and q has the right lifting property with respect to all cofibrations, by Lemma 11.2. The lift exists in the diagram

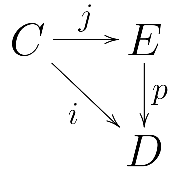
$$\begin{array}{ccc} A & \longrightarrow & X \\ & & \downarrow{\scriptstyle i} \\ {\scriptstyle j}\downarrow & \overset{\theta}{\nearrow} & Z \\ & & \downarrow{\scriptstyle q} \\ B & \longrightarrow & Y \end{array}$$

Also $\theta(B)$ is countable, so there is a countable subcomplex $D \subset Z$ with $\theta(B) \subset D$ such that the map $D \cap X \to D$ is a trivial cofibration. It follows that there is a factorization

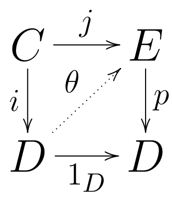
$$\begin{array}{ccccc} A & \longrightarrow & D \cap X & \longrightarrow & X \\ {\scriptstyle j}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & D & \longrightarrow & Y \end{array}$$

of the original diagram through a countable trivial cofibration.

Now suppose that $i : C \to D$ is a trivial cofibration. Then i has a factorization



such that p has the right lifting property with respect to all countable trivial cofibrations, and j is built from countable trivial cofibrations by pushout and composition. It follows that j is a weak equivalence, so that p is a weak equivalence. But the previous paragraph then implies that p has the right lifting property with respect to all countable cofibrations, and hence with respect to all cofibrations. The lift therefore exists in the diagram



so that i is a retract of j. It follows that if a map q has the right lifting property with respect to all countable trivial cofibrations, then it has the right lifting property with respect to all trivial cofibrations. □

Exercise: Find a different, maybe simpler proof for Lemma 11.6.

Theorem 11.7. *With the definitions of weak equivalence, cofibration and fibration given above the category* $s\mathbf{Set}$ *of simplicial sets satisfies the axioms for a closed model category.*

Proof. The axioms **CM1**, **CM2** and **CM3** are trivial to verify: **CM2** and the fact that the class of weak equivalences is closed under retraction are consequences of the corresponding statements for spaces.

Every map $f : X \to Y$ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & W \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle q} \\ & & Y \end{array}$$

such that j is a cofibration and q is a trivial fibration — this follows from Lemma 11.2 and a standard small object argument. The other half of the factorization axiom **CM5** is a consequence of Lemma 11.5 and Lemma 11.6. The axiom **CM4** is also a consequence of Lemma 11.2. □

Remark 11.8. Observe that, in the adjoint pair of functors

$$|\ | : s\mathbf{Set} \leftrightarrows \mathbf{CGHaus} : S$$

the realization functor (the left adjoint part) preserves cofibrations and trivial cofibrations. It's an immediate consequence that the singular functor S preserves fibrations and trivial fibrations. Adjunctions like this between closed model category are called *Quillen adjunctions* or *Quillen pairs.* We'll see later on, and this is a big result, that these functors form a Quillen equivalence.

Remark 11.9. We defined the weak equivalences of simplicial sets to be those maps whose realizations are weak equivalences of spaces. In this way, the model structure for $s\mathbf{Set}$, as it is described here, is *induced* from the model structure for **CGHaus** via the realization functor $|\ |$. Some people would say that the model structure on simplicial sets is obtained from that on spaces by transfer.

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