Lecture 005 (October 3, 2007)

12 Kan fibrations

Say that a map $p : X \to Y$ is a *Kan fibration* if it has the right lifting property with respect to all inclusions $\Lambda^n_k \subset \Delta^n$ of horns in simplices.

A fibration, as defined in Section 11 (Lecture 004), is obviously a Kan fibration since the induced map $|\Lambda^n_k| \to |\Delta^n|$ is a weak equivalence, so that the inclusion $\Lambda^n_k \subset \Delta^n$ is a trivial cofibration. The converse statement is also true: every Kan fibration is a fibration ... and that is one of the things that will concern us in this section and the next. The statement appears as Theorem 13.6 below.

As is traditional, we say that a simplicial set X for which the map $X \to *$ is a Kan fibration is a *Kan complex.*

Exercise: Suppose that C is a small category. Show that the nerve BC is a Kan complex if and only if C is a groupoid.

In particular, the ordinal number posets $\mathbf{n}$ *are not* groupoids if $n \geq 1$, so none of the corresponding simplices $\Delta^n = B\mathbf{n}$ are Kan complexes, and they aren't fibrant either.

The saturation of the set of cofibrations $\Lambda^n_k \subset \Delta^n$ is normally called the class of *anodyne extensions*. This class is otherwise characterized as the set of cofibrations which has the left lifting property with respect to all Kan fibrations. We shall need all of the combinatorial control of these things that we can find:

Lemma 12.1. *The following sets of cofibrations have the same saturations:*

- $\mathbf{A}_1 =$ *all maps* $\Lambda^n_k \subset \Delta^n$*,*
- $\mathbf{A}_2 =$ *all inclusions*

$$(\Delta^1 \times \partial\Delta^n) \cup (\{\epsilon\} \times \Delta^n) \subset \Delta^1 \times \Delta^n, \ \epsilon = 0, 1.$$

Proof. The saturation of $\mathbf{A}_2$ includes all maps

$$(\Delta^1 \times K) \cup (\{\epsilon\} \times L) \subset \Delta^1 \times L, \ \epsilon = 0, 1.$$

induced by inclusions $K \subset L$, since L is built from K by attaching cells.

The functor $r_k : \mathbf{n} \times \mathbf{1} \to \mathbf{n}$ specified by the picture

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & \cdots \arrow[r] & k \arrow[r] \arrow[d] & k \arrow[r] \arrow[d] & \cdots \arrow[r] & k \arrow[d] \\
0 \arrow[r] & 1 \arrow[r] & \cdots \arrow[r] & k \arrow[r] & k+1 \arrow[r] & \cdots \arrow[r] & n
\end{tikzcd}$$

and the functor $i : \mathbf{n} \to \mathbf{n} \times \mathbf{1}$ defined by $i(j) = (j, 1)$ together determine a retraction diagram

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d] & (\Lambda^n_k \times \Delta^1) \cup (\Delta^n \times \{0\}) \arrow[r] \arrow[d] & \Lambda^n_k \arrow[d] \\
\Delta^n \arrow[r] & \Delta^n \times \Delta^1 \arrow[r] & \Delta^n
\end{tikzcd}$$

(note the only content: $\Delta^n \times \{0\}$ is mapped into Λ^n_k) so that $\Lambda^n_k \subset \Delta^n$ is in the saturation of the family $\mathbf{A}_2$ if $k < n$. Similarly, the map $\Lambda^n_k \subset \Delta^n$ is a retraction of the map

$$(\Lambda^n_k \times \Delta^1) \cup (\Delta^n \times \{1\}) \subset \Delta^n \times \Delta^1$$

if $k > 0$. Thus, the saturation of $\mathbf{A}_1$ is contained in the saturation of $\mathbf{A}_2$.

The non-degenerate $(n+1)$-simplices of $h_i : \Delta^n \times \Delta^1$ are functors $\mathbf{n+1} \to \mathbf{n} \times \mathbf{1}$ defined by the pictures

$$\begin{tikzcd}
(0,0) \arrow[r] & (1,0) \arrow[r] & \cdots \arrow[r] & (i,0) \arrow[d] & & \\
 & & & (i,1) \arrow[r] & \cdots \arrow[r] & (i,n)
\end{tikzcd}$$

Let $(\Delta^n \times \Delta^1)^{(i)}$ be the subcomplex of $\Delta^n \times \Delta^1)$ generated by $\partial\Delta^n \times \Delta^1$ and the simplices $h_0, \ldots, h_i$. Let

$$(\Delta^n \times \Delta^1)^{(-1)} = (\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}).$$

Then $(\Delta^n \times \Delta^1)^{(n)} = \Delta^n \times \Delta^1$, and there are pushouts

$$\begin{tikzcd}
\Lambda^{n+1}_{i+2} \arrow[r] \arrow[d] & (\Delta^n \times \Delta^1)^{(i)} \arrow[d] \\
\Delta^{n+1} \arrow[r] & (\Delta^n \times \Delta^1)^{(i+1)}
\end{tikzcd}$$

It follows that the members of $\mathbf{A}_2$ are in the saturation of the set $\mathbf{A}_1$. □

Lemma 12.2. *Suppose that $i : K \to L$ is an anodyne extension and that $j : A \to B$ is a cofibration. Then the inclusion*

$$(K \times B) \cup (L \times A) \subset L \times B$$

is anodyne.

Proof. The class of cofibrations $K' \to L'$ such that

$$(K' \times B) \cup (L' \times A) \subset L' \times B$$

is anodyne is saturated, and includes all cofibrations

$$(\Delta^1 \times \partial\Delta^n) \cup (\{\epsilon\} \times \Delta^n) \subset \Delta^1 \times \Delta^n,\ \epsilon = 0, 1.$$

(see [2, I.4.6]). □

Corollary 12.3. *The cofibrations*

$$(\Lambda^n_k \times \Delta^m) \cup (\Delta^n \times \partial\Delta^m) \subset \Delta^n \times \Delta^m$$

are anodyne.

Here's something else that Lemma 12.2 buys you:

Corollary 12.4. *Suppose that $p : X \to Y$ is a Kan fibration such that Y is a Kan complex, and suppose that $i : A \to B$ is a cofibration. Then the induced map*

$$\mathbf{hom}(B,X) \xrightarrow{(i^*,p_*)} \mathbf{hom}(A,X) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(B,X)$$

is a Kan fibration. If either p is a trivial fibration or i is anodyne, then the map (i^*,p_*) is a trivial fibration.

Proof. Solutions to the lifting problem

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "j"'] & \mathbf{hom}(B,X) \arrow[d, "(i^*,p_*)"] \\
L \arrow[r] \arrow[ur, dotted] & \mathbf{hom}(A,X) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(B,X)
\end{tikzcd}$$

are equivalent to solutions to the lifting problem

$$\begin{tikzcd}
(L \times A) \cup (K \times B) \arrow[r] \arrow[d, "(i,j)_*"'] & X \arrow[d] \\
L \times B \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

by the exponential law, and the map $(i,j)_*$ is anodyne if either i or j is anodyne, by Lemma 12.2. Recall that p is a trivial fibration if and only if it has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$ (Lemma 11.2, Lecture 004). $\square$

Lemma 12.5. *Simplicial homotopy of maps $X \to Y$ is an equivalence relation if Y is a Kan complex.*

Proof. It's enough to show that simplicial homotopy classes of vertices $\Delta^0 \to Z$ is an equivalence relation if Z is a Kan complex. In effect, $\mathbf{hom}(X, Y)$ is a Kan complex by Corollary 12.4, and a path between vertices of $\mathbf{hom}(X, Y)$ is a simplicial homotopy between maps $X \to Y$.

The paths

$$x \xrightarrow{\omega_2} y \xrightarrow{\omega_0} z$$

define a map $(\omega_0, \ , \omega_2) : \Lambda^2_1 \to Z$ which extends to a 2-simplex $\sigma : \Delta^2 \to Z$. Then the 1-simplex $d_1\sigma$ is a path $x \to z$. It follows that the path relation is transitive.

Suppose that $\omega_2 : x \to y$ is a path as above, and let $x : x \to x$ denote the constant path (ie. degenerate 1-simplex) at x. Then there is a diagram

$$\begin{tikzcd}
\Lambda^2_0 \arrow[r, "{(\ ,x,\omega_2)}"] \arrow[d] & Z \\
\Delta^2 \arrow[ur, dotted, "\theta"'] &
\end{tikzcd}$$

and so there is a path $d_0\theta : y \to x$. The path relation is therefore symmetric.

The constant path $\Delta^1 \xrightarrow{s^0} \Delta^1 \xrightarrow{x} X$ is a path from x to x, so the relation is reflexive. $\square$

Write $\pi_0(Z)$ for the *path components*, aka. homotopy classes of vertices $\Delta^0 \to Z$ for a Kan complex of Z. The argument for Lemma 12.5 implies that there is a coequalizer

$$\begin{tikzcd} Z_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & Z_0 \arrow[r] & \pi_0(Z) \end{tikzcd}$$

in sets. This points a way to defining the set $\pi_0 X$ for arbitrary simplicial sets X: this set is defined by the coequalizer

$$\begin{tikzcd} X_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \arrow[r] & \pi_0(X) \end{tikzcd}$$

Exercise: Show that there is a natural bijection

$$\pi_0(X) \cong \pi_0(|X|)$$

for simplicial sets X.

Suppose that Y is a Kan complex, and let $x \in Y_0$ be a vertex. Then the map i^* in the pullback diagram

$$\begin{tikzcd} F_x \arrow[r] \arrow[d] & \mathbf{hom}(\Delta^n, Y) \arrow[d, "i^*"] \\ \Delta^0 \arrow[r, "x"'] & \mathbf{hom}(\partial\Delta^n, Y) \end{tikzcd}$$

is a Kan fibration by Corollary 12.4, and the vertices of the Kan complex F_x are diagrams

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "x"] \arrow[d, "i"'] & Y \\
\Delta^n \arrow[ur] &
\end{tikzcd}$$

or in other words simplices $\alpha : \Delta^n \to Y$ which restrict to the trivial map $\partial\Delta^n \to \Delta^0 \xrightarrow{x} Y$ on the boundary. The path components $\pi_0(F_x)$ are the simplicial homotopy classes of maps

$$(\Delta^n, \partial\Delta^n) \to (Y, x)$$

rel $\partial\Delta^n$. Write $\pi_n^s(Y, x)$ to denote this set of simplicial homotopy classes.

The set $\pi_n^s(Y, x)$ has the structure of a group for $n \geq 1$, and this group is abelian if $n \geq 2$. These are the *simplicial homotopy groups* of a Kan complex. The multiplication can be specified for $[\alpha], [\beta] \in \pi_n^s(Y, x)$ by

$$[\alpha] * [\beta] = [d_n\sigma],$$

where $\sigma : \Delta^{n+1} \to Y$ is a lifting

$$\begin{tikzcd}
\Lambda^{n+1}_n \arrow[r, "{(x,\dots,x,\beta,?,\alpha)}"] \arrow[d] & Y \\
\Delta^{n+1} \arrow[ur, dotted, "\sigma"'] &
\end{tikzcd}$$

Equivalently, the set $\pi_n^s(Y,x)$ can be identified with homotopy classes of maps

$$((\Delta^1)^{\times n}, \partial((\Delta^1)^{\times n})) \to (Y,x)$$

by the same (prismatic) argument as the corresponding result for topological spaces, and then $\pi_n^s(Y,x)$ is an automorphism group in a combinatorially defined version of the fundamental groupoid ($\pi^s Z$ is generally defined for Kan complexes Z). In this case, one clearly sees two or more multiplications satisfying an interchange law with common identity if $n \geq 2$.

Suppose that $p : X \to Y$ is a Kan fibration such that Y (hence X) is a Kan complex, and define the fibre F over a vertex $y \in Y$ by the pullback diagram

$$\begin{tikzcd}
F \arrow[r, "i"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^0 \arrow[r, "y"'] & Y
\end{tikzcd}$$

Suppose that x is a vertex of the Kan complex F. There is a boundary homomorphism

$$\partial : \pi_{n+1}^s(Y,y) \to \pi_n^s(F,x)$$

which is defined for $[\alpha] \in \pi_{n+1}^s(Y,y)$ by setting $\partial([\alpha]) = [d_0\theta]$, where θ is a choice of lifting making the diagram

$$\begin{tikzcd}
\Lambda^{n+1}_0 \arrow[r, "x"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^{n+1} \arrow[r, "\alpha"'] \arrow[ur, dotted, "\theta"] & Y
\end{tikzcd}$$

commute. The same constructions and arguments as for Lemma 5.3 (Lecture 002) are still in play in this context, giving

Lemma 12.6. *Suppose that $p : X \to Y$ is a Kan fibration such that Y is a Kan complex, and suppose that the Kan complex F is the fibre over a vertex $y \in Y$. Then we have the following:*

1) *For each vertex $x \in F$ there is a sequence of pointed sets*

$$\dots \pi_n^s(F,x) \xrightarrow{i_*} \pi_n^s(X,x) \xrightarrow{p_*} \pi_n^s(Y,p(x)) \xrightarrow{\partial} \pi_{n-1}^s(F,x) \to \dots$$
$$\dots \pi_1^s(Y,f(x)) \xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(X) \xrightarrow{p_*} \pi_0(Y)$$

which is exact in the sense that $\ker = \operatorname{im}$ *everywhere.*

2) *There is a group action*

$$* : \pi_1^s(Y,p(x)) \times \pi_0(F) \to \pi_0(F)$$

such that $\partial([\alpha]) = [\alpha] * [x]$, and such that $i_*[z] = i_*[w]$ if and only if there is an ele-

ment $[\beta] \in \pi_1(Y, p(x))$ *such that* $[\beta] * [z] = [w]$.

Now here's a combinatorial analogue of Lemma 5.2 of Lecture 002:

Lemma 12.7. *Suppose that* $p : X \to Y$ *is a Kan fibration and that* Y *is a Kan complex. Suppose that* p *induces a bijection* $\pi_0(X) \cong \pi_0(Y)$, *and that it induces isomorphisms* $\pi_n^s(X, x) \cong \pi_n^s(Y, p(x))$ *for all* $n \geq 1$ *and all vertices* x *of* X. *Then* p *is a trivial fibration of* $s\mathbf{Set}$.

Proof. Show that p has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$, $n \geq 0$. The argument is the same as for Lemma 5.2. $\square$

Say that a *combinatorial weak equivalence* is a map $f : X \to Y$ between Kan complexes which induces an isomorphism in all possible simplicial homotopy groups, meaning that f induces a bijection $\pi_0(X) \cong \pi_0(Y)$ and isomrphisms $\pi_n^s(X, x) \cong \pi_n^s(Y, f(x))$ for all vertices $x \in X$ and all $n \geq 1$.

In this language, Lemma 12.7 says that a map p which is a Kan fibration and a combinatorial weak equivalence between Kan complexes must also be a trivial fibration.

13 Simplicial sets and spaces

Here's a major theorem, due to Quillen:

Theorem 13.1. *The realization of a Kan fibration is a Serre fibration.*

Proof. This will only be a brief sketch — the details can be found, for example, [2, I.10].

The idea is to use the theory of minimal fibrations to show that every Kan fibration $p : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "p"'] & Z \arrow[d, "q"] \\ & Y \end{tikzcd}$$

where g is a trivial fibration (ie. has the right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$) and q is a minimal Kan fibration.

Garbriel and Zisman show [1], [2] that the realization of a minimial fibration $q : Z \to Y$ is a Serre fibration: the idea is that every pullback $q^{-1}(\sigma)$ of a simplex $\sigma : \Delta^n \to Y$ is isomorphic over Δ^n to a simplicial set $F \times \Delta^n$, where F is a fibre over some vertex Δ^n, and it follows that the realization of q is locally a projection, hence a Serre fibration.

The trivial fibration g sits in a diagram

$$\begin{tikzcd}
X \arrow[r, "1_X"] \arrow[d, "{(1_X,g)}"'] & X \arrow[d, "g"] \\
X \times Z \arrow[r, "pr"'] \arrow[ur, dotted] & Z
\end{tikzcd}$$

and is therefore a retract of a projection. $\square$

For the record, a Kan fibration $p : X \to Y$ is said to be *minimal* if, given simplices $\alpha, \beta : \Delta^n \to Y$ (with common boundary and such that $p(\alpha) = p(\beta)$), the existence of a diagram

$$\begin{tikzcd}
\partial\Delta^n \times \Delta^1 \arrow[r, "pr"] \arrow[d, "i\times 1"'] & \partial\Delta^n \arrow[d] \\
\Delta^n \times \Delta^1 \arrow[r, "h"] \arrow[d, "pr"'] & X \arrow[d, "p"] \\
\Delta^n \arrow[r] & Y
\end{tikzcd}$$

where h is a homotopy from α to β forces $\alpha = \beta$. Every Kan fibration has a minimal Kan fibration as a strong fibrewise deformation retract, and every fibrewise weak equivalence of minimal fibrations is an isomorphism. Details can be found in [2, I.10].

The Milnor Theorem is a first consequence of Quillen's theorem:

Theorem 13.2. *Suppose that Y is a Kan complex and that $\eta : Y \to S(|Y|)$ is the adjunction homomorphism. Then η is a combinatorial weak equivalence.*

We need an extra formality for the proof of Theorem 13.2. If Y is a Kan complex, then the map $\partial\Delta^1 \subset \Delta^1$ induces a Kan fibration

$$\mathbf{hom}(\Delta^1, Y) \xrightarrow{(p_0,p_1)} Y \times Y \cong \mathbf{hom}(\partial\Delta^1, Y),$$

and the induced maps p_0, p_1 are trivial fibrations, by Corollary 12.4. Take a vertex $x \in Y$ and define the simplicial set P_xY by the pullback

$$\begin{tikzcd} P_xY \arrow[r, "i"] \arrow[d, "p_{0*}"'] & \mathbf{hom}(\Delta^1, Y) \arrow[d, "p_0"] \\ \Delta^0 \arrow[r, "x"'] & Y \end{tikzcd}$$

Then the induced map p_{0*} is a trivial fibration, so that P_xY is contractible. At the same time, there is a pullback diagram

$$\begin{tikzcd} P_xY \arrow[r, "i"] \arrow[d, "{(p_{0*},p_1 i)}"'] & \mathbf{hom}(\Delta^1, Y) \arrow[d, "{(p_0,p_1)}"] \\ \Delta^0 \times Y \arrow[r, "{(x,1_Y)}"'] & Y \times Y \end{tikzcd}$$

so that the composite $\pi = p_1 i : P_xY \to Y$ is a Kan fibration. Write ΩY for the fibre of π over

$x \in Y$. Then we have the Kan fibre sequence

$$\Omega Y \to P_x Y \xrightarrow{\pi} Y$$

This is the *path-loop fibre sequence* for the Kan complex Y.

Proof of Theorem 13.2. It is easily seen that the map $\eta : Y \to S(|Y|)$ induces a bijection $\pi_0(Y) \cong \pi_0(S(|Y|))$.

The induced maps

$$S(|\Omega Y|) \to S(|P_x Y|) \to S(|Y|)$$

form a Kan fibre sequence on account of Theorem 13.1 and the exactness of the realization functor (Lemma 10.5). Furthermore, the Kan complex $S(|P_x Y|)$ is contractible. It follows that there is a commutative diagram of functions

$$\begin{tikzcd}
\pi_1^s(Y,x) \arrow[r, "\eta_*"] \arrow[d, "\partial"', "\cong"] & \pi_1^s(S(|Y|),x) \arrow[d, "\cong"', "\partial"] \\
\pi_0(\Omega Y) \arrow[r, "\cong", "\eta_*"'] & \pi_0(S(|\Omega Y|))
\end{tikzcd}$$

The map $\eta_* : \pi_1^s(Y,x) \to \pi_1^s(S(|Y|),x)$ is therefore an isomorphism. Inductively, all induced maps $\pi_n^s(Y,x) \to \pi_n^s(S(|Y|),x)$ are isomorphisms, and this argument holds for all vertices x of Y. $\square$

Corollary 13.3. *There are natural isomorphisms*

$$\pi_n^s(Y, x) \cong \pi_n(|Y|, x)$$

at all vertices x for all Kan complexes Y.

Proof. The adjunction isomorphism

$$[(\Delta^n, \partial\Delta^n), (S(X), x)] \cong [(|\Delta^n|, |\partial\Delta^n|), (X, x)]$$

gives an isomorphism

$$\pi_n^s(S(X), x) \cong \pi_n(X, x)$$

for an arbitrary space X. □

Lemma 13.4. *Suppose that $p : X \to Y$ is a Kan fibration and a weak equivalence. Then p is a trivial fibration.*

Proof. The class of maps which are both Kan fibrations and weak equivalences is stable under pullback. In effect, given a pullback diagram

$$\begin{tikzcd}
Z \times_Y X \arrow[r] \arrow[d, "p_*"'] & X \arrow[d, "p"] \\
Z \arrow[r] & Y
\end{tikzcd}$$

the realization $|p|$ is a trivial Serre fibration by Theorem 13.1, so that $|p_*|$ is also a trivial Serre fibration since realization preserves pullbacks.

It is therefore enough to show that if $p : X \to \Delta^n$ is a Kan fibration and a weak equivalence then p is a trivial fibration.

As in the proof of Theorem 13.1, p has a factorization

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[dr, "p"'] & F \times \Delta^n \arrow[d, "pr"] \\
& \Delta^n
\end{tikzcd}$$

where g is a trivial fibration and the projection pr is minimal. But then pr is a weak equivalence, so that all homotopy groups of the space $|F|$ vanish, and Theorem 13.2 implies that all simplicial homotopy groups of F vanish. It follows from Lemma 12.7 that all lifting problems

$$\begin{tikzcd}
\partial\Delta^m \arrow[r] \arrow[d] & F \times \Delta^n \arrow[d, "pr"] \\
\Delta^m \arrow[r] \arrow[ur, dotted] & \Delta^n
\end{tikzcd}$$

have solutions. Finish by using Lemma 11.2. □

Remark 13.5. You don't need minimal fibrations for the last step of the proof of Lemma 13.4. It is enough to show instead that the Kan fibration $p : X \to \Delta^n$ is fibrewise homotopy equivalent to a projection $F \times \Delta^n \to \Delta^n$, and this can be achieved by using an elementary result [2, I.10.6].

Theorem 13.6. *Every Kan fibration is a fibration.*

Proof. Suppose that $i : A \to B$ is a trivial cofibration. Then i has a factorization

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[dr, "i"'] & Z \arrow[d, "p"] \\ & B \end{tikzcd}$$

such that j is an anodyne extension and p is a Kan fibration, by a standard small object argument. Then j is a weak equivalence, so that p is a weak equivalence, and is therefore a trivial fibration by Lemma 13.4. The lifting therefore exists in the diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & Z \arrow[d, "p"] \\ B \arrow[r, "1_B"'] \arrow[ur, dotted] & B \end{tikzcd}$$

so that i is a retract of an anodyne extension and is therefore an anodyne extension. Every Kan fibration therefore has the right lifting property with respect to all trivial cofibrations, so every Kan fibration is a fibration. $\square$

Suppose that $f : X \to Y$ is a morphism of Kan complexes (aka. fibrant simplicial sets). Form the diagram

$$\begin{tikzcd}
X \times_Y \mathbf{hom}(\Delta^1, Y) \arrow[r, "f_*"] \arrow[d, "p_{0*}"'] & \mathbf{hom}(\Delta^1, Y) \arrow[r, "p_1"] \arrow[d, "p_0"] & Y \\
X \arrow[r, "f"'] & Y &
\end{tikzcd}$$

where the maps p_0 and p_1 are the trivial fibration pieces of the path object

$$\begin{tikzcd}
& \mathbf{hom}(\Delta^1, Y) \arrow[d, "{(p_0,p_1)}"] \\
Y \cong \mathbf{hom}(\Delta^0, Y) \arrow[ur, "s"] \arrow[r, "\Delta"'] & \mathbf{hom}(\partial\Delta^1, Y) \cong Y \times Y
\end{tikzcd}$$

for Y which are induced by the simplicial set maps

$$\begin{tikzcd}
& \Delta^1 \arrow[dl] \\
\Delta^0 & \partial\Delta^1 \arrow[l] \arrow[u]
\end{tikzcd}$$

(as defined before the the proof of Theorem 13.2.) Then the map $\pi := p_1 f_*$ is a fibration and p_{0*} is a trivial fibration. The map sf defines a section s_* of p_{0*}, and $\pi s_* = p_1 sf = f$. Thus, every map $f : X \to Y$ between fibrant simplicial sets has a functorial factorization

$$\begin{tikzcd}
X \arrow[r, "s_*"] \arrow[dr, "f"'] & X \times_Y \mathbf{hom}(\Delta^1, Y) \arrow[d, "\pi"] \\
& Y
\end{tikzcd} \tag{1}$$

such that π is a fibration and s_* is a section of a trivial fibration. This construction is an abstraction of the classical replacement of a map by a fibration.

Theorem 13.7. *The adjunction maps* $\eta : X \to S(|X|)$ *and* $\epsilon : |S(Y)| \to Y$ *are weak equivalences, for all simplicial sets* X *and spaces* Y*, respectively.*

Proof. Every combinatorial weak equivalence $f : X \to Y$ between Kan complexes is a weak equivalence. In effect, every map which is a fibration and a combinatorial weak equivalence is a weak equivalence by Lemma 12.7, and then one finishes by replacing the map f with a fibration as above. It follows from Theorem 13.2 that the adjunction map $\eta : X \to S(|X|)$ is a weak equivalence if X is fibrant. Finally, choose a fibrant model for an arbitrary simplicial set X, meaning a weak equivalence $j : X \to Z$ such that Z is fibrant. Then in the diagram

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "\simeq"'] & S(|X|) \arrow[d, "\simeq"] \\ Z \arrow[r, "\simeq", "\eta"'] & S(|Z|) \end{tikzcd}$$

the indicated maps are weak equivalences, so that

$\eta : X \to S(|X|)$ is a weak equivalence too.

Suppose that Y is a space. Then from the triangle identity

$$\begin{tikzcd} S(Y) \arrow[r, "\eta"] \arrow[dr, "1"'] & S(|S(Y)|) \arrow[d, "s(\epsilon)"] \\ & S(Y) \end{tikzcd}$$

one sees that $S(\epsilon)$ is a weak equivalence of Kan complexes, so that $\epsilon : |S(Y)| \to Y$ is a weak equivalence of spaces. □

As a result of Theorem 13.7 the realization and singular functors

$$|\ | : s\mathbf{Set} \leftrightarrows \mathbf{CGHaus} : S$$

give a classic example of a Quillen equivalence. In particular we have the following:

Corollary 13.8. *The realization and singular functors induce an adjoint equivalence*

$$|\ | : \mathrm{Ho}(s\mathbf{Set}) \leftrightarrows \mathrm{Ho}(\mathbf{CGHaus}) : S.$$

Here's a final result for this section, which gives the closed "simplicial' model structure for the simplicial set category:

Lemma 13.9. *Suppose that $p : X \to Y$ is a fibration and that $i : A \to B$ is a cofibration.*

Then the induced map

$$\mathbf{hom}(B, X) \xrightarrow{(i^*, p_*)} \mathbf{hom}(A, X) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(B, X)$$

is a fibration. This map is a trivial fibration if either i or p is a weak equivalence.

Proof. This result is a consequence of Corollary 12.4, Theorem 13.6, and the claim appearing in the proof of Theorem 13.6 that every trivial cofibration is an anodyne extension. □

Roughly speaking (see [2] for a full definition), a *closed simplicial model category* is a closed model category **M** together with an internal function space construction with exponential law such that the following axiom holds:

SM7: Suppose that $p : X \to Y$ is a fibration and that $i : A \to B$ is a cofibration. Then the induced map

$$\mathbf{hom}(B, X) \xrightarrow{(i^*, p_*)} \mathbf{hom}(A, X) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(B, X)$$

is a fibration. This map is a trivial fibration if either i or p is a weak equivalence.

The category **CGHaus** of compactly generated Hausdorff spaces also has a closed simplicial model category structure, with the usual mapping space construction. The statement **SM7** for that example follows from the observation that two cofibrations $i : A \to B$ and $j : C \to D$ induce a cofibration, which is trivial if either i or j is trivial (exercise).

References

[1] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.

[2] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics.* Birkhäuser Verlag, Basel, 1999.