On the commutativity problem in Banach algebra.

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Abstract

Let A be a complex Banach algebra with unity. As it was shown by Williams [?] the elements $\mathbf{a}, \mathbf{b} \in A$ commutate if and only if $\sup_{\lambda \in \mathbf{C}} \| \exp(\lambda \mathbf{b}) \mathbf{a} \exp(-\lambda \mathbf{b}) \| < \infty$. This result allows us to obtain the von Neiman-Fuglede-Putnam theorem for the normal elements in the complex Banach algebra. In the present paper the results by Williams [?] and Khasbardar et al., [?] are refined based on the papers by Gorin and Karahanyan [?, ?, ?], and the abstract result of Picard theorem is obtained in this context.

Let A be a complex Banach algebra with unity $\mathbf{1}$ and $\|\mathbf{1}\| = 1$, $\|\mathbf{ab}\| \leq \|\mathbf{a}\|\|\mathbf{b}\|$ for all elements $\mathbf{a}, \mathbf{b} \in A$. We remind [?, ?] that a linear functional $\varphi : A \to \mathbf{C}$ is called a *normalized state* if $\|\varphi\| = \varphi(\mathbf{1}) = 1$. A set of all normalized states forms a weak-star compact convex subset in the space conjugate to A, which is denoted in what follows by $P(A)$. The set $V(\mathbf{a}) = \{\varphi(\mathbf{a}) : \varphi \in P(A)\}$ is called as *(algebraic) numerical range* of the $\mathbf{a}$ element in A. In particular, if A is a subalgebra in the operator algebra $BL(\mathbf{H})$ of all bounded linear operators defined in the complex Hilbert space in $\mathbf{H}$ and the T operator ($T \in A$), then its numerical range $V(T)$ coincides with the close of a general numerical range of the operator T [?]. We note that an element $\mathbf{h} \in A$ is called Hermitian if $V(\mathbf{h}) \in R$, were R is the field of reals. This condition is equivalent to the $\| \exp(it\mathbf{h})\| = 1$ for all $t \in R$. The set $H(A)$ of all Hermitian elements is a closed R -linear subspace in A. We also note that an element $\mathbf{h} \in A$ is called *quasi-Hermitian*[?] if $\| \exp(it\mathbf{h})\| = o(\mid t \mid^{\frac{1}{2}})$ when $\mid t \mid \to \infty$, $t \in R$. This condition provides $sp(\mathbf{h}) \subset R$, where $sp(\mathbf{h})$ is the spectrum of the $\mathbf{h}$ -element. However, the numerical range $V(\mathbf{h})$ may be not contained in the real axis. It should be noted that for any element $\mathbf{a} \in A$ there takes place inclusion $sp(\mathbf{a}) \subset V(\mathbf{a})$. If the element $\mathbf{a} = \mathbf{h} + i\mathbf{k}$, where $\mathbf{h},\ \mathbf{k} \in H(A)$, then the element $\mathbf{a}$ is called Hermitian *decomposable*, and $\mathbf{a}^+ = \mathbf{h} - i\mathbf{k}$ is the Hermitian conjugate versus $\mathbf{a}$. In case when $[\mathbf{h}, \mathbf{k}] = \mathbf{hk} - \mathbf{kh} = 0$, then the element $\mathbf{a} = \mathbf{h} + i\mathbf{k}$ is called *Hermitian normal* element. When $\mathbf{h}, \mathbf{k}$ are a quasi-Hermitian elements and $[\mathbf{h}, \mathbf{k}] = 0$, then the element $\mathbf{a}$ is called *quasi-normal.*

Let J be a closed bi-ideal in A. Then the factor-algebra A/J is a Banach algebra with respect to the factor-norm $||| \bullet |||$ and the element $\tilde{\mathbf{a}} = \mathbf{a} + J = \pi_J(\mathbf{a}) \in A/J$, where $\pi_J :\ A \to A/J$ is a canonical homomorphism generated by J -ideal.

Let $\{\mathbf{a}_n\}_0^\infty,\ \{\mathbf{b}_n\}_0^\infty$ be such a sequences of elements in Banach algebra A, that $\overline{\lim\limits_{n\to\infty}} \|\mathbf{a}_n\|^{\frac{1}{n}} = \rho_a < \infty$, $\overline{\lim\limits_{n\to\infty}} \|\mathbf{b}_n\|^{\frac{1}{n}} = \rho_b < \infty$, and let the element $\mathbf{c} \in A$. Then the function $f(\lambda) = \mathbf{A}(\lambda)\mathbf{c}\mathbf{B}(\lambda) = \sum\limits_0^\infty \frac{\mathbf{c}_n \lambda^n}{n!}$, where $\mathbf{A}(\lambda) = \sum\limits_0^\infty \frac{\mathbf{a}_n \lambda^n}{n!}$, $\mathbf{B}(\lambda) = \sum\limits_0^\infty \frac{\mathbf{b}_n \lambda^n}{n!}$, is an A -valued entire function of exponential type $\sigma_f \leq \rho_a + \rho_b$, where $\mathbf{c}_n = \sum\limits_{p=0}^n \binom{\mathrm{n}}{\mathrm{p}} \mathbf{a}_p \mathbf{c} \mathbf{b}_{n-p}$. If the elements $\mathbf{a}, \mathbf{b}, \mathbf{c} \in A$, then one writes $\mathbf{ca} = \mathbf{bc}(modJ)$ if $\mathbf{ca} - \mathbf{bc} \in J$. In particular, if $\mathbf{b} = \mathbf{a}$ then the condition $\mathbf{ca} = \mathbf{ac}(modJ)$ means that the $\mathbf{a}$ and $\mathbf{c}$ elements are commutative over the J -ideal module.

Theorem 1. Let A be a complex Banach algebra with unity, J is a closed bi-ideal in A and $\{\mathbf{a}_n\}_0^\infty,\ \{\mathbf{b}_n\}_0^\infty \in \mathbf{A}$, $\overline{\lim\limits_{n\to\infty}} \|\mathbf{a}_n\|^{\frac{1}{n}} < \infty$, $\overline{\lim\limits_{n\to\infty}} \|\mathbf{b}_n\|^{\frac{1}{n}} < \infty$. Then, the elements $\mathbf{c}_n = \sum\limits_{p=0}^n \binom{\mathrm{p}}{\mathrm{n}} \mathbf{a}_p \mathbf{c} \mathbf{b}_{n-p} \in J$ for all $n \geq 1$ if and only if $|||\tilde{f}(\lambda)||| = o(|\lambda|)$ for $|\lambda| \to \infty,\ \lambda \in \mathbf{C}$.

Proof: Let $\pi_J :\ A \to A/J$ is a canonical homomorphism generated by J -ideal. Since, $\mathbf{c}_n \in J$, then $\tilde{\mathbf{c}}_n = \tilde{0}$. Then one has $\tilde{f}(\lambda) = \sum\limits_0^\infty \frac{\tilde{\mathbf{c}}_n \lambda^n}{n!} = \tilde{\mathbf{c}}_0$, i.e., $|||\tilde{f}(\lambda)||| = |||\tilde{\mathbf{c}}_0|||$, and hence $\frac{|||\tilde{f}(\lambda)|||}{|\lambda|} \to 0,\ |\lambda| \to \infty$.

Let conversly assume that $|||\tilde{f}(\lambda)||| = o(|\lambda|)$ for $|\lambda| \to \infty,\ \lambda \in \mathbf{C}$. Then for the A/J -valued entire function $\tilde{f}(\lambda) = \sum\limits_0^\infty \frac{\tilde{\mathbf{c}}_n \lambda^n}{n!}$, as far as the Liouville theorem holds $\tilde{f}(\lambda) \equiv \mathbf{c}_0$. Hence, for all $n \geq 1$ one has $\tilde{\mathbf{c}}_n = 0$, and therefore $\mathbf{c}_n \in J$. The proof is completed.

Let us consider a case when $\mathbf{a}_n = \mathbf{a}^n$ and $\mathbf{b}_n = \mathbf{b}^n$, where $\mathbf{a}, \mathbf{b} \in A$. Then the following holds:

Theorem 2. Let A be a complex Banach algebra with unity, J is a closed bi-ideal in A and the elements $\mathbf{a}, \mathbf{b}, \mathbf{c} \in A$. Then $\mathbf{ca} = \mathbf{bc}(modJ)$ if and only if $||| \exp(\lambda\tilde{\mathbf{b}})\tilde{\mathbf{c}} \exp(-\lambda\tilde{\mathbf{a}})||| = o(|\lambda|)$, $|\lambda| \to \infty$, $\lambda \in \mathbf{C}$.

Proof. Let $\pi_J :\ A \to A/J$ is a canonical homomorphism generated by J -ideal. Since, $\mathbf{ca} - \mathbf{bc} \in J$, then for any natural n one has: $\tilde{\mathbf{c}}\tilde{\mathbf{a}}^n = \tilde{\mathbf{b}}^n\tilde{\mathbf{c}}$, and hence $\tilde{\mathbf{c}} \exp(\lambda\tilde{\mathbf{a}}) = \exp(\lambda\tilde{\mathbf{b}})\tilde{\mathbf{c}}$ for any $\lambda \in \mathbf{C}$. Therefore $\tilde{\mathbf{c}} = \exp(\lambda\tilde{\mathbf{b}})\tilde{\mathbf{c}} \exp(-\lambda\tilde{\mathbf{a}})$, and hence $||| \exp(\lambda\tilde{\mathbf{b}})\tilde{\mathbf{c}} \exp(-\lambda\tilde{\mathbf{a}})||| = o(|\lambda|)$, $|\lambda| \to \infty$, $\lambda \in \mathbf{C}$.

Let conversly assume that $\||\exp(\lambda\tilde{\mathbf{b}})\tilde{\mathbf{c}}\exp(-\lambda\tilde{\mathbf{a}})\|| = o(|\lambda|)$, $|\lambda| \to \infty$, $\lambda \in \mathbf{C}$. Then for A/J -valued entire function $\tilde{f}(\lambda) = \exp(\lambda\tilde{\mathbf{b}})\tilde{\mathbf{c}}\exp(-\lambda\tilde{\mathbf{a}})$ as far as the Liouville theorem holds one has $\tilde{f}(\lambda) \equiv \tilde{\mathbf{c}}$. Then for all natural n one has $\tilde{\mathbf{c}}\tilde{\mathbf{a}}^n = \tilde{\mathbf{b}}^n\tilde{\mathbf{c}}$, and hence $\mathbf{ca} = \mathbf{bc}(modJ)$. The proof is completed.

Corollary 3. Let $M, N, T \in BL(X)$, where X is a complex Banach (in particular, Hilbertian) space, and $J \subset BL(X)$ is a closed bi-ideal. Then $MT = TN(modJ)$ if and only if $\||\exp(\lambda\tilde{M})\tilde{T}\exp(-\lambda\tilde{N})\|| = o(|\lambda|)$, $|\lambda| \to \infty$, $\lambda \in \mathbf{C}$.

The combination of Theorem 2. with the generalized theorem of Von Neiman-Fuglede-Putnam [**?**, **?**] gives

*Corollary 4. Let $\mathbf{a}, \mathbf{b} \in A$ be quasinormal elements, and $\mathbf{c} \in A$, $J \in A$ is a closed bi-ideal. If $\||\exp(\lambda\tilde{M})\tilde{T}\exp(-\lambda\tilde{N})\|| = o(|\lambda|)$, $|\lambda| \to \infty$, $\lambda \in \mathbf{C}$, then $\mathbf{a}^+\mathbf{c} = \mathbf{cb}^+(modJ)$[**?**].*

Corollary 5. Let A be a complex Banach algebra with unity, and $\mathbf{a}, \mathbf{b} \in A$ be quasinormal elements and $\mathbf{c} \in A$. If $\||\exp(\lambda\mathbf{a})\mathbf{c}\exp(-\lambda\mathbf{b})\|| = o(|\lambda|)$, $|\lambda| \to \infty$, $\lambda \in \mathbf{C}$. Then $\mathbf{a}^+\mathbf{c} = \mathbf{c}\,\mathbf{b}^+$.

In the case of $J = \{0\}$ the Corollaries 4,5 refine the corresponding results of [**?**, **?**].

Let $D : A \to A$ be a continuous A-derivation (i.e., $D \in BL(A)$ and $D(\mathbf{ab}) = (D\mathbf{a})\mathbf{b} + \mathbf{a}(D\mathbf{b})$, where $\mathbf{a}, \mathbf{b} \in A$). Then in the factor-algebra A/J the A/J-derivation $D_J : A/J \to A/J$ is induced, which is defined according to the formula $D_J(\tilde{\mathbf{a}}) = \pi_J(D\mathbf{a}) = \tilde{D\mathbf{a}}$. It is obvious that $D_J \in BL(A/J)$. We denote as $aut(A)$ the group of automorphisms of the algebra A. Then, for all the fixed $\lambda \in \mathbf{C}$ the operator $\exp(\lambda D) \in aut(A)$, and therefore $\exp(\lambda D_J) \in aut(A/J)$.

Theorem 6. Let A be a complex Banach algebra with unity, J is a closed bi-ideal in A and $D : A \to A$ is a continuous A-derivation. Then $D\mathbf{a} \in J$, where $\mathbf{a} \in A$, if and only if $\||(\exp\lambda D_J)(\tilde{\mathbf{a}})\|| = o(|\lambda|)$, $|\lambda| \to \infty$, $\lambda \in \mathbf{C}$.

The proof of this theorem is similar to those of theorems 1 and 2.

Corollary 7. Let A be a complex Banach algebra with unity and $D : A \to A$ is a continuous A-derivation. Then the element $\mathbf{a} \in ker(D)$, where $ker(D)$ is the kernel of the operator D, if and only if $\|(\exp\lambda D)(\mathbf{a})\| = o(|\lambda|)$, $|\lambda| \to \infty$, $\lambda \in \mathbf{C}$.

In the same context it is interesting the following result, which is a generalization of Picard's theorem for entire functions.

Theorem 8. Let A be a complex Banach algebra with unity, J is a closed bi-ideal in A and $D : A \to A$ is a continuous A-derivation. If for the element $\mathbf{a} \in A$ one has $D\mathbf{a} \notin J$, then $\bigcup_{\lambda \in \mathbf{C}} \bigvee((\exp\lambda D_J)(\tilde{\mathbf{a}})) = \mathbf{C}$.

Proof. Let $\varphi \in P(A/J)$. Let us consider an entire function $f_\varphi(\lambda) = \varphi(\exp(\lambda D_J)(\tilde{\mathbf{a}}))$. Then the range of the f_φ -function is in the set $U_J = \bigcup_{\lambda \in \mathbf{C}} \bigvee((\exp\lambda D_J)(\tilde{\mathbf{a}}))$. Let us assume that $\mathbf{C} \setminus U_J$ contains at least two points. Then according to the Picard's theorem for the entire functions one has $f_\varphi(\lambda) \equiv const$, and hence, for all natural $n \geq 1$ there takes place $\varphi(D_J^n(\tilde{\mathbf{a}})) = 0$, and in particular, $D_J(\tilde{\mathbf{a}}) = 0$, i.e., $D\mathbf{a} \in J$. However it contradicts to the conditions of the theorem. So, $C \setminus U_J$ contains at most one point. Let us make sure that $U_J = C$. Since for any fixed

$\lambda \in \mathbf{C}$ the operator $\exp(\lambda D_J) \in aut(A/J)$, then for any element $\tilde{\mathbf{a}} \in A/J$ one has $sp(\tilde{\mathbf{a}}) = sp((\exp \lambda D_J)(\tilde{\mathbf{a}}))$.

Let $\zeta_0 \in sp(\tilde{\mathbf{a}})$ and $\zeta \in \mathbf{C}$ is a complex number. Then on the line passing through the points ζ_0 and ζ there exists such a point ζ_1, that $\zeta_1 \in \bigvee((\exp \lambda D_J)(\tilde{\mathbf{a}}))$ for some $\lambda \in \mathbf{C}$. However, the segment $[\zeta_0, \zeta_1] \subset \bigvee((\exp \lambda D_J)(\tilde{\mathbf{a}}))$, and since $\bigvee((\exp \lambda D_J)(\tilde{\mathbf{a}}))$ is a convex set, then $\zeta \in \bigvee((\exp \lambda D_J)(\tilde{\mathbf{a}}))$, and hence, $U_J = \mathbf{C}$. The proof is completed.

In the case when the ideal $J = \{0\}$ one has:

Corollary 9. Let A be a complex Banach algebra with unity and $D : A \to A$ is a continuous A-derivation. If the element $\mathbf{a} \notin ker(D)$, then $\bigcup_{\lambda \in \mathbf{C}} \bigvee((\exp \lambda D)(\mathbf{a})) = \mathbf{C}$.

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