Problems connected with sub-Riemannian geometry

1. (B. Uribe) **Is it true that a straightest line on a non-holonomic surface is a shortest line?**

On a two-dimensional surface Σ the geodesics can be characterized in the following way.

a) A geodesic is a straightest line. This means that if $\vec{\rho} = \vec{\rho}(s)$ is the equation of a geodesic on Σ given with respect to the natural parameter (the arc length), then $\vec{\rho}''(s)$ is collinear to the normal vector of Σ.

b) A geodesic is a shortest line. This means that, for any two sufficiently closed points, the geodesic is the curve of minimal length among all the curves joining these points.

For a holonomic surface a) is equivalent to b). However, for a non-holonomic surface a) gives an ODE of second order for $\vec{\rho}(s)$ and b) gives an ODE of third order for $\vec{\rho}(s)$. These equations look different, however it is possible that any solution of the second order equation is a solution to the third order equation, this would mean that **a straightest line is a shortest line**.

At the same time, even if this is not true in general, one can find classes of non-holonomic surfaces for which a) is equivalent b), or implies b).

2. **Can the total curvature be obtained from the Riemannian curvature tensor?**

It is known that the total curvature of a holonomic surface is expressed in terms of the curvature tensor of the Riemannian connection induced on the surface, namely it coincides with the scalar curvature. For a non-holonomic surface, we have two types of the total curvature, the partial linear connection induced on the surface, and the prolongation of this connection. We can construct the curvature tensor of this connection and of its prolongation. The question is: can the total curvature (of first or second type) be obtained from these curvature tensors?

3. **Flows of non-holonomic surfaces**

A flow of non-holonomic surfaces in the three-dimensional Euclidean space E_3 is a family Δ_t of 2-dimensional distributions with induced metrics g_t. One can consider a "Ricci flow" of non-holonomic surfaces, i. e. $\frac{d}{dt}g = Kg$, where K is the curvature of first or second type. One can ask if the Ricci flow Δ_t converges to a non-holonomic surface Δ, and, if it converges, what are the properties of Δ and of the limiting metric on Δ; in particular, can Δ be holonomic?

Another question is if one can find natural geometric conditions on a flow Δ_t of non-holonomic surfaces under which this flow converges to a holonomic surface.

And, finally, what is a Ricci flow of sub-Riemannian manifolds?