1. Distributions on manifolds

Let M be a smooth manifold. Assume that for each $p \in M$ a subspace $\Delta(p) \subset T_pM$ is assigned such that 1) $\dim \Delta(p) = k$ is constant, 2) for each $q \in M$ there exists a neigbourhood U with vector fields $E_1, E_2, \ldots, E_k \in \mathfrak{X}(U)$ such that, for each $q \in U$, $\Delta(q) = span\{E_1(q), E_2(q), \ldots, E_k(q)\}$. We say that the dimension of the distribution Δ equals k.

A distribution Δ on a manifold M is called *totally integrable* if for each $p \in M$ there exists a submanifold Σ passing through p such that $T_q\Sigma = \Delta(q)$, for any $q \in \Sigma$. This submanifold is called an integral submanifold of Δ. A totally integrable distibution is called a foliation, and the integral submanifolds are called the leaves.

Locally, a distribution Δ can be given either by vector fields E_a, $a = 1, \ldots, k$, spanning the subspaces $\Delta(p) \subset T_pM$, or by differential forms $\omega^1, \omega^2, \ldots, \omega^{n-k}$ such that, for any $V \in T_pM$, $V \in \Delta(p)$ if and only if $\omega^\alpha(V) = 0$, $\alpha = 1, \ldots, n-k$

The following theorem gives necessary and sufficient conditions for a distribution be completely integrable.

Theorem 1 (Frobenius). *Let Δ be a distribution on a manifold M locally given by a frame field E_a, $a = 1, \ldots, k$, or by the fields of 1-forms ω^α. The following conditions are equivalent:*

a) The distribution Δ is completely integrable;

b) $[E_a, E_b] = Q^c_{ab}E_c$, where Q^c_{ab} are functions;

c) $d\omega^\gamma = Q^\gamma_\beta \wedge \omega^\beta$, where Q^γ_β are 1-forms.

Example 1.1. Let a distribution Δ on $\mathbb{R}^3$ be given by the 1-form $\omega = dz - xdx - ydy$. Then $d\omega = 0$, and Δ is completely integrable because this distribution consists of panes tangent to hyperbolic paraboloids of the family $z = \frac{1}{2}(x^2 + y^2)$.

Example 1.2. Let a distribution Δ on $\mathbb{R}^3$ be given by the 1-form $\omega = dy - zdx$. Then one can take the local frame spanning Δ as follows: $E_1 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}$, $E_2 = \frac{\partial}{\partial z}$. Then $[E_1, E_2] = -\frac{\partial}{\partial y}$, and this vector field does not lie in Δ, because $\omega(\frac{\partial}{\partial y}) = -1 \neq 0$. Thus this distribution is not completely integrable. This means that one cannot find a surface tangent to Δ.

2. Non-holonomic surface in E^3

Let E_3 be a three-dimensional Euclidean space, and Δ be a distribution on E_3. We denote by $(\cdot, \cdot)$ the scalar product in E_3. This distribution can be given by a unit vector field $\vec{n}$ normal to Δ. The integrability condition for Δ can be formulated in terms of $\vec{n}$.

Theorem 2. Δ *is integrable if and only if* $(\vec{n}, \operatorname{rot}\vec{n}) = 0$