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Generalized complex geometry

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Abstract

Generalized complex geometry is a new kind of geometrical structure which contains complex and symplectic geometry as its extremal special cases. In this thesis, we explore novel phenomena exhibited by this geometry, such as the natural action of a B-field. We provide many examples of generalized complex structures, including some on manifolds which admit no known complex or symplectic structure. We prove a generalized Darboux theorem which yields a local normal form for the geometry. We show that there is a well-behaved elliptic deformation theory and establish the existence of a Kuranishi-type moduli space.

We then introduce a Riemannian metric and define the concept of a generalized Kähler manifold. We prove that generalized Kähler geometry is equivalent to a certain bi-Hermitian geometry first discovered by physicists in the context of supersymmetric sigma-models. We then use this theorem together with our deformation result to solve an outstanding problem in 4-dimensional bi-Hermitian geometry: we prove that there exists a Riemannian metric on $\mathbb{C}P^2$ which admits exactly two distinct orthogonal complex structures with equal orientation.

In addition, we introduce the concept of a generalized complex submanifold, and show that these sub-objects correspond precisely with the predictions of physicists concerning D-branes in the special cases of complex and symplectic manifolds.

Contents

Chapter 1

Introduction

A central theme of this thesis is that classical geometrical structures which appear, at first glance, to be completely different in nature, may actually be special cases of a more general unifying structure. Of course, there is wide scope for such generalization, as we may consider structures defined by sections of any number of natural bundles present on manifolds. What must direct us in deciding which tensor structures to study is the presence of natural integrability conditions.

Good examples of such conditions include the closure of a symplectic form, the Einstein or special holonomy constraint on a Riemannian metric, the vanishing of the Nijenhuis tensor of a complex structure, and the Jacobi identity for a Poisson bivector, among many others.

This thesis describes a way of unifying complex and symplectic geometry by taking seriously the idea that both of these structures should be thought of, not as linear operations on the tangent bundle of a manifold, but actually on the sum of the tangent and cotangent bundles, $T \oplus T^*$. Since the smooth sections of $T \oplus T^*$ have a natural bracket operation called the *Courant bracket*, there are canonical integrability conditions for certain linear structures on $T \oplus T^*$. Indeed, any complex or symplectic structure determines a maximal isotropic sub-bundle of $(T \oplus T^*) \otimes \mathbb{C}$; the requirement that this sub-bundle be Courant involutive actually specializes to the usual integrability conditions for these two structures. This was one of the observations which led Hitchin [19] to define a generalized complex structure as an almost complex structure $\mathcal{J}$ on $T \oplus T^*$ whose $+i$-eigenbundle L is Courant involutive. This new geometrical structure is, in a sense, the complex analogue of a Dirac structure, a concept introduced by Courant and Weinstein [14],[15] to unify Poisson and symplectic geometry.

We begin, in Chapter 2, with a study of the natural split-signature orthogonal structure which exists on the real vector bundle $T \oplus T^*$. A spin bundle for this orthogonal bundle is shown always to exist and to be isomorphic to $\wedge^\bullet T^*$, the bundle of differential forms. The correspondence between maximal isotropic subspaces and pure spinors then leads to the fact that a generalized complex structure is determined by a canonical line sub-bundle of the complex differential forms. In the case of a complex manifold, this line bundle is the usual canonical line bundle. In the symplectic case, however, this line bundle is generated by $e^{i\omega}$, where ω is the symplectic form.

We proceed, in Chapter 3, to describe and study the Courant bracket, which, while it is not a Lie bracket, does restrict, on involutive maximal isotropic sub-bundles, to be a Lie bracket, and thus endows the bundle L with the structure of a Lie algebroid. L acquires not only a Lie bracket on its sections, but also an exterior derivative operator $d_L : C^\infty(\wedge^k L^*) \to C^\infty(\wedge^{k+1} L^*)$. The Courant integrability of L may also be phrased in terms of a condition on the differential forms defining it, and we determine this condition. But perhaps the most important feature of the Courant bracket is that, unlike the Lie bracket of vector fields, it admits more symmetries than just diffeomorphisms.

The extra symmetries are called B-field transformations and are generated by closed 2-forms $B \in \Omega^2(M)$. This means that given any structure naturally defined in terms of the Courant bracket, like generalized complex structures, a B-field transform produces another one. This action of the 2-forms agrees precisely with the 2-form gauge freedom studied by physicists in the context of sigma models. We conclude the chapter with an investigation of the fact that the Courant bracket itself may be deformed by a real closed 3-form H, and we describe what this means in the language of gerbes.

The first two chapters have been organized to contain all the necessary algebraic and differential-geometric machinery for the rest of the thesis. In Chapter 4, we come to the subject at hand: generalized complex structures themselves. We show that topologically, the obstruction to the existence of a generalized complex structure is the same as that for an almost complex structure or a nondegenerate 2-form. We then describe the algebraic conditions on differential forms which makes them generators for generalized complex structures. This then allows us to produce exotic examples of generalized complex structures; indeed we exhibit examples on manifolds which admit no known complex or symplectic structure. We are even able to give an example of a family of generalized complex structures which interpolates between a complex and a symplectic structure, thus connecting the moduli spaces of these two structures.

Still in Chapter 4, we prove a local structure theorem for generalized complex manifolds, analogous to the Darboux theorem in symplectic geometry and the Newlander-Nirenberg theorem in complex geometry. We show that at each point of a $2n$-dimensional generalized complex manifold, the structure is characterized algebraically by an integer k, called the *type*, which may vary along the manifold and take values anywhere from $k = 0$ to $k = n$. It is lower semi-continuous, and a point where it is locally constant is called a *regular* point of the generalized complex manifold. Our local structure theorem states that near any regular point of type k, the generalized complex manifold is equivalent to a product of a complex space of dimension k with a symplectic space. It is crucial to note, however, that this equivalence is obtained not only by using diffeomorphisms but also B-field transformations. This is consistent with the fact that, as symmetries of the Courant bracket, B-field transformations should be considered on a par with diffeomorphisms. We end that chapter by defining *twisted* generalized complex structures, thus enabling us to deform all our work by a real closed 3-form H.

Chapter 5 contains our main analytical result: the development of a Kuranishi deformation space for compact generalized complex manifolds. The deformation theory is governed by the differential complex $(C^\infty(\wedge^k L^*), d_L)$ mentioned above, which we show is elliptic, and therefore has finite-dimensional cohomology groups $H^k_L(M)$ on a compact manifold M. In particular, integrable deformations correspond to sections $\varepsilon \in C^\infty(\wedge^2 L^*)$ satisfying the Maurer-Cartan equation

$$d_L\varepsilon + \tfrac{1}{2}[\varepsilon, \varepsilon] = 0.$$

There is an analytic obstruction map $\Phi : H^2_L(M) \to H^3_L(M)$, and if this vanishes then there is a locally complete family of deformations parametrized by an open set in $H^2_L(M)$. In the case that we are deforming a complex structure, this cohomology group turns out to be

$$H^0(M, \wedge^2 T) \oplus H^1(M, T) \oplus H^2(M, \mathcal{O}).$$

This is familiar as the "extended deformation space" of Barannikov and Kontsevich [3], for which a geometrical interpretation has been sought for some time. Here it appears naturally as the deformation space of a complex structure in the generalized setting.

In chapter 6, we introduce a Riemannian metric G on $T \oplus T^*$ in such a way that it is compatible with two commuting generalized complex structures $\mathcal{J}_1, \mathcal{J}_2$. This enriched geometry generalizes classical Kähler geometry, where the compatible complex and symplectic structures play the role of $\mathcal{J}_1$ and $\mathcal{J}_2$. In studying this geometry we discover the result that it is equivalent to a geometry first discovered by Gates, Hull and Roček [42] in their study of supersymmetric sigma-models. Our proof of this equivalence occurs in section 6.4. The latter geometry is a bi-Hermitian geometry with an extra condition relating the pair of Hermitian 2-forms. The general subject of bi-Hermitian geometry has been studied in depth by Apostolov, Gauduchon, and Grantcharov [1] in the four-dimensional case, and in their paper they state that the main unsolved problem in the field is to determine whether or not there exist bi-Hermitian structures on complex surfaces which admit no hyperhermitian structure, for example, $\mathbb{C}P^2$. Using our equivalence theorem together with the deformation theorem of chapter 5.1, we are able to prove that a bi-hermitian structure on $\mathbb{C}P^2$ exists. That is, we show that there exists a Riemannian metric on $\mathbb{C}P^2$ admitting exactly two distinct orthogonal complex structures.

Chapter 6 ends with the definition of *twisted* generalized Kähler geometry, and we provide a family of examples of twisted generalized Kähler manifolds: the compact even-dimensional semi-simple Lie groups. We then define a further restriction on Kähler geometry analogous to the Calabi-Yau constraint: we define generalized Calabi-Yau metric geometry.

Chapter 7 introduces the concept of generalized complex submanifold. The requirement that the property of being such a submanifold should be invariant under B-field transformations renders the definition highly non-trivial; it turns out that the geometry of $T \oplus T^*$ demands that the submanifold carry a 2-form F; indeed if (N, H) is a manifold together with a closed 3-form H (the 'twist'), we say that the pair (M, F) of a submanifold $M \subset N$ and a 2-form $F \in \Omega^2(M)$ is a generalized submanifold of (N, H) if $H|_M = dF$. We explore the gerbe interpretation of this statement, and see how, in special cases, F should be interpreted as the curvature of a unitary line bundle on M. This situation would sound very familiar to physicists studying D-branes, which in certain cases are represented by submanifolds equipped with line bundles. While in the case of complex manifolds, a generalized complex submanifold is simply a complex submanifold equipped with a unitary holomorphic line bundle (or more generally a closed $(1, 1)$-form), in the symplectic case we obtain not only Lagrangian submanifolds equipped with flat line bundles, but also a special class of co-isotropic submanifolds, equipped with a line bundle with constrained curvature. These are precisely the co-isotropic A-branes recently discovered by Kapustin and Orlov [22]. Finally, to conclude this chapter, we indicate how, in the case of generalized Calabi-Yau metric geometry, one defines the analog of calibrated special Lagrangian submanifolds.

For many years physicists and mathematicians have studied the mysterious links between complex and symplectic geometry predicted by mirror symmetry. Therefore, the explicit unification of these two structures would seem an important step in understanding how they are connected. Indeed, this thesis is full of formulae and concepts which appear in some form or other in the realm of mirror symmetry. In the final, very speculative, chapter of the thesis we propose a vague picture of how mirror symmetry might be phrased in the language of generalized complex geometry.

Chapter 2

Linear algebra of $V \oplus V^*$

Let V be a real vector space of dimension m, and let V^* be its dual space. Then $V \oplus V^*$ is endowed with the following natural symmetric and skew-symmetric bilinear forms:

$$\langle X+\xi, Y+\eta\rangle_+ = \frac{1}{2}(\xi(Y)+\eta(X))$$
$$\langle X+\xi, Y+\eta\rangle_- = \frac{1}{2}(\xi(Y)-\eta(X)),$$

where $X, Y \in V$ and $\xi, \eta \in V^*$. Both bilinear forms are nondegenerate, and it is the symmetric one which is ubiquitous in this thesis; for this reason we usually denote it by $\langle,\rangle$ and refer to it as 'the inner product'. This symmetric inner product has signature (m,m) and therefore defines the non-compact orthogonal group $O(V \oplus V^*) \cong O(m,m)$. In addition to these bilinear forms, $V \oplus V^*$ has a canonical orientation, as follows. The highest exterior power can be decomposed as

$$\wedge^{2m}(V \oplus V^*) = \wedge^m V \otimes \wedge^m V^*,$$

and there is a natural pairing between $\wedge^k V^*$ and $\wedge^k V$ given by

$$(v^*, u) = \det(v_i^*(u_j)),$$

where $v^* = v_1^* \wedge \cdots \wedge v_k^* \in \wedge^k V^*$ and $u = u_1 \wedge \cdots \wedge u_k \in \wedge^k V$. Therefore we have a natural identification $\wedge^{2m}(V \oplus V^*) = \mathbb{R}$, and the number $1 \in \mathbb{R}$ defines a canonical orientation on $V \oplus V^*$. The Lie group preserving the symmetric bilinear form together with the canonical orientation is of course the special orthogonal group $SO(V \oplus V^*) \cong SO(m,m)$.

In this section we study the behaviour of maximal isotropic subspaces of $V \oplus V^*$ and their description using pure spinors. Further details can be found in the main reference for this classical material, Chevalley's monograph [13], which contains a chapter dealing exclusively with bilinear forms of signature (m,m). Also, we complexify the situation and consider the *real index* of a complex maximal isotropic subspace.

2.1 Symmetries of $V \oplus V^*$

For all that follows, it is crucial to understand certain special symmetries of $V \oplus V^*$. The Lie algebra of the special orthogonal group $SO(V \oplus V^*)$ is defined as usual:

$$\mathfrak{so}(V \oplus V^*) = \left\{T \;\middle|\; \langle Tx, y\rangle + \langle x, Ty\rangle = 0 \ \ \forall\, x, y \in V \oplus V^*\right\}$$

Using the splitting $V \oplus V^*$ we can decompose as follows:

$$T = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix}, \tag{2.1}$$

where $A \in \mathrm{End}(V)$, $B : V \longrightarrow V^*$, $\beta : V^* \longrightarrow V$, and where B and β are skew, i.e. $B^* = -B$ and $\beta^* = -\beta$. Therefore we may view B as a 2-form in $\wedge^2 V^*$ via $B(X) = i_X B$ and similarly we may regard β as an element of $\wedge^2 V$, i.e. a bivector. This corresponds to the observation that $\mathfrak{so}(V \oplus V^*) = \wedge^2(V \oplus V^*) = \mathrm{End}(V) \oplus \wedge^2 V^* \oplus \wedge^2 V$.

By exponentiation, we obtain certain important orthogonal symmetries of $T \oplus T^*$ in the identity component of $SO(V \oplus V^*)$.

Example 2.1 (B-transform). First let B be as above, and consider

$$\exp(B) = \begin{pmatrix} 1 & \\ B & 1 \end{pmatrix}, \tag{2.2}$$

an orthogonal transformation sending $X + \xi \mapsto X + \xi + i_X B$. It is useful to think of $\exp(B)$ a *shear* transformation, which fixes projections to T and acts by shearing in the T^* direction. We will sometimes refer to this as a B-transform.

Example 2.2 (β-transform). Similarly, let β be as above, and consider

$$\exp(\beta) = \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix}, \tag{2.3}$$

an orthogonal transformation sending $X + \xi \mapsto X + \xi + i_\xi \beta$. Again, it is useful to think of $\exp(\beta)$ a shear transformation, which fixes projections to T^* and acts by shearing in the T direction. We will sometimes refer to this as a β-transform.

Example 2.3 ($GL(V)$ action). If we choose $A \in \mathfrak{so}(V \oplus V^*)$ as above, then since

$$\exp(A) = \begin{pmatrix} \exp A & \\ & (\exp A^*)^{-1} \end{pmatrix}, \tag{2.4}$$

we see that we have a distinguished diagonal embedding of $GL^+(V)$ into the identity component of $SO(V \oplus V^*)$. Of course, we can extend this to a map

$$P \mapsto \begin{pmatrix} P & \\ & {P^*}^{-1} \end{pmatrix} \tag{2.5}$$

of the full $GL(V)$ into $SO(V \oplus V^*)$. Note that $SO(V \oplus V^*)$ has two connected components, and the two connected components of $GL(V)$ do map into different components of $SO(V \oplus V^*)$.

2.2 Maximal isotropic subspaces

A subspace $L < V \oplus V^*$ is *isotropic* when $\langle X, Y \rangle = 0$ for all $X, Y \in L$. Since we are in signature (m, m), the maximal dimension of such a subspace is m, and if this is the case, L is called maximal isotropic. Maximal isotropic subspaces of $V \oplus V^*$ are also called *linear Dirac structures* (see [14]). Note that V and V^* are examples of maximal isotropics. The space of maximal isotropics is disconnected into two components, and elements of these are said to have *even* or *odd* parity (sometimes called helicity), depending on whether they share their connected component with V or not, respectively. This situation becomes more transparent after studying the following two examples.

Example 2.4. Let $E \leq V$ be any subspace. Then consider the subspace

$$E \oplus \mathrm{Ann}(E) < V \oplus V^*,$$

where $\mathrm{Ann}(E)$ is the annihilator of E in V^*. Then this is a maximally isotropic subspace.

Example 2.5. Let $E \leq V$ be any subspace, and let $\varepsilon \in \wedge^2 E^*$. Regarding ε as a skew map $E \longrightarrow E^*$ via $X \mapsto i_X\varepsilon$, consider the following subspace, analogous to the graph of ε:

$$L(E,\varepsilon) = \left\{ X + \xi \in E \oplus V^* \;:\; \xi\big|_E = \varepsilon(X) \right\}.$$

Then if $X+\xi, Y+\eta \in L(E,\varepsilon)$, we check that

$$\begin{aligned}\langle X+\xi, Y+\eta\rangle &= \tfrac{1}{2}(\xi(Y) + \eta(X)) \\ &= \tfrac{1}{2}(\varepsilon(Y,X) + \varepsilon(X,Y)) = 0,\end{aligned}$$

showing that $L(E,\varepsilon)$ is a maximal isotropic subspace.

Note that the second example specializes to the first by taking $\varepsilon = 0$. Furthermore note that $L(V,0) = V$ and $L(\{0\},0) = V^*$. It is not difficult to see that every maximal isotropic is of this form:

Proposition 2.6. *Every maximal isotropic in $V \oplus V^*$ is of the form $L(E,\varepsilon)$.*

Proof. Let L be a maximal isotropic and define $E = \pi_V L$, where π_V is the canonical projection $V \oplus V^* \to V$. Then since L is maximal isotropic, $L \cap V^* = \mathrm{Ann}(E)$. Finally note that $E^* = V^*/\mathrm{Ann}(E)$, and define $\varepsilon : E \to E^*$ via $\varepsilon : e \mapsto \pi_{V^*}(\pi_V^{-1}(e) \cap L) \in V^*/\mathrm{Ann}(E)$. Then $L = L(E,\varepsilon)$. $\square$

The integer $k = \dim \mathrm{Ann}(E) = m - \dim \pi_V(L)$ is an invariant associated to any maximal isotropic in $V \oplus V^*$.

Definition 2.7. *The* type *of a maximal isotropic $L(E,\varepsilon)$, is the codimension k of its projection onto V.*

Since a B-transform preserves projections to V, it does not affect E:

$$\exp B(L(E,\varepsilon)) = L(E, \varepsilon + i^*B),$$

where $i : E \hookrightarrow V$ is the natural inclusion. Hence B-transforms do not change the type of the maximal isotropic. In fact, we see that by choosing B and E accordingly, we can obtain any maximal isotropic as a B-transform of $L(E,0)$.

On the other hand, β-transforms do modify projections to V, and therefore may change the dimension of E. To see how this occurs more clearly, we express the maximal isotropic as a generalized graph from $V^* \to V$, i.e. define $F = \pi_{V^*}L$ and $\gamma \in \wedge^2 F^*$ given by $\gamma(f) = \pi_V(\pi_{V^*}^{-1}(f) \cap L)$. As before, define

$$L(F,\gamma) = \left\{ X + \xi \in V \oplus F \;:\; X\big|_F = \gamma(\xi) \right\}.$$

Then, as happened in the B-field case,

$$\exp \beta(L(F,\gamma)) = L(F, \gamma + j^*\beta),$$

where this time $j : F \hookrightarrow V^*$ is the inclusion. Now, the projection $E = \pi_V L(F, \gamma)$ always contains $L \cap V = Ann(F)$, and if we take the quotient of E by this subspace we obtain the image of γ in $F^* = V/\operatorname{Ann}(F)$:

$$\frac{E}{L \cap V} = \frac{E}{\operatorname{Ann}(F)} = \operatorname{Im}(\gamma).$$

Therefore, we obtain the dimension of E as a function of γ:

$$\dim E = \dim L \cap V + \operatorname{rk} \gamma.$$

Because γ is a skew form, its rank is even. A β-transform sends $\gamma \mapsto \gamma + j^*\beta$, which also has even rank, and therefore we see that a β-transform, which is in the identity component of $SO(V \oplus V^*)$, can be used to change the dimension of E, and hence the type of $L(E, \varepsilon)$, by an even number.

Proposition 2.8. *Maximal isotropics $L(E, \varepsilon)$ of even parity are precisely those of even type; those of odd type have odd parity. The generic even maximal isotropics are those of type 0, whereas the generic odd ones are of type 1. The least generic type is $k = m$, of which there is only one maximal isotropic: V^*.*

Before we move on to the description of maximal isotropics using pure spinors, we indicate that alternative splittings for $V \oplus V^*$ should be considered:

Remark 2.9. *While the maximal isotropics V and V^* are distinguished in the vector space $V \oplus V^*$, all our results about the linear algebra of $V \oplus V^*$ are portable to the situation where L and L' are any maximal isotropics in $V \oplus V^*$ such that $L \cap L' = 0$. Then the inner product defines an isomorphism $L' \cong L^*$, and we obtain $V \oplus V^* = L \oplus L^*$.*

2.3 Spinors for $V \oplus V^*$: exterior forms

Let $CL(V \oplus V^*)$ be the Clifford algebra defined by the relation

$$v^2 = \langle v, v \rangle, \quad \forall v \in V \oplus V^*, \tag{2.6}$$

The Clifford algebra has a natural representation on $S = \wedge^\bullet V^*$ given by

$$(X + \xi) \cdot \varphi = i_X \varphi + \xi \wedge \varphi, \tag{2.7}$$

where $X + \xi \in V \oplus V^*$ and $\varphi \in \wedge^\bullet V^*$. We verify that this defines an algebra representation:

$$\begin{aligned} (X + \xi)^2 \cdot \varphi &= i_X(i_X \varphi + \xi \wedge \varphi) + \xi \wedge (i_X \varphi + \xi \wedge \varphi) \\ &= (i_X \xi)\varphi \\ &= \langle X + \xi, X + \xi \rangle \varphi, \end{aligned}$$

as required. This representation is the standard spin representation, and so we see that for $V \oplus V^*$ there is a canonical choice of spinors: the exterior algebra on V^*. Since in signature (m, m) the volume element ω of a Clifford algebra satisfies $\omega^2 = 1$, the spin representation decomposes into the ± 1 eigenspaces of ω (the positive and negative helicity spinors):

$$S = S^+ \oplus S^-,$$

and this is easily seen to be equivalent to the decomposition

$$\wedge^\bullet V^* = \wedge^{\mathrm{ev}} V^* \oplus \wedge^{\mathrm{odd}} V^*.$$

While the splitting $S = S^+ \oplus S^-$ is not preserved by the whole Clifford algebra, $S^\pm$ are irreducible representations of the spin group, which sits in the Clifford algebra as

$$Spin(V \oplus V^*) = \{v_1 \cdots v_r \mid v_i \in V \oplus V^*, \langle v_i, v_i \rangle = \pm 1 \text{ and } r \text{ even}\},$$

and which is a double cover of $SO(V \oplus V^*)$ via the homomorphism

$$\begin{aligned} &\rho : Spin(V \oplus V^*) \longrightarrow SO(V \oplus V^*) \\ &\rho(x)(v) = xvx^{-1} \quad x \in Spin(V \oplus V^*),\ v \in V \oplus V^*. \end{aligned}$$

Earlier we described certain symmetries of $V \oplus V^*$ by exponentiating elements of $\mathfrak{so}(V \oplus V^*)$. Since $\mathfrak{so}(V \oplus V^*) = \wedge^2(V \oplus V^*)$ sits naturally inside the Clifford algebra, we can see how its components act on the spin representation. Note that the derivative of ρ, given by

$$d\rho_x(v) = xv - vx = [x, v], \quad x \in \mathfrak{so}(V \oplus V^*), \quad v \in V \oplus V^*,$$

must be the usual representation of $\mathfrak{so}(V \oplus V^*)$ on $V \oplus V^*$.

Let $\{e_i\}$ be a basis for V and $\{e^i\}$ be the dual basis. We calculate the action of B- and β-transforms on S, as well as the more complicated case of the $GL(V)$ action:

Example 2.10 (B-transform). If $B = \frac{1}{2}B_{ij}e^i \wedge e^j$, $B_{ij} = -B_{ji}$ is a 2-form acting on $V \oplus V^*$ via $X + \xi \mapsto i_X B$, then its image in the Clifford algebra is $\frac{1}{2}B_{ij}e^j e^i$, since

$$e^i \wedge e^j : e_i \mapsto e^j,$$

and

$$\begin{aligned} d\rho_{e^j e^i}(e_i) &= (e^j e^i)e_i - e_i(e^j e^i) \\ &= (e^i e_i + e_i e^i)e^j = e^j. \end{aligned}$$

Its spinorial action on an exterior form $\varphi \in \wedge^\bullet V^*$ is

$$B \cdot \varphi = \frac{1}{2}B_{ij}e^j \wedge (e^i \wedge \varphi) = -B \wedge \varphi.$$

Therefore, exponentiating, we obtain

$$e^{-B}\varphi = (1 - B + \tfrac{1}{2}B \wedge B + \cdots) \wedge \varphi. \tag{2.8}$$

Example 2.11 (β-transform). If $\beta = \frac{1}{2}\beta^{ij}e_i \wedge e_j$, $\beta^{ij} = -\beta^{ji}$ is a 2-vector acting on $V \oplus V^*$ via $X + \xi \mapsto i_\xi \beta$, then its image in the Clifford algebra is $\frac{1}{2}\beta^{ij}e^j e^i$, and its spinorial action on a form φ is

$$\beta \cdot \varphi = \frac{1}{2}\beta^{ij} i_{e_j}(i_{e_i}\varphi) = i_\beta \varphi.$$

Therefore, exponentiating, we obtain

$$e^\beta \varphi = (1 + i_\beta + \tfrac{1}{2}i_\beta^2 + \cdots)\varphi. \tag{2.9}$$

Next we must understand the inverse image of the diagonally embedded $GL(V) \subset SO(V \oplus V^*)$ under the covering ρ. The group $SO(V \oplus V^*) \cong SO(n,n)$ has maximal compact subgroup $S(O(n) \times O(n))$ and hence has two connected components, each with fundamental group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Hence the double cover $Spin(n,n)$, which has maximal compact $S(Pin(n,0) \times_{\mathbb{Z}_2} Pin(0,n))$, has two connected components, each with fundamental group $\mathbb{Z}_2$. Note that the image of the injective map

$\rho_* : \pi_1(Spin(n,n)) \longrightarrow \pi_1(SO(n,n))$ is the diagonal subgroup $\{1,(\sigma,\sigma)\}$, where σ is the nontrivial element of $\pi_1(SO(n))$. Now, consider the natural inclusion of $GL(V)$ in $SO(V \oplus V^*)$ via

$$P \mapsto \begin{pmatrix} P & \\ & P^{*^{-1}} \end{pmatrix}.$$

Topologically, this maps the maximal compact of $GL(V)$, which is a disjoint union of two copies of $SO(n)$, diagonally into $S(O(n)\times O(n))$. This means that upon restriction to $GL(V) \subset SO(V \oplus V^*)$, the Spin double cover is a *trivial* covering. Therefore there are two distinguished $GL(V)$ subgroups of $Spin(V \oplus V^*)$ (depending on which branch is chosen for the non-identity component of $GL(V)$) which we denote $GL(V)_1$ and $GL(V)_2$, both mapping isomorphically to $GL(V)$ via the covering map. Their intersection is the connected group $GL^+(V)$. We are interested to see how these general linear groups act on $\wedge^\bullet V^*$ via the spin representation.

Example 2.12 ($GL^+(V)$ action). The action of exponentiating elements of $\mathrm{End}(V) < \mathfrak{so}(V \oplus V^*)$ is slightly more complicated, and becomes crucial when constructing the spinor bundle later on.

If $A = A_i^j e^i \otimes e_j$ is an endomorphism of V then as an element of $\mathfrak{so}(V \oplus V^*)$ it acts on $V \oplus V^*$ via $X + \xi \mapsto A(X) - A^*(\xi)$. That is to say, $e^i \otimes e_j$ maps $e_i \mapsto e_j$ and $e^j \mapsto -e^i$. Its image in the Clifford algebra is $\frac{1}{2}A_i^j(e_je^i - e^ie_j)$, and its spinorial action on φ is

$$\begin{aligned} A\cdot\varphi &= \tfrac{1}{2}A_i^j(e_je^i - e^ie_j)\cdot\varphi \\ &= \tfrac{1}{2}A_i^j(i_{e_j}(e^i \wedge \varphi) - e^i \wedge i_{e_j}\varphi) \\ &= \tfrac{1}{2}A_i^j\delta_j^i\varphi - A_i^je^i \wedge i_{e_j}\varphi \\ &= -A^*\varphi + \tfrac{1}{2}(\mathrm{Tr}\,A)\varphi, \end{aligned}$$

where $\varphi \mapsto -A^*\varphi = -A_i^je^i \wedge i_{e_j}\varphi$ is the usual action of $End(V)$ on $\wedge^\bullet V^*$. Hence, by exponentiation, the spinorial action of $GL^+(V)$ on $\wedge^\bullet V^*$ is by

$$g\cdot\varphi = \sqrt{\det g}(g^*)^{-1}\varphi,$$

i.e. as a $GL^+(V)$ representation the spinor representation decomposes as

$$S = \wedge^\bullet V^* \otimes (\det V)^{1/2}.$$

As we have seen, the $GL^+(V)$ action may be extended to a full $GL(V)$ action in two natural ways, which we now describe.

Example 2.13. Let $A_\pm = \pm\lambda e^1 \otimes e_1 + e^2 \otimes e_2 + \cdots + e^n \otimes e_n$, where λ is a positive real number. Then clearly $A_\pm \in GL^\pm(V)$. We now describe the elements in $Spin(V \oplus V^*)$ covering these.

Using the Clifford algebra norm, we see that $N(e_1 + e^1) = 1$ and that $N(\lambda^{-1/2}e_1 \pm \lambda^{1/2}e^1) = \pm 1$ (Note that we take positive square roots only.) Hence we form the $Spin(V \oplus V^*)$ elements

$$\begin{aligned} a_\pm &= (e_1 + e^1)(\lambda^{-1/2}e_1 \pm \lambda^{1/2}e^1) \\ &= \lambda^{-1/2}e^1e_1 \pm \lambda^{1/2}e_1e^1. \end{aligned}$$

We now check that these elements actually cover $A_\pm$. Note that $a_\pm^{-1} = \lambda^{1/2}e^1e_1 \pm \lambda^{-1/2}e_1e^1$:

$$\begin{aligned} \rho(a_\pm)(e_1) &= a_\pm e_1 a_\pm^{-1} \\ &= (\lambda^{-1/2}e^1e_1 \pm \lambda^{1/2}e_1e^1)e_1(\lambda^{1/2}e^1e_1 \pm \lambda^{-1/2}e_1e^1) \\ &= \pm\lambda e_1, \end{aligned}$$

as required. Hence the two elements of $Spin(V \oplus V^*)$ which cover the element A_- are $\pm a_-$. By convention, let us say that a_- is in $GL^-(V)_1 \subset Spin(V \oplus V^*)$ and that $-a_-$ is in $GL^-(V)_2 \subset Spin(V \oplus V^*)$. Now we may see how they act via the Spin representation. We take the unique Koszul decomposition of the general form $\varphi = e^1 \wedge \varphi_0 + \varphi_1$ where $i_{e_1}\varphi_0 = i_{e_1}\varphi_1 = 0$.

$$\begin{aligned} a_- \cdot \varphi &= (\lambda^{-1/2} e^1 e_1 - \lambda^{1/2} e_1 e^1) \cdot \varphi \\ &= \lambda^{-1/2} e^1 \wedge \varphi_0 - \lambda^{1/2}(\varphi - e^1 \wedge \varphi_0) \\ &= -\lambda^{1/2}(-\lambda^{-1} e^1 \wedge \varphi_0 + \varphi_1). \\ &= -(-\det A_-)^{1/2} A_-^{*^{-1}} \varphi \end{aligned}$$

Hence we deduce that the action of $g \in GL^-(V)$ is as follows:

$$g \cdot \varphi = \begin{cases} -(-\det g)^{1/2} g^{*^{-1}} \varphi & \text{if } g \in GL^-(V)_1, \\ (-\det g)^{1/2} g^{*^{-1}} \varphi & \text{if } g \in GL^-(V)_2. \end{cases} \tag{2.10}$$

2.4 The bilinear form on spinors

There is a bilinear form on spinors which behaves well under the spin representation. We describe it here, following the treatment of Chevalley [13]. For $V \oplus V^*$ this bilinear form coincides with the Mukai pairing of forms [35].

Since we have the splitting $V \oplus V^*$ into maximal isotropics, the exterior algebras $\wedge^\bullet V$ and $\wedge^\bullet V^*$ are subalgebras of $CL(V \oplus V^*)$. In particular, $\det V$ is a distinguished line inside $CL(V \oplus V^*)$, and it generates a left ideal with the property that upon choosing a generator $f \in \det V$, every element of the ideal has a unique representation as sf, $s \in \wedge^\bullet V^*$. This defines an action of the Clifford algebra on $\wedge^\bullet V^*$ by

$$(\rho(x)s)f = xsf \quad \forall x \in CL(V \oplus V^*),$$

which is the same action as that defined by (2.7).

Having expressed the spin representation in this way, we proceed to write down the bilinear form. Let α be the main antiautomorphism of the Clifford algebra, i.e. that determined by the tensor map $v_1 \otimes \cdots \otimes v_k \mapsto v_k \otimes \cdots \otimes v_1$. Now let $s, t \in \wedge^\bullet V^*$ be spinors and consider the bilinear expression

$$\begin{gathered} S \otimes S \xrightarrow{(,)} \det V^* \\ (s,t) = (\alpha(s) \wedge t)_{\text{top}}, \end{gathered}$$

where $()_{\text{top}}$ indicates taking the top degree component of the form.

We can express $(,)$ in the following way, using any generator $f \in \det V$:

$$\begin{aligned} (i_f(s,t))f &= (i_f(\alpha(s) \wedge t))f \\ &= (\rho(\alpha(f))(\alpha(s)t))f \\ &= \alpha(f)\alpha(s)tf \\ &= \alpha(sf)tf. \end{aligned} \tag{2.11}$$

From this description the behaviour of under the action of the Clifford algebra becomes clear. Let $v \in V \oplus V^*$:

$$\begin{aligned} (i_f(v \cdot s, v \cdot t))f &= \alpha(vsf)vtf \\ &= \alpha(sf)\alpha(v)vtf \\ &= \langle v, v\rangle i_f(s,t)f, \end{aligned}$$

showing that $(v \cdot s, v \cdot t) = \langle v, v\rangle (s,t)$, so that in particular $(g \cdot s, g \cdot t) = \pm(s,t)$ for any $g \in Spin(V \oplus V^*)$.

Proposition 2.14. *The bilinear form* $S \otimes S \xrightarrow{(,)} \det V^*$ *is invariant under the identity component of Spin:*

$$(x \cdot s, x \cdot t) = (s,t) \quad \forall\, x \in Spin_0(V \oplus V^*).$$

For example, $(\exp B \cdot s, \exp B \cdot s) = (s,t)$*, for any* $B \in \wedge^2 V^*$*.*

This bilinear form is non-degenerate, and can be symmetric or skew-symmetric depending on the dimension of V:

$$(s,t) = (-1)^{m(m-1)/2}(t,s).$$

Proposition 2.15. *The nondegenerate bilinear form* $(,)$ *is symmetric for* $m \equiv 0$ *or* $1 \pmod 4$ *and skew-symmetric otherwise.*

The behaviour of $(,)$ with respect to the decomposition $S = S^+ \oplus S^-$ depends on $m = \dim V$ in the following way, which is clear from the description of β in terms of wedge product.

Proposition 2.16. *If* $m \equiv 0 \pmod 2$ *then is zero on* $S^+ \times S^-$ *(and hence* $S^- \times S^+$*); if* $m \equiv 1 \pmod 2$ *then is zero on* $S^+ \times S^+$ *and* $S^- \times S^-$*.*

Example 2.17. Suppose V is 4-dimensional; then is symmetric, and the even spinors are orthogonal to the odd spinors. The inner product of even spinors $\rho = \rho_0 + \rho_2 + \rho_4$ and $\sigma = \sigma_0 + \sigma_2 + \sigma_4$ is given by

$$\begin{aligned}(\rho, \sigma) &= ((\rho_0 - \rho_2 + \rho_4) \wedge (\sigma_0 + \sigma_2 + \sigma_4))_4 \\ &= \rho_0\sigma_4 - \rho_2\sigma_2 + \rho_4\sigma_0.\end{aligned}$$

The inner product of odd spinors $\rho = \rho_1 + \rho_3$ and $\sigma = \sigma_1 + \sigma_3$ is given by

$$\begin{aligned}(\rho, \sigma) &= ((\rho_1 - \rho_3) \wedge (\sigma_1 + \sigma_3))_4 \\ &= \rho_1\sigma_3 - \rho_3\sigma_1.\end{aligned}$$

2.5 Pure spinors

Let φ be any nonzero spinor. Then we define its *null space* $L_\varphi < V \oplus V^*$, as follows:

$$L_\varphi = \{v \in V \oplus V^* \ : \ v \cdot \varphi = 0\}, \tag{2.12}$$

and it is clear that L_φ depends equivariantly on φ under the spin representation:

$$L_{g\cdot\varphi} = \rho(g)L_\varphi \quad \forall\, g \in Spin(V \oplus V^*). \tag{2.13}$$

The key property of null spaces is that they are isotropic: if $v, w \in L_\varphi$, then

$$2\langle v, w\rangle\varphi = (vw + wv) \cdot \varphi = 0, \tag{2.14}$$

implying that $\langle v, w\rangle = 0 \ \forall\, v, w \in L_\varphi$.

Definition 2.18. *A spinor* φ *is called* pure *when* L_φ *is maximally isotropic, i.e. has dimension* m*.*

Example 2.19. Let $1 \in \wedge^\bullet V^*$ be the unit spinor. Then the null space is

$$\{X + \xi \in V \oplus V^* \ : \ (i_X + \xi\wedge)1 = 0\} = V,$$

and $V < V \oplus V^*$ is maximally isotropic, equal to $L(V, 0)$ as we saw in section 2.2. Hence 1 is a pure spinor. Of course, we may apply any spin transformation to 1 to obtain more pure spinors; for instance, let $B \in \wedge^2 V^*$ and form $\varphi = e^B \wedge 1 = e^B$, which has maximal null space

$$N_\varphi = \{X - i_X B \ : \ X \in V\} = L(V, -B).$$

Example 2.20. Let $\theta \in \wedge^\bullet V^*$ be a nonzero 1-form. Then its null space is

$$\{X + \xi \in V \oplus V^* \ : \ (i_X + \xi\wedge)\theta = 0\} = \ker\theta \oplus \langle\theta\rangle = L(\ker\theta, 0),$$

which is also maximal isotropic. Hence θ, and therefore $e^B\theta$ for any $B \in \wedge^2 V^*$, is a pure spinor.

We will refer to Chevalley [13] for the main properties of pure spinors, and summarize the results here. Every maximal isotropic subspace of $V \oplus V^*$ is represented by a unique pure line in the spin bundle S. This line must lie in S^+ for even maximal isotropics and in S^- for odd ones. The cone of pure spinors in $\mathbb{P}(S^+)$ and $\mathbb{P}(S^-)$ is defined by a set of quadratic equations; in the even case these are the equations relating the different degree components of e^B (the generic element), and in the odd case they are the equations relating the different degree components of $e^B\theta$, for θ a 1-form.

The intersection properties of maximal isotropics can also be obtained from the pure spinors, using the bilinear form $(,)$:

Proposition 2.21 ([13], III.2.4.). *Maximal isotropics L, L' satisfy $\dim L \cap L' = 0$ if and only if their pure spinor representatives φ, φ' satisfy*

$$(\varphi, \varphi') \neq 0.$$

The bilinear form also provides us with the operation of "squaring" the spinor, as follows. The bilinear form determines an isomorphism

$$S \longrightarrow S^* \otimes \det V^*,$$

and, tensoring with $\mathrm{id} : S \to S$, we obtain an isomorphism

$$S \otimes S \xrightarrow{\Phi} S \otimes S^* \otimes \det V^* = \wedge^\bullet(V \oplus V^*) \otimes \det V^*.$$

Chevalley shows that this map takes a pure spinor line to the determinant line of the maximal isotropic it defines:

Proposition 2.22 ([13], III.3.2.). *Let $U_L < \wedge^\bullet V^*$ be a pure spinor line representing the maximal isotropic $L < V \oplus V^*$. Then Φ determines an isomorphism*

$$U_L \otimes U_L \xrightarrow{\Phi} \det L \otimes \det V^*,$$

where $\det L < \wedge^m(V \oplus V^*)$ is the determinant line.

In the remainder of this section, we will provide an expression for the pure spinor line associated to any maximal isotropic $L(E, \varepsilon)$.

Lemma 2.23. *Let $E \leq V$ be any subspace of codimension k. The maximal isotropic $L(E, 0) = E \oplus \mathrm{Ann}(E)$ is associated to the pure spinor line $\det(\mathrm{Ann}(E)) < \wedge^k V^*$.*

Proof. If $\varphi = \theta_1 \wedge \cdots \wedge \theta_k$ is a nonzero element of $\det(\mathrm{Ann}(E))$, then clearly $(X + \xi) \cdot \varphi = 0$ if and only if $X \in E$ and $\xi \in \mathrm{Ann}(E)$, as required. $\square$

As we saw in section 2.2, any maximal isotropic $L(E, \varepsilon)$ may be expressed as the B-transform of $L(E, 0)$ for B chosen such that $i^*B = \varepsilon$. Although ε is not in $\wedge^2 V^*$, we may abuse notation and write

$$L(E, \varepsilon) = \exp(\varepsilon)(L(E, 0)), \tag{2.15}$$

where in this equation ε is understood to represent any 2-form $B \in \wedge^2 V^*$ such that $i^*B = \varepsilon$, where $i : E \hookrightarrow V \otimes \mathbb{C}$ is the inclusion. Passing now to the spinorial description of maximal isotropics, and making use of the previous lemma, we obtain a description of any pure spinor:

Proposition 2.24. *Let $L(E, \varepsilon)$ be any maximal isotropic. Then the pure spinor line U_L defining it is given by*

$$U_L = \exp(\varepsilon)(\det \mathrm{Ann}(E)). \tag{2.16}$$

To be more precise, let $(\theta_1, \ldots, \theta_k)$ be a basis for $\mathrm{Ann}(E)$, and let $B \in \wedge^2 V^*$ be any 2-form such that $i^*B = \varepsilon$. Then the following spinor represents the maximal isotropic $L(E, \varepsilon)$:

$$\varphi_L = c \exp(B) \theta_1 \wedge \cdots \wedge \theta_k, \quad c \neq 0, \tag{2.17}$$

and any pure spinor can be expressed this way. Note that even maximal isotropics are represented by even forms and odd maximal isotropics by odd forms.

2.6 Complexification and the real index

The natural inner product $\langle , \rangle$ extends by complexification to $(V \oplus V^*) \otimes \mathbb{C}$, and all of our results concerning maximal isotropics and spinors for $V \oplus V^*$ can be extended by complexification to $(V \oplus V^*) \otimes \mathbb{C}$. We summarize our results in this new context.

Proposition 2.25. *Let V be a real vector space of dimension m. A maximal isotropic subspace $L < (V \oplus V^*) \otimes \mathbb{C}$ of type $k \in \{0, \ldots, m\}$ is specified equivalently by the following data:*

- *A complex subspace $L < (V \oplus V^*) \otimes \mathbb{C}$, maximal isotropic with respect to $\langle , \rangle$, and such that $E = \pi_{V \otimes \mathbb{C}} L$ has complex dimension $m - k$;*

- *A complex subspace $E < V \otimes \mathbb{C}$ such that $\dim_{\mathbb{C}} E = m - k$, together with a complex 2-form $\varepsilon \in \wedge^2 E^*$;*

- *A complex spinor line $U_L < \wedge^\bullet (V^* \otimes \mathbb{C})$ generated by*

$$\varphi_L = c \exp(B + i\omega) \theta_1 \wedge \cdots \wedge \theta_k, \tag{2.18}$$

where $(\theta_1, \ldots, \theta_k)$ are linearly independent complex 1-forms in $V^* \otimes \mathbb{C}$, B and ω are the real and imaginary parts of a complex 2-form in $\wedge^2 (V^* \otimes \mathbb{C})$, and $c \in \mathbb{C}$ is a nonzero scalar.

A new ingredient which appears when considering the complexified situation is the complex conjugate, which acts on all associated structures L, E, and U_L. We use it to define the concept of *real index*, introduced in [26].

Definition 2.26. *Let $L < (V \oplus V^*) \otimes \mathbb{C}$ be a maximal isotropic subspace. Then $L \cap \overline{L}$ is real, i.e. the complexification of a real space: $L \cap \overline{L} = K \otimes \mathbb{C}$, for $K < V \oplus V^*$. The real index r of the maximal isotropic L is defined by*

$$r = \dim_{\mathbb{C}} L \cap \overline{L} = \dim_{\mathbb{R}} K.$$

For example, a real maximal isotropic $L < V \oplus V^*$ has real index m.

Since the parity of L is determined by its intersection with the real subspace $V \otimes \mathbb{C} < (V \oplus V^*) \otimes \mathbb{C}$, it is clear that L and $\overline{L}$ must have the same parity, implying that $\dim L \cap \overline{L} \equiv m \pmod 2$, i.e.

$$r \equiv m \pmod 2, \tag{2.19}$$

showing that the real index must be even or odd depending on the dimension of V.

2.7 Functorial property of Dirac structures

Maximal isotropics (linear Dirac structures) have the interesting functorial property, noticed by Weinstein and used profitably in [9], that they can be pulled back and pushed forward along any linear map $f : V \to W$ of vector spaces. In particular, if $L < V \oplus V^*$ is a maximal isotropic then

$$f_* L = \{ f(X) + \eta \in W \oplus W^* \; : \; X + f^*\eta \in L \} \tag{2.20}$$

is a maximal isotropic subspace of $W \oplus W^*$. Similarly, if $M < W \oplus W^*$ is maximal isotropic then

$$f^* M = \{ X + f^*\eta \in V \oplus V^* \; : \; f(X) + \eta \in M \} \tag{2.21}$$

is a maximal isotropic subspace of $V \oplus V^*$.

2.8 The spin bundle for $T \oplus T^*$

In this section we will transport our algebraic work on $V \oplus V^*$ to a manifold. Let M be a smooth manifold of real dimension m, with tangent bundle T. Then consider the direct sum of the tangent and cotangent bundles $T \oplus T^*$. This bundle is endowed with the same canonical bilinear forms and orientation we described on $V \oplus V^*$. Therefore, while we are aware of the fact that $T \oplus T^*$ is associated to a $GL(m)$ principal bundle, we may also view it as having natural structure group $SO(m, m)$.

It is well-known that an oriented bundle with Euclidean structure group $SO(n)$ admits spin structure if and only if the second Stiefel-Whitney class vanishes, i.e. $w_2(E) = 0$. The situation for bundles with metrics of indefinite signature is more complicated, and was worked out by Karoubi in [23]. We summarize his results:

If an orientable bundle E has structure group $SO(p, q)$, we can always reduce the structure group to its maximal compact subgroup $S(O(p) \times O(q))$. This reduction is equivalent to the choice of a maximal positive definite subbundle $E^+ < E$, which allows us to write E as the direct sum $E = E^+ \oplus E^-$, where $E^- = (E^+)^\perp$ is negative definite.

Proposition 2.27 ([23], 1.1.26). *The $SO(p, q)$ bundle E admits $Spin(p, q)$ structure if and only if $w_2(E^+) = w_2(E^-)$.*

In the special case of $T \oplus T^*$, which has signature (m, m), the positive and negative definite bundles $E^\pm$ project isomorphically via $\pi_T : T \oplus T^* \to T$ onto the tangent bundle. Hence the condition $w_2(E^+) = w_2(E^-)$ is always satisfied for $T \oplus T^*$, yielding the following result.

Proposition 2.28. *The $SO(m, m)$ bundle $T \oplus T^*$ always admits $Spin(m, m)$ structure.*

As usual, the exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin(m, m) \longrightarrow SO(m, m) \longrightarrow 1$$

informs us that the set of spin structures is an affine space under $H^1(M, \mathbb{Z}_2)$, the group of real line bundles, which act on the associated spinor bundles by tensor product.

While we have given an argument for the existence of spin structure using characteristic classes, it is clear from the results of section 2.3 that the spin bundles can be constructed. In that section we showed that there are two subgroups of $Spin(V \oplus V^*)$, namely $GL(V)_1$ and $GL(V)_2$, which map isomorphically to $GL(V) \subset SO(V \oplus V^*)$ under the spin homomorphism. Therefore we can use the principal $GL(m)$ principal bundle of frames in T to form an associated $Spin(m, m)$ principal bundle, and this can be done in two different ways, producing two spin structures. Let the associated spinor bundles be S_1 and S_2. Since these bundles are associated to the original $GL(m)$ bundle of frames, by decomposing the spin representation according to $GL(m)_1$ and $GL(m)_2$ we can express S_1 and S_2 in terms of well-known associated bundles to the frame bundle.

In section 2.3 we did just this, and obtained the result that the spin representation decomposes under $GL(V)_1$ as

$$S = \wedge^\bullet V^* \otimes \det V |\det V|^{-1/2},$$

and under $GL(V)_2$ as

$$S = \wedge^\bullet V^* \otimes |\det V|^{1/2}.$$

Note that $|\det V|^{1/2}$ indicates the representation of $GL(V)$ sending $g \mapsto |\det g|^{1/2}$. Alternatively we could think of $|\det V|^{1/2}$ as the 1-dimensional vector space of 1/2-densities on V^*, i.e. maps $v : \det V^* \to \mathbb{R}$ such that $v(\lambda\omega) = |\lambda|^{1/2} v(\omega)$ for all $\lambda \in \mathbb{R}$ and $\omega \in \det V^*$.

Therefore when we form the spin bundles S_1 and S_2 by association, we obtain

$$\begin{aligned} S_1 &= \wedge^\bullet T^* \otimes \det T |\det T|^{-1/2} \\ S_2 &= \wedge^\bullet T^* \otimes |\det T|^{1/2}. \end{aligned}$$

Since $|\det T|^{1/2}$ is isomorphic to the trivial bundle, we see that there is always a choice of spin structure such that the spin bundle is (non-canonically) isomorphic to the exterior algebra $\wedge^\bullet T^*$. In any case, we will be primarily interested not with sections of the spin bundle itself but of its projectivisation. Therefore we can use the fact that the projectivisation of any spin bundle for $T \oplus T^*$ is canonically isomorphic to the projectivised differential forms:

$$\mathbb{P}(S) = \mathbb{P}(\wedge^\bullet T^*). \tag{2.22}$$

The bilinear form is inherited by the spin bundle, and further by the projectively isomorphic $\wedge^\bullet T^*$, in the form

$$(\varphi, \psi) = (\alpha(\varphi) \wedge \psi)_{\text{top}},$$

which is invariant under the action of $Spin_0(T \oplus T^*)$ and is covariant under diffeomorphisms. '

In previous sections we have studied the correspondence between maximal isotropics in $V \oplus V^*$ and certain lines in the spin bundle. In the same way, maximal isotropic subbundles of $T \oplus T^*$

correspond to sections of the projectivised bundle of differential forms, or equivalently line subbundles of $\wedge^\bullet T^*$. If we are fortunate and the line bundle is trivial, then it is possible to represent the maximal isotropic subbundle by a global differential form. For example, the maximal isotropic $T < T \oplus T^*$ is represented by the line generated by the differential form $1 \in \wedge^\bullet T^*$, whereas the maximal isotropic T^*, represented by the line $\det T^* < \wedge^\bullet T^*$, cannot be given by a global form if M is non-orientable.

Chapter 3

The Courant bracket

The Courant bracket is a natural bracket operation on the smooth sections of $T \oplus T^*$. It was first introduced in its present form by T. Courant [14], in the context of his work with Weinstein [15]. It also was implicit in the contemporaneous work of Dorfman [16]. Courant and Weinstein used it to define a new geometrical structure called a *Dirac* structure, which successfully unifies Poisson geometry and presymplectic geometry (the geometry defined by a real closed 2-form) by expressing each structure as a maximal isotropic subbundle of $T \oplus T^*$. The integrability condition, namely that the subbundle be closed under the Courant bracket, specializes to the usual integrability conditions in the Poisson and presymplectic cases. In the same way, we will use the Courant bracket to define the integrability of generalized complex structures.

We begin this section with an introduction to Lie algebroids, a class of vector bundles with structure that closely resembles that of the tangent bundle. Lie algebroids are particularly useful for at least two reasons. First, they provide a sufficiently general framework to accommodate a unified treatment of many kinds of geometry, including Poisson, foliated, (pre)symplectic, and as we shall show, complex and CR geometry, in addition to many new (generalized) types of geometry. Second, they provide a way to handle, in a smooth (non-singular) fashion, structures which at first glance appear to acquire singularities at certain loci in the manifold.

We then introduce the Courant bracket on $T \oplus T^*$, describing its basic properties, and showing that it fails to fit into the framework of Lie algebroid theory. Indeed, when its properties are systematized, we obtain the axioms of a Courant algebroid, first introduced in [30]. We follow the treatment in Roytenberg's thesis [41] for this material. The Courant algebroid structure of $T \oplus T^*$ is useful to us for two main reasons. First, it provides a source of new Lie algebroids by restriction to subbundles (as in the case of Dirac structures, mentioned above). Second, its natural group of symmetries includes not only diffeomorphisms, but also closed 2-forms, which act in a way familiar to physicists as the action of the B-field. As a consequence, all geometrical structures defined in terms of the Courant bracket can be transformed naturally by a B-field.

Finally, we investigate the notion of 'twisting' the Courant bracket, and the relation of this to the theory of gerbes with connection.

Throughout this section we will make extensive use of the following identities relating Lie derivative and interior product:

$$\mathcal{L}_X = i_X d + d i_X, \quad \mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y], \quad i_{[X,Y]} = [\mathcal{L}_X, i_Y],$$

where X, Y are vector fields. We follow the convention that for any differential form ρ, the interior product is a contraction with the first argument: $i_X \rho = \rho(X, \cdots)$.

3.1 Lie algebroids

A Lie algebroid, first defined by Pradines in [39] and explored in depth by Mackenzie in [31], is a vector bundle L on a smooth manifold M, equipped with a Lie bracket $[,]$ on $C^\infty(L)$ and a smooth bundle map $a : L \to T$, called the *anchor*. The anchor must induce a Lie algebra homomorphism $a : C^\infty(L) \to C^\infty(T)$, i.e.

$$a([X,Y]) = [a(X), a(Y)] \quad \forall X, Y \in C^\infty(L), \tag{3.1}$$

and the following Leibniz rule must be satisfied:

$$[X, fY] = f[X,Y] + (a(X)f)Y \quad \forall X, Y \in C^\infty(L), f \in C^\infty(M). \tag{3.2}$$

Example 3.1 (The tangent bundle). The tangent bundle is itself a Lie algebroid, taking the identity map as anchor. It is useful to think of a Lie algebroid as a generalization of the tangent bundle.

Example 3.2 (Foliations). Any integrable sub-bundle of the tangent bundle defines a Lie algebroid, choosing a to be the inclusion map.

Example 3.3 (The Atiyah sequence). Let $\pi : P \to M$ be a principal G-bundle on the manifold M. Then G-invariant vector fields on P are given by sections of the vector bundle $TP/G \to M$. This bundle has a Lie algebroid structure defined by the Lie bracket on $C^\infty(TP)$ and the surjective anchor π_*, which defines an exact sequence of vector bundles on M:

$$0 \longrightarrow \mathfrak{g} \longrightarrow TP/G \xrightarrow{\pi_*} T \longrightarrow 0\ ,$$

where $\mathfrak{g}$ is the adjoint bundle associated to P.

The notion of Lie algebroid can obviously be complexified, defining a *complex* Lie algebroid by requiring L to be a complex bundle and $a : L \to T \otimes \mathbb{C}$ a complex map, satisfying complexified conditions (3.1), (3.2). While the theory of Dirac structures uses real Lie algebroids, in this thesis complex Lie algebroids are particularly important. We will produce many examples of complex Lie algebroids in what follows, but we have some immediately at hand.

Example 3.4 (Complex structures). If M is a complex manifold then $T_{1,0} < T \otimes \mathbb{C}$ is a complex bundle closed under the Lie bracket. Using the inclusion map as anchor, $T_{1,0}$ is a complex Lie algebroid.

Example 3.5 (CR structures). A CR (Cauchy-Riemann) structure on a real $2n-1$-dimensional manifold is a complex $n-1$-dimensional sub-bundle $L < T \otimes \mathbb{C}$ which satisfies $\dim L \cap \overline{L} = 0$ and which is closed under the Lie bracket. L is then a complex Lie algebroid, with the inclusion map as anchor.

We now define several structures which exist naturally on Lie algebroids by generalizing the well-known case of the tangent bundle. We begin with the fact that the Lie bracket on vector fields has a natural $\mathbb{Z}$-graded extension to multivector fields $C^\infty(\wedge^\bullet T)$, called the Schouten bracket. This can be generalized to the context of Lie algebroids.

Definition 3.6. *Let L be a Lie algebroid. The Schouten bracket acting on sections $X_1 \wedge \cdots \wedge X_p \in C^\infty(\wedge^p L)$, $Y_1 \wedge \cdots \wedge Y_q \in C^\infty(\wedge^q L)$ is as follows:*

$$[X_1 \wedge \cdots \wedge X_p, Y_1 \wedge \cdots \wedge Y_q] = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \hat{X}_i \cdots \wedge X_p \wedge Y_1 \wedge \cdots \hat{Y}_j \cdots \wedge Y_q,$$

and $[X, f] = -[f, X] = a(X)f$ for $X \in C^\infty(L)$ and $f \in C^\infty(M)$. This bracket makes $C^\infty(\wedge^\bullet L)$ into a graded Lie algebra where the degree k component is $C^\infty(\wedge^{k+1} L)$. That is,

$$[A, B] = -(-1)^{(a-1)(b-1)}[B, A] \quad \text{and}$$
$$[A, [B, C]] = [[A, B], C] + (-1)^{(a-1)(b-1)}[B, [A, C]],$$

for all $A \in C^\infty(\wedge^a L), B \in C^\infty(\wedge^b L), C \in C^\infty(\wedge^c L)$.

Furthermore, if $A \in C^\infty(\wedge^a L)$, then $ad_A = [A, \cdot]$ is a derivation of degree $a - 1$ of the exterior multiplication on $C^\infty(\wedge^\bullet L)$:

$$ad_A(B \wedge C) = ad_A(B) \wedge C + (-1)^{(a-1)b} B \wedge ad_A(C).$$

The two interacting graded algebra structures of $C^\infty(\wedge^\bullet L)$, namely the exterior product and the Schouten bracket, make it into a Poisson superalgebra[1].

In addition to the Schouten bracket of vector fields, smooth manifolds are equipped with the exterior derivative operator d, a derivation of degree 1 of the algebra of differential forms. The exterior derivative operator can be defined in terms of the Lie bracket, and for this reason we may generalize it to Lie algebroids as follows:

Definition 3.7. *The Lie algebroid derivative d_L is a first order linear operator from $C^\infty(\wedge^k L^*)$ to $C^\infty(\wedge^{k+1} L^*)$ defined by*

$$\begin{aligned} d_L \sigma(X_0, \ldots, X_k) = &\sum_i (-1)^i a(X_i) \sigma(X_0, \ldots, \hat{X}_i, \ldots, X_k) \\ &+ \sum_{i<j} (-1)^{i+j} \sigma([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k), \end{aligned}$$

where $\sigma \in C^\infty(\wedge^k L^*)$ and $X_i \in C^\infty(L)$. Its principal symbol $s(d_L) : T^* \otimes \wedge^k L^* \to \wedge^{k+1} L^*$ is given by $a^* : T^* \to L^*$ composed with wedge product, i.e.

$$s_\xi(d_L) = a^*(\xi) \wedge \cdot \quad ,$$

where $\xi \in T^*$. The operator d_L satisfies $d_L^2 = 0$ due to the Jacobi identity for $[,]$, and therefore $(C^\infty(\wedge^\bullet L^*), d_L)$ is a natural differential complex associated with any Lie algebroid.

By analogy, we define interior product and Lie derivative for Lie algebroids:

Definition 3.8. *Let $X \in C^\infty(L)$. Then the interior product i_X is the degree -1 derivation on $C^\infty(\wedge^\bullet L^*)$ defined by $i_X \sigma = \sigma(X, \cdots)$, and the Lie derivative $\mathcal{L}_X^L$ is defined by the Cartan formula*

$$\mathcal{L}_X^L = d_L i_X + i_X d_L.$$

[1] Since the two graded structures differ in parity, it is an 'odd Poisson superalgebra', otherwise known as a *Gerstenhaber algebra.*

So far, we have described the intrinsic algebraic structures present in Lie algebroids. It is important to understand what these structures imply for the underlying geometry of the manifold. In particular, as described by Courant [14], every Lie algebroid induces a generalized foliation in the sense of Sussmann [46], which we now explain.

A foliated manifold M is one which has been expressed as a disjoint union of subsets called *leaves*. A leaf is a connected submanifold (injective immersion) $l \subset M$ such that any point $p \in l$ has a neighbourhood $U \subset M$ where the connected component of p in $l \cap U$ is an embedded submanifold of M. A usual foliation is one where all leaves have the same dimension, whereas a generalized foliation allows the dimension of the leaves to vary. The second main result in Sussmann's paper [46] describes necessary and sufficient conditions on a distribution Δ for it to be integrable into such a generalized foliation (by which is meant that at any point $m \in M$, the tangent space to the leaf through m is precisely $\Delta(m)$). Essentially, Sussmann's theorem is a more powerful version of the classical Frobenius integrability theorem. We needn't use Sussmann's full theorem, but only a corollary of his main result:

Theorem 3.9 ([46], Theorem 8.1.). *Let Δ_D be a distribution spanned by a collection $D \subset C^\infty(T)$ of smooth vector fields. This is called a* smooth distribution*. Such a distribution is said to be of* finite type *if, for every $m \in M$, there exist smooth vector fields $X^1, \dots, X^n \in D$ such that*

- *$X^1(m), \dots, X^n(m)$ span $\Delta_D(m)$ and*
- *For every $X \in D$, there exists a neighbourhood U of m and smooth functions $c_k^i \in C^\infty(U)$ such that for all i,*

$$[X, X^i] = \sum_k c_k^i X^k.$$

If Δ_D is of finite type, then it is integrable to a generalized foliation as described above.

Any real Lie algebroid L with anchor a produces a distribution $\Delta_D = a(L)$, and since it is the image of a smooth bundle map, it is spanned by the smooth vector fields $D = a(C^\infty(L))$. Hence it is a smooth distribution. Furthermore, for any point $m \in M$ we may choose a local basis of sections $X^1, \dots, X^n \in C^\infty(U, L)$ in some neighbourhood of m. Then $a(X^1), \dots, a(X^n)$ certainly span Δ_D in U and by the Lie algebroid property (3.1), we see that

$$[a(X^i), a(X^j)] = a([X^i, X^j]) = a(\sum_k c_k^{ij} X^k) = \sum_k c_k^{ij} a(X^k),$$

for some $c_k^{ij} \in C^\infty(U)$, implying that Δ_D is of finite type. Hence by the theorem, we conclude that a real Lie algebroid induces a generalized foliation on the manifold M. Furthermore, since the rank of the smooth bundle map a is a lower semi-continuous function (every point has a neighbourhood in which the rank *does not decrease*), we conclude that the dimension of the leaf is also a lower semi-continuous function.

Proposition 3.10 ([14], Theorem 2.1.3.). *If L is a real Lie algebroid on M with anchor a, then $\Delta = a(L)$ is a smooth integrable distribution in the sense of Sussmann, implying that M can be expressed locally as a disjoint union of embedded submanifolds (called leaves) such that at any point $m \in M$, the tangent space to the leaf through m is precisely $\Delta(m)$. Furthermore, the dimension of the leaf,* $\dim \Delta$*, is a lower semi-continuous function on the manifold.*

If L is a complex Lie algebroid, then let $E < T \otimes \mathbb{C}$ denote the image under the anchor, i.e. $E = a(L)$. The complex distribution E induces two real distributions $\Delta \subset \Theta \subset T$, defined by $E + \overline{E} = \Theta \otimes \mathbb{C}$ and $E \cap \overline{E} = \Delta \otimes \mathbb{C}$. While Θ need not be integrable, or even involutive, certainly Δ is involutive, but we haven't enough information to decide its Sussmann integrability. To establish an analogous result to the previous proposition we will need an extra assumption.

In particular, consider the case where $E + \overline{E} = T \otimes \mathbb{C}$. Because of this, the real bundle map $i(a - \overline{a}) : L \to T$ is surjective, hence the kernel is a smooth real sub-bundle $K < L$:

$$K = \left\{ X \in L \ : \ a(X) = \overline{a(X)} \right\}.$$

Projecting K to T via the anchor a, we obtain precisely the distribution Δ. As a result, we see that Δ is a smooth distribution. It is easy to check now that it is of finite type: for any point $m \in M$, let $X^1, \ldots, X^n$ be a local basis of sections for $C^\infty(U, K)$ in some neighbourhood U of m. Then $a(X^1), \ldots, a(X^n)$ span Δ in U. Furthermore, since $a([X^i, X^j]) = [a(X^i), a(X^j)] = \overline{[a(X^i), a(X^j)]} = \overline{a([X^i, X^j])}$, we see that $[X^i, X^j] \in C^\infty(U, K)$, and therefore $[X^i, X^j] = \sum_k c_k^{ij} X^k$, giving

$$[a(X^i), a(X^j)] = \sum_k c_k^{ij} a(X^k),$$

which implies that Δ is of finite type. Hence we obtain the following result.

Proposition 3.11. *Let L be a complex Lie algebroid on M with anchor a, and such that $E + \overline{E} = T \otimes \mathbb{C}$, where $E = a(L)$. Let Δ be the real distribution defined by $\Delta \otimes \mathbb{C} = E \cap \overline{E}$. Then Δ is a smooth integrable distribution in the sense of Sussmann, defining a generalized foliation of M. Furthermore, the dimension of the leaf,* $\dim \Delta$*, is a lower semi-continuous function on the manifold.*

Besides the generalized foliation, a complex Lie algebroid of this type induces a *transverse complex structure* on this foliation in a sense which we now describe.

A complex distribution $E < T \otimes \mathbb{C}$ of constant complex codimension k on a real n-manifold M is *integrable* if, in some neighbourhood U of each point $m \in M$, there exist complex functions $f_1, \ldots, f_k \in C^\infty(U, \mathbb{C})$ such that $\{df_1, \ldots, df_k\}$ are linearly independent at each point in U and annihilate all complex vector fields lying in E. By the Newlander-Nirenberg theorem [36], we know that E is integrable if the following conditions are satisfied:

- E is involutive (closed under Lie bracket), and
- $\dim E \cap \overline{E}$ is constant, and
- $E + \overline{E}$ is involutive as well.

In this situation, the functions $f_1, \ldots f_k$ are complex coordinates transverse to the foliation determined by $E \cap \overline{E}$. In other words, every point $m \in M$ has a neigbourhood isomorphic, as a smooth manifold with complex distribution, to an open set in $\mathbb{R}^{n-2k} \times \mathbb{C}^k$, which has natural complex distribution spanned by $\{\partial/\partial x_1, \ldots, \partial/\partial x_{n-2k}, \partial/\partial z_1, \ldots, \partial/\partial z_k\}$.

In our situation of a complex Lie algebroid satisfying $E + \overline{E} = T \otimes \mathbb{C}$, if we restrict our attention to points $m \in M$ which are *regular*, in the sense that the leaf dimension is constant in a neighbourhood U of m, then the above conditions are satisfied, and we obtain a transversal complex structure (at regular points).

Proposition 3.12. *Let L be a complex Lie algebroid on the real n-manifold M with anchor a, and such that $E + \overline{E} = T \otimes \mathbb{C}$, where $E = a(L)$. Let $m \in M$ be a regular point for the Lie algebroid, i.e. a point where $k = \dim E \cap \overline{E}$ is locally constant. Then in some neighbourhood U of m, there exist complex functions $z_1, \ldots, z_k \in C^\infty(U, \mathbb{C})$ such that $\{dz_1, \ldots, dz_k\}$ are linearly independent at each point in U and annihilate all complex vector fields lying in E, i.e. we have a transverse complex structure to the foliation, at regular points.*

Remark 3.13. *Note that a transverse complex structure on a foliation implies that any smooth section of the foliation in a domain of regular points inherits an integrable complex structure. The coordinates $\{z_i\}$ can be restricted to the section and serve as complex coordinates.*

Before we proceed to study the Courant bracket, which will provide us with many more examples of complex Lie algebroids, we provide the definition of a structure called a *Lie bialgebroid*, which was introduced by Mackenzie and Xu [32] as the infinitesimal object corresponding to a Poisson groupoid. The Lie algebroids we will study will appear naturally in bialgebroid pairs, and this will be crucially important when we study the deformation theory of generalized complex structures.

Definition 3.14 (Lie bialgebroid). *Let L be a Lie algebroid and suppose its dual bundle L^* also has the structure of a Lie algebroid. Then (L, L^*) is a Lie bialgebroid if the Lie algebroid derivative $d_L : C^\infty(L^*) \to C^\infty(\wedge^2 L^*)$ is a derivation of the Schouten bracket $[,]_{L^*}$ on $C^\infty(L^*)$, in the sense that*

$$d_L[X, Y] = [d_L X, Y] + [X, d_L Y].$$

The Lie bialgebroid condition is self-dual, in the sense that (L, L^*) is a Lie bialgebroid if and only if (L^*, L) is. Furthermore, the Lie bialgebroid condition is equivalent to requiring that d_L is a derivation of degree 1 of the graded algebra $(C^\infty(\wedge^\bullet(L^*)), [,]_{L^*})$. Both these facts are proven in [27].

Example 3.15. The most obvious Lie bialgebroid is simply (T, T^*), where we take the usual Lie algebroid structure on the tangent bundle T and the trivial structure on T^* (zero bracket and anchor). Then the exterior derivative is certainly a derivation of the trivial bracket.

3.2 The Courant bracket and Courant algebroids

The Courant bracket is a skew-symmetric bracket defined on smooth sections of $T \oplus T^*$, given by

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi),$$

where $X + \xi, Y + \eta \in C^\infty(T \oplus T^*)$.

Note that on vector fields the Courant bracket reduces to the Lie bracket $[X, Y]$; in other words, if $\pi : T \oplus T^* \to T$ is the natural projection,

$$\pi([A, B]) = [\pi(A), \pi(B)], \tag{3.3}$$

for any $A, B \in C^\infty(T \oplus T^*)$. On the other hand, on 1-forms the Courant bracket vanishes. We will address this basic asymmetry of the bracket in section 3.5.

The Courant bracket is not a Lie bracket, since it fails to satisfy the Jacobi identity. Therefore, although the projection map π acts as a Lie algebroid anchor, $(T \oplus T^*, [,])$ is not a Lie algebroid. However it is interesting to examine how it fails to be a Lie algebroid. The Jacobiator is a trilinear operator which measures the failure to satisfy the Jacobi identity:

$$\mathrm{Jac}(A, B, C) = [[A, B], C] + [[B, C], A] + [[C, A], B],$$

where $A, B, C \in C^\infty(T \oplus T^*)$. The Jacobiator can be usefully expressed as the derivative of a quantity which we will call the *Nijenhuis* operator, for reasons which will become clear later. For this reason, one can say that the Courant bracket satisfies the Jacobi identity *up to an exact term*. We now prove this and two other basic properties of the Courant bracket; these results are implicit in [30]; we provide proofs here which will be useful for later development.

Proposition 3.16.

$$\mathrm{Jac}(A, B, C) = d\left(\mathrm{Nij}(A, B, C)\right), \tag{3.4}$$

where Nij *is the Nijenhuis operator:*

$$\mathrm{Nij}(A, B, C) = \tfrac{1}{3}\left(\langle [A, B], C\rangle + \langle [B, C], A\rangle + \langle [C, A], B\rangle\right).$$

Here $\langle , \rangle$ *is the inner product on* $T \oplus T^*$ *introduced in the previous section.*

Proof. To prove this result, we introduce the Dorfman bracket operation $\circ$ on $T \oplus T^*$ which is not skew, but whose skew symmetrization is the Courant bracket:

$$(X + \xi) \circ (Y + \eta) = [X, Y] + \mathcal{L}_X \eta - i_Y d\xi.$$

The difference between the two brackets is as follows:

$$[A, B] = A \circ B - d\langle A, B\rangle,$$

and of course $[A, B] = \frac{1}{2}(A \circ B - B \circ A)$. The advantage of the Dorfman bracket is that it satisfies a kind of Leibniz rule[2]:

$$A \circ (B \circ C) = (A \circ B) \circ C + B \circ (A \circ C),$$

which is easily proved, setting $A = X + \xi,\ B = Y + \eta,\ C = Z + \zeta$:

$$\begin{aligned}
&(A \circ B) \circ C + B \circ (A \circ C) \\
&\qquad = [[X, Y], Z] + [Y, [X, Z]] + \mathcal{L}_{[X,Y]}\zeta - i_Z d(\mathcal{L}_X \eta - i_Y d\xi) + \mathcal{L}_Y(\mathcal{L}_X \zeta - i_Z d\xi) - i_{[X,Z]} d\eta \\
&\qquad = [X, [Y, Z]] + \mathcal{L}_X \mathcal{L}_Y \zeta - \mathcal{L}_X i_Z d\eta - \mathcal{L}_Y i_Z d\xi + i_Z d i_Y d\xi \\
&\qquad = [X, [Y, Z]] + \mathcal{L}_X(\mathcal{L}_Y \zeta - i_Z d\eta) - i_{[Y,Z]} d\xi \\
&\qquad = A \circ (B \circ C),
\end{aligned}$$

as required. Now note that

$$\begin{aligned}
[[A, B], C] &= [A, B] \circ C - d\langle [A, B], C\rangle \\
&= (A \circ B - d\langle A, B\rangle) \circ C - d\langle [A, B], C\rangle \\
&= (A \circ B) \circ C - d\langle [A, B], C\rangle,
\end{aligned}$$

where we have used the fact that $\mu \circ C = 0$ whenever μ is a closed 1-form. Finally, we express the Jacobiator as follows (c.p. indicates cyclic permutations):

$$\begin{aligned}
\mathrm{Jac}(A, B, C) &= [[A, B], C] + \text{c.p.} \\
&= \tfrac{1}{4}((A \circ B) \circ C - C \circ (A \circ B) - (B \circ A) \circ C + C \circ (B \circ A) + \text{c.p.}) \\
&= \tfrac{1}{4}(A \circ (B \circ C) - B \circ (A \circ C) - C \circ (A \circ B) - B \circ (A \circ C) + A \circ (B \circ C) + C \circ (B \circ A) + \text{c.p}) \\
&= \tfrac{1}{4}(A \circ (B \circ C) - B \circ (A \circ C) + \text{c.p.}) \\
&= \tfrac{1}{4}((A \circ B) \circ C + \text{c.p.}) \\
&= \tfrac{1}{4}([[A, B], C] + d\langle [A, B], C\rangle + \text{c.p.}) \\
&= \tfrac{1}{4}(\mathrm{Jac}(A, B, C) + 3d(\mathrm{Nij}(A, B, C))),
\end{aligned}$$

[2] In fact, the Dorfman bracket makes $C^\infty(T \oplus T^*)$ into a *Loday* algebra.

which implies that $\text{Jac}(A, B, C) = d(\text{Nij}(A, B, C))$, as required. $\square$

The next proposition describes the failure of the Courant bracket to satisfy the second Lie algebroid axiom (3.2).

Proposition 3.17. *Let $f \in C^\infty(M)$. Then the Courant bracket satisfies*

$$[A, fB] = f[A, B] + (\pi(A)f)B - \langle A, B\rangle df. \tag{3.5}$$

Proof. Let $A = X + \xi$ and $B = Y + \eta$, so that

$$\begin{aligned}
[X + \xi, f(Y + \eta)] &= [X, fY] + \mathcal{L}_X f\eta - \mathcal{L}_{fY}\xi - \tfrac{1}{2}d(i_X(f\eta) - i_{fY}\xi) \\
&= f[X + \xi, Y + \eta] + (Xf)Y + (Xf)\eta - (i_Y\xi)df - \tfrac{1}{2}(i_X\eta - i_Y\xi)df \\
&= f[X + \xi, Y + \eta] + (Xf)(Y + \eta) - \langle X + \xi, Y + \eta\rangle df,
\end{aligned}$$

as required. $\square$

We see yet again that the Courant bracket differs from being a Lie algebroid by exact terms. Both properties (3.4) and (3.5) of the Courant bracket demonstrate that it is intimately linked to the natural inner product $\langle , \rangle$, a fact which we will employ frequently. There is a further property which highlights the relationship between $[,]$ and $\langle , \rangle$, which we will find useful:

Proposition 3.18. *Differentiation of the natural inner product can be expressed in terms of the Courant bracket thus:*

$$\pi(A)\langle B, C\rangle = \langle [A, B] + d\langle A, B\rangle, C\rangle + \langle B, [A, C] + d\langle A, C\rangle\rangle. \tag{3.6}$$

Proof. In terms of the Dorfman bracket, we wish to prove that

$$\pi(A)\langle B, C\rangle = \langle A \circ B, C\rangle + \langle B, A \circ C\rangle;$$

set $A = X + \xi,\ B = Y + \eta,\ C = Z + \zeta$. Then we have

$$\begin{aligned}
\langle A \circ B, C\rangle + \langle B, A \circ C\rangle &= \tfrac{1}{2}(i_{[X,Y]}\zeta + i_Z(\mathcal{L}_X\eta - i_Y d\xi) + i_{[X,Z]}\eta + i_Y(\mathcal{L}_X\zeta - i_Z d\xi)) \\
&= \tfrac{1}{2}(L_X i_Y\zeta + L_X i_Z\eta) \\
&= \tfrac{1}{2}i_X d(i_Y\zeta + i_Z\eta) \\
&= \pi(A)\langle B, C\rangle,
\end{aligned}$$

as required. $\square$

The fundamental properties (3.3), (3.4), (3.5), and (3.6) make $(T \oplus T^*,\ \langle , \rangle,\ [,],\ \pi)$ into the motivating example of a *Courant algebroid*, the definition of which is the first main result of the paper [30].

Definition 3.19 ([30], Definition 2.1). *A Courant algebroid is a vector bundle E equipped with a nondegenerate symmetric bilinear form $\langle , \rangle$ as well as a skew-symmetric bracket $[,]$ on $C^\infty(E)$, and with a smooth bundle map $\pi : E \to T$ called the* anchor*. This induces a natural differential operator $\mathcal{D} : C^\infty(M) \to C^\infty(E)$ via the definition $\langle \mathcal{D}f, A\rangle = \frac{1}{2}\pi(A)f$ for all $f \in C^\infty(M)$ and $A \in C^\infty(E)$. These structures must be compatible in the following sense:*

C1) $\pi([A, B]) = [\pi(A), \pi(B)] \quad \forall A, B \in C^\infty(E)$, *C2)* $\mathrm{Jac}(A,B,C) = \mathcal{D}\left(\mathrm{Nij}(A,B,C)\right) \quad \forall A,B,C \in C^\infty(E)$,

C3) $[A, fB] = f[A,B] + (\pi(A)f)B - \langle A,B\rangle \mathcal{D}f, \quad \forall A,B \in C^\infty(E),\ f \in C^\infty(M)$,

C4) $\pi \circ \mathcal{D} = 0$, *i.e.* $\langle \mathcal{D}f, \mathcal{D}g\rangle = 0 \quad \forall f,g \in C^\infty(M)$.

C5) $\pi(A)\langle B,C\rangle = \langle [A,B] + \mathcal{D}\langle A,B\rangle, C\rangle + \langle B, [A,C] + \mathcal{D}\langle A,C\rangle\rangle \quad \forall A,B,C \in C^\infty(E)$,

where the Jacobiator $\mathrm{Jac}(\cdot,\cdot,\cdot)$ *and the Nijenhuis operator* $\mathrm{Nij}(\cdot,\cdot,\cdot)$ *are as defined before.*

Remark 3.20. *The definition of* $\mathcal{D}$ *implies that it satisfies a Leibniz rule, i.e.* $\mathcal{D}(fg) = f\mathcal{D}(g) + \mathcal{D}(f)g$. *In [47] it is noted that this Leibniz property, together with axioms* $(C1)$ *and* $(C5)$*, imply not only the definition of* $\mathcal{D}$ *but also axioms* $(C3)$ *and* $(C4)$.

Remark 3.21. *As with Lie algebroids, we define the concept of* complex *Courant algebroid in the obvious way.*

3.3 Symmetries of the Courant bracket; the B-field

The Lie bracket of smooth vector fields is a canonically defined structure on a manifold; that is to say, it is invariant under diffeomorphisms. In fact, there are no other symmetries of the tangent bundle preserving the Lie bracket.

Proposition 3.22. *Let* (f,F) *be an automorphism of the tangent bundle* $\pi : TM \longrightarrow M$ *of a smooth manifold* M*, i.e. a pair of diffeomorphisms* $f : M \longrightarrow M$, $F : TM \longrightarrow TM$ *such that the diagram*

$$\begin{tikzcd} TM \arrow[r, "F"] \arrow[d, "\pi"'] & TM \arrow[d, "\pi"] \\ M \arrow[r, "f"] & M \end{tikzcd}$$

commutes and F *is a linear map on each fibre. Suppose also that* F *preserves the Lie bracket, i.e.* $F([X,Y]) = [F(X),F(Y)]$ *for all vector fields* X,Y*. Then* F *must equal* f_**, the derivative of* f.

Proof. Note that (f, f_*) is an automorphism of the tangent bundle preserving the Lie bracket. Therefore, setting $G = f_*^{-1} \circ F$, the pair (Id, G) is also an automorphism preserving the Lie bracket. In particular, for any vector fields X,Y and $h \in C^\infty(M)$ we have $G([hX,Y]) = [G(hX), G(Y)]$, or expanding,

$$G([hX,Y]) = G(h[X,Y] - Y(h)X) = hG([X,Y]) - Y(h)G(X),$$

while, on the other hand,

$$\begin{aligned}[G(hX), G(Y)]) &= h[G(X),G(Y)] - G(Y)(h)G(X) \\ &= hG([X,Y]) - G(Y)(h)G(X),\end{aligned}$$

so that $Y(h)G(X) = G(Y)(h)G(X)$ for all X,Y,h. This can only hold when $G(Y) = Y$ for all vector fields Y, i.e. $G = Id$, yielding finally that $F = f_*$. □

In the case of $T \oplus T^*$, however, the situation is not so simple. While the Courant bracket and natural inner product are invariant under diffeomorphisms, they have an additional symmetry, which we call a *B-field transformation*. Let B be a smooth 2-form and view it as a map $T \longrightarrow T^*$ via interior product $X \mapsto i_X B$. This is the natural Lie algebra action of $\wedge^2 T^* < \mathfrak{so}(T \oplus T^*)$ on $T \oplus T^*$. Then the invertible bundle map given by exponentiating B, namely

$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} : \; X + \xi \mapsto X + \xi + i_X B$$

is orthogonal, since $B^* = -B$ implies that $(e^B)^* e^B = e^{B-B} = 1$. It is useful to think of e^B a *shear* transformation, which fixes projections to T and acts by shearing in the T^* direction.

Proposition 3.23. *The map e^B is an automorphism of the Courant bracket if and only if B is closed, i.e. $dB = 0$.*

Proof. Let $X + \xi, Y + \eta \in C^\infty(T \oplus T^*)$ and let B be a smooth 2-form. Then

$$\begin{aligned}
&[e^B(X+\xi), e^B(Y+\eta)] \\
&\quad = [X + \xi + i_X B, Y + \eta + i_Y B] \\
&\quad = [X+\xi, Y+\eta] + [X, i_Y B] + [i_X B, Y] \\
&\quad = [X+\xi, Y+\eta] + L_X i_Y B - \frac{1}{2} d i_X i_Y B - L_Y i_X B + \frac{1}{2} d i_Y i_X B \\
&\quad = [X+\xi, Y+\eta] + L_X i_Y B - i_Y L_X B + i_Y i_X dB \\
&\quad = [X+\xi, Y+\eta] + i_{[X,Y]} B + i_Y i_X dB \\
&\quad = e^B([X+\xi, Y+\eta]) + i_Y i_X dB.
\end{aligned}$$

Therefore we see that e^B is an automorphism of the Courant bracket if and only if $i_Y i_X dB = 0$ for all X, Y, which happens precisely when $dB = 0$. $\square$

A natural question we may ask at this point is whether B-field transforms and diffeomorphisms are the only orthogonal automorphisms of the Courant bracket.

Proposition 3.24. *Let (f, F) be an orthogonal automorphism of the direct sum $T \oplus T^*$ for a smooth manifold M. Suppose also that F preserves the Courant bracket, i.e. $F([A,B]) = [F(A), F(B)]$ for all sections $A, B \in C^\infty(T \oplus T^*)$. Then F must be the composition of a diffeomorphism of M and a B-field transformation. To be more precise, the group of orthogonal Courant automorphisms of $T \oplus T^*$ is the semidirect product of Diff(M) and $\Omega^2_{closed}(M)$.*

Proof. Note that if f is a diffeomorphism, the map $f_c = \left(\begin{smallmatrix} f_* & 0 \\ 0 & (f^*)^{-1} \end{smallmatrix}\right)$ is an orthogonal automorphism of $T \oplus T^*$ preserving the Courant bracket. Therefore, setting $G = f_c^{-1} \circ F$, the pair (Id, G) is also an orthogonal automorphism preserving the Courant bracket. In particular, for any sections $A, B \in C^\infty(T \oplus T^*)$ and $h \in C^\infty(M)$ we have $G([hA, B]) = [G(hA), G(B)]$, or expanding,

$$\begin{aligned}
G([hA, B]) &= G(h[A,B] - (B_T h)A - \langle A, B\rangle dh) \\
&= hG([A,B]) - (B_T h)G(A) - \langle A, B\rangle G(dh),
\end{aligned}$$

while, on the other hand,

$$\begin{aligned}
[G(hA), G(B)]) &= h[G(A), G(B)] - (G(B)_T h)G(A) - \langle G(A), G(B)\rangle dh \\
&= hG([A,B]) - (G(B)_T h)G(A) - \langle G(A), G(B)\rangle dh.
\end{aligned}$$

Setting these equal and using orthogonality, we obtain

$$(B_T h)G(A) + \langle A, B\rangle G(dh) = (G(B)_T h)G(A) + \langle A, B\rangle dh.$$

Choose $A = X, B = Y$, where $X, Y \in C^\infty(T)$ so that $\langle A, B\rangle = 0$. Then we have that $Y(h)G(X) = (G(Y)_T h)G(X)$ for all X, Y, h. This can only hold when $G(Y)_T = Y$ for all vector fields Y, implying that $G = \left(\begin{smallmatrix} 1 & * \\ * & * \end{smallmatrix}\right)$. With this in mind, the previous equation becomes

$$\langle A, B\rangle G(dh) = \langle A, B\rangle dh,$$

which implies that $G = \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$. Orthogonality then forces $G = \left(\begin{smallmatrix} 1 & 0 \\ B & 1 \end{smallmatrix}\right) = e^B$ where B is a skew 2-form, and to preserve the Courant bracket B must be closed. Hence we have that $F = f_c \circ e^B$, as required. $\square$

Remark 3.25. *All our results concerning symmetries of the Courant algebroid* $(T \oplus T^*,\ \langle,\rangle,\ [,],\ \pi)$ *hold for the complexified situation, but although any closed complex 2-form will act as a symmetry, we restrict the terminology "B-field" to only those 2-forms which are* real.

3.4 Dirac structures

It is clear from our investigation of the Courant bracket that it fails to be a Lie algebroid due to exact terms *involving the inner product* $\langle,\rangle$. For this reason, if we were to find a sub-bundle $L < (T \oplus T^*) \otimes \mathbb{C}$ which was involutive (closed under the Courant bracket) as well as being isotropic, then the anomalous terms would vanish, and $(L, [,], \pi)$ would define a Lie algebroid. Furthermore, we could take the image of such a sub-bundle under a B-field symmetry, obtaining another Lie algebroid. Even beyond this, there may be orthogonal transformations of $T \oplus T^*$ which, while they may not be symmetries of the entire Courant structure, may take L to a Lie algebroid nonetheless. In these ways, we will manufacture many Lie algebroids as sub-bundles of $(T \oplus T^*) \otimes \mathbb{C}$.

In fact, the Courant bracket itself places a tight constraint on which proper sub-bundles may be involutive a priori:

Proposition 3.26. *If $L < T \oplus T^*$ is involutive then L must either be an isotropic subbundle, or a bundle of type $\Delta \oplus T^*$ for Δ a nontrivial involutive sub-bundle of T. Similarly for the complex case $L < (T \oplus T^*) \otimes \mathbb{C}$.*

Proof. Suppose that $L < T \oplus T^*$ is involutive, but not an isotropic subbundle, i.e. there exists $X + \xi \in C^\infty(L)$ which is not null at some point $m \in M$, i.e. $\xi(X)_m \neq 0$. Then for any $f \in C^\infty(M)$,

$$[X + \xi, f(X + \xi)] = (Xf)(X + \xi) - \xi(X)df,$$

implying that $df_m \in L$ for all f, i.e. $T^*_m \leq L_m$. Since T^*_m is isotropic, this inclusion must be proper, i.e. $L_m = \Delta_m \oplus T^*_m$, where $\Delta_m = \ker \pi_{T^*}|_L : L_m \to T^*_m$. Hence the rank of L must exceed the maximal dimension of an isotropic sub-bundle, which for a real n-manifold is simply n. This implies that $T^*_m < L_m$ at every point m, and hence that $\Delta = \ker \pi_{T^*}|_L : L \to T^*$ is a smooth sub-bundle of T, which must itself be involutive. Hence $L = \Delta \oplus T^*$, as required. $\square$

In the maximal isotropic case, there is a remarkable equivalence between certain natural conditions on L and the involutive condition.

Proposition 3.27. *Let L be a maximal isotropic sub-bundle of $T \oplus T^*$ (or its complexification). Then the following are equivalent:*

- *L is involutive,*
- $\mathrm{Nij}\,\big|_L = 0$,

- $\mathrm{Jac}\big|_L = 0$.

Proof. If L is involutive then it is clear that $\mathrm{Nij}|_L = 0$ and since $\mathrm{Jac}(A,B,C) = d(\mathrm{Nij}(A,B,C))$, it is clear that this implies $\mathrm{Jac}|_L = 0$ as well. What remains to show is that $\mathrm{Jac}|_L = 0$ implies that L is involutive.

Suppose then that $\mathrm{Jac}|_L = 0$ but that L is not involutive, so that there exist $A, B, C \in C^\infty(L)$ such that $\langle [A,B], C\rangle \neq 0$. Then for all $f \in C^\infty(M)$,

$$\begin{aligned} 0 = \mathrm{Jac}(A,B,fC) &= d(\mathrm{Nij}(A,B,fC)) \\ &= \tfrac{1}{3}\langle [A,B], C\rangle df, \end{aligned}$$

which is a contradiction. Hence L must be involutive. $\square$

Definition 3.28 (Dirac structure). *A real, maximal isotropic sub-bundle $L < T \oplus T^*$ is called an almost* Dirac *structure. If L is involutive, then the almost Dirac structure is said to be integrable, or simply a* Dirac *structure. Similarly, a maximal isotropic and involutive complex sub-bundle $L < (T \oplus T^*) \otimes \mathbb{C}$ is called a* complex Dirac *structure. All the Lie algebroids we will be considering will be Dirac structures. Note that this definition still works if $T \oplus T^*$ is replaced with any real or complex Courant algebroid. Thus we may speak of Dirac structures in an arbitrary Courant algebroid.*

Remark 3.29. *Note that for an isotropic sub-bundle L, the restricted Nijenhuis operator* $\mathrm{Nij}|_L$ *is actually tensorial and is a section of $\wedge^3 L^*$. Hence the above theorem indicates that the integrability of a Dirac structure L is determined by the vanishing of a tensor field* $\mathrm{Nij}|_L$*. In particular, any almost Dirac structure on a 2-dimensional surface Σ is integrable for this reason.*

We will now provide several main examples of Dirac structures.

Example 3.30 (Symplectic geometry). The tangent bundle T is itself maximal isotropic and involutive, hence defines a Dirac structure. To this basic Dirac structure we can apply any closed (possibly complex) 2-form $\omega \in \Omega^2_{cl}(M)$ to obtain another involutive maximal isotropic. Indeed, the maximal isotropic subspace

$$e^\omega(T) = \{X + i_X\omega \ : \ X \in T\}$$

is involutive if and only if $d\omega = 0$ (see Proposition 3.23). Immediately we see that pre-symplectic geometry, i.e. the geometry defined by a closed 2-form, can be described by a Dirac structure.

Example 3.31 (Poisson geometry). Similarly, the cotangent bundle T^* is maximal isotropic and involutive (the Courant bracket vanishes on T^*), defining a Dirac structure. We may apply a bivector field $\beta \in C^\infty(\wedge^2 T)$ to this basic Dirac structure, obtaining

$$L_\beta = e^\beta(T^*) = \{i_\xi\beta + \xi \ : \ \xi \in T^*\}.$$

Since $\mathrm{Nij}|_{L_\beta}$ is tensorial, it suffices to check the involutivity of sections of the form $i_\xi\beta + \xi$ where $\xi = df$ for $f \in C^\infty(M)$. The bivector β determines a bracket on functions

$$\{f, g\} = \beta(df, dg),$$

and it is a straightforward calculation that

$$\mathrm{Nij}(i_{df}\beta + df, i_{dg}\beta + dg, i_{dh}\beta + dh) = \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\},$$

showing that $\mathrm{Nij}|_{L_\beta} = 0$ if and only if the bracket $\{,\}$ satisfies the Jacobi identity, that is, β is a Poisson structure, which is equivalent to $[\beta,\beta] = 0$. In this case we see that while β is not a symmetry of the Courant bracket, it does take T^* to an integrable Dirac structure.

Example 3.32 (Foliated geometry). Let $\Delta < T$ be a smooth distribution of constant rank. Then form the maximal isotropic subbundle

$$\Delta \oplus \mathrm{Ann}(\Delta) < T \oplus T^*.$$

This almost Dirac structure is Courant involutive if and only if Δ is an integrable distribution. By the theorem of Frobenius, this produces a foliation on the manifold. From this point of view, a foliation can also be described by a (real) Dirac structure.

Remark 3.33. *Note that any involutive sub-bundle of type $\Delta \oplus T^*$ contains a maximal isotropic involutive sub-bundle $\Delta \oplus \mathrm{Ann}(\Delta)$.*

Example 3.34 (Complex geometry). An almost complex structure $J \in \mathrm{End}(T)$ determines a complex distribution, given by the $-i$-eigenbundle $T_{0,1} < T \otimes \mathbb{C}$ of J. Forming the maximal isotropic space

$$L_J = T_{0,1} \oplus \mathrm{Ann}(T_{0,1}) = T_{0,1} \oplus T^*_{1,0},$$

we see that if L is Courant involutive, then since the vector component of the Courant bracket is simply the Lie bracket, this implies $T_{0,1}$ is Lie involutive, i.e. J is integrable. Conversely, if J is integrable, then, letting $X + \xi, Y + \eta \in C^\infty(T_{0,1} \oplus T^*_{1,0})$, we have

$$[X + \xi, Y + \eta] = [X, Y] + i_X \overline{\partial}\eta - i_Y \overline{\partial}\xi,$$

which is clearly a section of $T_{0,1} \oplus T^*_{1,0}$. Hence L is Courant involutive if and only if J is integrable. In this way, integrable complex structures can also be described by (complex) Dirac structures.

Although the preceding examples are only a few of the possible Dirac structures, they demonstrate that four completely separate classical geometrical structures, with very different-looking integrability conditions, are unified when considered as Dirac structures. As we shall see, there are many additional advantages to describing these geometries in this way. We have already encountered one: the fact that from this point of view, there is a natural action of $\Omega^2_{\mathrm{cl}}(M)$ on the geometries. Another is the fact that under suitable conditions, almost Dirac structures may be pushed forward and pulled back along smooth maps $f : M \to N$ between manifolds.

3.5 Lie bialgebroids and the Courant bracket

In the preceding section, we observed that the Courant bracket interpolates between the condition $d\omega = 0$ for a 2-form and the condition $[\beta, \beta] = 0$ for a bivector. This curious behaviour is related to the asymmetry of the Courant bracket, which we now address.

It was observed in [30] that Courant algebroids can be constructed out of Lie bialgebroids via a generalization of the Drinfel'd double construction. In particular, given a Lie bialgebroid (L, L^*), one can define an inner product and bracket on the sections of $L \oplus L^*$ making this bundle into a Courant algebroid, and both L, L^* into Dirac structures in this Courant algebroid. In the same paper, the converse is shown: given any two Dirac structures L, L' in a Courant algebroid which are transversal, then the inner product may be used to identify $L' = L^*$, and (L, L^*) is a Lie bialgebroid.

With this in mind, it is clear that the asymmetrical form of the Courant bracket on $T \oplus T^*$ is related to the asymmetry of the Lie bialgebroid (T, T^*). By choosing a different pair of transverse Dirac structures $L, L' < (T \oplus T^*) \otimes \mathbb{C}$, the Courant bracket may appear more even-handed with respect to L and L^*.

Theorem 3.35 ([30], Theorem 2.5). *Let (L, L^*) be a Lie bialgebroid. Then we have the following inner product $\langle , \rangle$ on the bundle $L \oplus L^*$:*

$$\langle A+\alpha, B+\beta\rangle = \tfrac{1}{2}(\alpha(B)+\beta(A)).$$

We define also the following skew-symmetric bracket operation on $C^\infty(L \oplus L^*)$:

$$\begin{aligned}[A+\alpha, B+\beta] = [A,B] + \mathcal{L}_\alpha B - \mathcal{L}_\beta A - \tfrac{1}{2}d_{L^*}(i_A\beta - i_B\alpha) \\ + [\alpha,\beta] + \mathcal{L}_A\beta - \mathcal{L}_B\alpha + \tfrac{1}{2}d_L(i_A\beta - i_B\alpha),\end{aligned}$$

where here the Lie derivative and interior product operators are as in Definition 3.8. If a, a_* are the anchors for L, L^*, we define the bundle map

$$\pi = a + a_* : L \oplus L^* \to T.$$

With these definitions, the structure $(L \oplus L^*, [,], \langle , \rangle, \pi)$ is a Courant algebroid. The operator $\mathcal{D}$: $C^\infty(M) \to C^\infty(L \oplus L^*)$ clearly becomes

$$\mathcal{D} = d_L + d_{L^*}.$$

Theorem 3.36 ([30], Theorem 2.6). *Let $(E, [,], \langle , \rangle, \pi)$ be a Courant algebroid, and let $L, L' < E$ be Dirac sub-bundles transverse to each other, i.e. $E = L \oplus L'$. Then $L' = L^*$ using the inner product, and (L, L') is a Lie bialgebroid. Applying the construction in the previous theorem to (L, L'), we recover the original Courant algebroid structure on E.*

Given a splitting of a Courant algebroid $E = L \oplus L^*$ into a Lie bialgebroid, we could attempt to create new Dirac structures as we did for $T \oplus T^*$ in examples 3.30 and 3.31, as graphs of elements in $\wedge^2 L$ or $\wedge^2 L^*$. Without loss of generality, let $\varepsilon \in \wedge^2 L^*$ and consider the maximal isotropic sub-bundle

$$L_\varepsilon = \{A + i_A\varepsilon \ : \ A \in L\}.$$

The condition for the involutivity of the almost Dirac structure L_ε is the final main result of [30]:

Theorem 3.37 ([30], Theorem 6.1). *The almost Dirac structure L_ε, for $\varepsilon \in \wedge^2 L^*$, is integrable if and only if ε satisfies the generalized Maurer-Cartan equation*

$$d_L\varepsilon + \tfrac{1}{2}[\varepsilon, \varepsilon] = 0. \tag{3.7}$$

Here $d_L : C^\infty(\wedge^k L^*) \to C^\infty(\wedge^{k+1} L^*)$ and $[,]$ is the Lie algebroid bracket on L^*.

From this result we can finally understand how the Courant bracket allows us to interpolate between $d\omega = 0$ and $[\beta, \beta] = 0$: due to the asymmetric bialgebroid structure on (T, T^*), we see that $[,]$ vanishes on T^* while $d_{T^*} = 0$ on T. Hence the Maurer-Cartan equation on a 2-form $B \in \wedge^2 T^*$ is simply $dB = 0$, whereas on a bivector $\beta \in \wedge^2 T$ it is $\frac{1}{2}[\beta, \beta] = 0$.

The results of this section will be particularly important when we study the deformation theory of generalized complex structures: finding solutions to the Maurer-Cartan equation will correspond to finding integrable deformations of the generalized complex structure. In this way, we solve an open problem stated in [30], namely, to find an interpretation of equation (3.7) in terms of the deformation theory of a geometric structure. Similar problems arise in the work of Barannikov and Kontsevich [3].

3.6 Pure spinors and integrability

Since each maximal isotropic sub-bundle $L < (T \oplus T^*) \otimes \mathbb{C}$ corresponds to a pure line sub-bundle of the spin bundle $U < \wedge^\bullet T^* \otimes \mathbb{C}$, it stands to reason that the involutivity of L corresponds to some integrability condition on U. We now determine this condition.

Recall that $L < (T \oplus T^*) \otimes \mathbb{C}$ is the annihilator of U under Clifford multiplication:

$$L = \{X + \xi \in (T \oplus T^*) \otimes \mathbb{C} \ : \ (X + \xi) \cdot U = 0\}.$$

The Clifford algebra $CL(V \oplus V^*)$, $\dim V = m$ is a $\mathbb{Z}/2\mathbb{Z}$-graded, $\mathbb{Z}$-filtered algebra with the following filtered direct summands:

$$\begin{aligned} \mathbb{R} = CL^0 < CL^2 < \cdots < CL^{2m} = CL^+(V \oplus V^*) \\ V \oplus V^* = CL^1 < CL^3 < \cdots < CL^{2m-1} = CL^-(V \oplus V^*), \end{aligned}$$

where CL^{2k} is spanned by products of even numbers of not more than $2k$ elements of $V \oplus V^*$, and CL^{2k-1} is spanned by products of odd numbers of not more than $2k-1$ elements. The Clifford multiplication respects this graded filtration structure.

By Clifford multiplication by U, we obtain filtrations of the even and odd exterior forms (here $2n$ is the real dimension of the manifold):

$$U = U_0 < U_2 < \cdots < U_{2n} = \wedge^{ev/odd} T^* \otimes \mathbb{C}, \tag{3.8}$$

$$L^* \cdot U = U_1 < U_3 < \cdots < U_{2n-1} = \wedge^{odd/ev} T^* \otimes \mathbb{C}, \tag{3.9}$$

where ev/odd is chosen according to the parity of U itself, and U_k is defined as $CL^k \cdot U$. Note that, using the inner product, we have the canonical isomorphism $L^* = ((T \oplus T^*) \otimes \mathbb{C})/L$, and so U_1 is isomorphic to $L^* \otimes U_0$.

Theorem 3.38. *The almost Dirac structure L is Courant involutive if and only if the exterior derivative d satisfies*

$$d(C^\infty(U_0)) \subset C^\infty(U_1),$$

i.e. L is involutive if and only if, for any local trivialization ρ of U, there exists a section $X + \xi \in C^\infty((T \oplus T^*) \otimes \mathbb{C})$ such that

$$d\rho = i_X \rho + \xi \wedge \rho. \tag{3.10}$$

Note that this condition is invariant under rescaling of ρ by a smooth function.

Proof. Let L be the almost Dirac structure, and let ρ be a trivialization of U over some open set. Then we prove in the next paragraph that

$$[A, B] \cdot \rho = A \cdot B \cdot d\rho, \tag{3.11}$$

for any sections $A, B \in C^\infty(L)$. Hence L is involutive if and only if $A \cdot B \cdot d\rho = 0 \ \forall A, B \in C^\infty(L)$, which is true if and only if $d\rho$ is in $C^\infty(U_1)$, as elements of U_k are precisely those which are annihilated by $k+1$ elements in L.

We now prove the identity. Since L is isotropic, the Courant and Dorfman brackets agree when restricted to L; we use the latter for simplicity. Let $A = X + \xi$ and $B = Y + \eta$, so that $i_X \rho = -\xi \wedge \rho$

and $i_Y\rho = -\eta\wedge\rho$. Then

$$\begin{aligned} i_{[X,Y]}\rho &= [\mathcal{L}_X, i_Y]\rho \\ &= \mathcal{L}_X(-\eta\wedge\rho) - i_Y(d(-\xi\wedge\rho) + i_X d\rho) \\ &= -\mathcal{L}_X\eta\wedge\rho - \eta\wedge(i_X d\rho + d(-\xi\wedge\rho)) - i_Y(-d\xi\wedge\rho + \xi\wedge d\rho + i_X d\rho) \\ &= (-\mathcal{L}_X\eta + i_Y d\xi)\wedge\rho - (i_Y + \eta\wedge)(i_X + \xi\wedge)d\rho \\ &= (-\mathcal{L}_X\eta + i_Y d\xi)\wedge\rho + A\cdot B\cdot d\rho, \end{aligned}$$

showing that

$$[A,B]\cdot\rho = (A\circ B)\cdot\rho = A\cdot B\cdot d\rho,$$

as required. ☐

Remark 3.39. *For many examples of Dirac structures, it is possible to choose local trivializations ρ of the pure spinor line which are closed: $d\rho = 0$, which obviously satisfies the above integrability condition. However this is not always the case; the more general integrability condition stated above is the appropriate one.*

It should be mentioned that if L is integrable, then the exterior derivative, which takes U to $U_1 = L^*\otimes U$, is an example of a *Lie algebroid connection* on the line bundle U with respect to the Lie algebroid L. The notion of Lie algebroid connection is developed in [17].

Definition 3.40 (Lie algebroid connection). *Let L be a Lie algebroid, E a vector bundle and $D : C^\infty(E) \longrightarrow C^\infty(L^*\otimes E)$ a linear operator such that*

$$D(fs) = (d_L f)\otimes s + fDs$$

for any $f\in C^\infty(M)$ and $s\in C^\infty(E)$. Then D is called a generalized connection, or Lie algebroid connection.

If D is such a connection, it can be extended in the usual way to a sequence $D_L : C^\infty(\wedge^k L^*\otimes E) \longrightarrow C^\infty(\wedge^{k+1}L^*\otimes E)$ via the rule:

$$D_L(\mu\otimes s) = d_L\mu\otimes s + (-1)^{|\mu|}\mu\wedge Ds,$$

and $D_L^2 \in C^\infty(\wedge^2 L^*\otimes \mathrm{End}(E))$ is then the curvature of the connection.

In this way we obtain a natural connection structure on the spinor line U of any Dirac structure. This will be particularly useful when we consider generalized complex structures in the next section.

Before we proceed to our final remarks concerning the Courant bracket, we should give a simple application of the above integrability condition to products of Dirac structures.

Proposition 3.41. *Let (M_1, L_1), (M_2, L_2) be two manifolds equipped with Dirac structures. Then $\pi_1^* L_1 \oplus \pi_2^* L_2$, where π_i are the canonical projections, is a Dirac structure on $M_1\times M_2$.*

Proof. Choose trivializations ρ_1, ρ_2 for the spinor lines U_1, U_2 in open sets around points $m_1\in M_1, m_2\in M_2$ respectively. Then form $\rho = \pi_1^*\rho_1\wedge\pi_2^*\rho_2$, defined in a neighbourhood of (m_1, m_2). The annihilator of this spinor is the maximal isotropic space $\pi_1^* L_1\oplus\pi_2^* L_2$, where π^* is the pull-back of Dirac structures defined in Equation 2.21. Furthermore, if we have $d\rho_i = \alpha_i\cdot\rho_i$ for $\alpha_1,\alpha_2\in C^\infty(T\oplus T^*)$, then we see that

$$d\rho = (\alpha_1 \pm \alpha_2)\cdot\rho,$$

showing that ρ satisfies the integrability condition (3.10), implying the integrability of the Dirac structure. ☐

3.7 The twisted Courant bracket

As was noticed by Ševera, and developed by him and Weinstein in [48], the Courant bracket on $T \oplus T^*$ can be 'twisted' by a real[3], closed 3-form H, in the following way: given a 3-form H, define another bracket $[,]_H$ on $T \oplus T^*$, by

$$[X+\xi, Y+\eta]_H = [X+\xi, Y+\eta] + i_Y i_X H.$$

Then, defining Nij_H and Jac_H using the usual formulae but replacing $[,]$ with $[,]_H$, one calculates that if $A = X+\xi$,$B = Y+\eta$, and $C = Z+\zeta$, then

$$\mathrm{Nij}_H(A,B,C) = \mathrm{Nij}(A,B,C) + H(X,Y,Z),$$

and that

$$\mathrm{Jac}_H(A,B,C) = d(\mathrm{Nij}_H(A,B,C)) + i_Z i_Y i_X dH.$$

We conclude that $[,]_H$ defines a Courant algebroid structure on $T\oplus T^*$ (using the same inner product and anchor) if and only if the extraneous term vanishes, i.e. $dH = 0$.

Therefore, given any closed 3-form H, one can study maximal isotropic sub-bundles of $T \oplus T^*$ which are involutive with respect to $[,]_H$: these are called *twisted Dirac structures*. Reexamining Proposition 3.23, we obtain the following relationship between 2-forms and the twisted Courant brackets.

Proposition 3.42. *If b is a 2-form then we have*

$$[e^b(W), e^b(Z)]_H = e^b[W,Z]_{H+db} \quad \forall\, W,Z \in C^\infty(T\oplus T^*),$$

showing that e^b is a symmetry of $[,]_H$ if and only if $db = 0$. These are the B-field transforms.

Proof. See Proposition 3.23. □

Note that the tangent bundle T is not involutive with respect to the bracket $[,]_H$ unless $H = 0$. If H is exact, i.e. there exists $b \in \Omega^2(M)$ such that $db = H$, then clearly the bundle

$$e^{-b}(T) = \{X - i_X b \;:\; X \in T\}$$

is closed with respect to $[,]_H$. In general, a sub-bundle L is involutive for $[,]_H$ if and only if $e^{-b}L$ is involutive for $[,]_{H+db}$. Using this observation, one reduces the study of H-Dirac structures, for H exact, to the study of ordinary Dirac structures. For H such that $[H] \in H^3(M,\mathbb{R})$ is nonzero, H-Dirac strucures represent geometries genuinely separate from ordinary Dirac structures. The main object of study in [48] is H-twisted Poisson structure.

Example 3.43 (Twisted Poisson geometry). Let $H \in \Omega^3_{cl}(M)$, and let $\beta \in C^\infty(\wedge^2 T)$ be such that

$$[\beta,\beta] = \beta^*(H),$$

where on the right hand side we are pulling back by $\beta : T^* \to T$. Then β is called a H-twisted Poisson structure, i.e. the graph $L_\beta = \{i_\xi\beta + \xi \;:\; \xi \in T^*\}$ is involutive with respect to the H-twisted Courant bracket. See [48] for details.

[3]None of our results depend on H being real, however since $[,]$ is a real quantity, it is reasonable to restrict ourselves to real twistings.

Modifying the integrability condition of a maximal isotropic sub-bundle by introducing a twist must also modify the integrability condition on the level of the spinor line $U < \wedge^{\bullet} T^* \otimes \mathbb{C}$. We now determine what this is.

Proposition 3.44. *The almost Dirac structure L is involutive with respect to the H-twisted Courant bracket if and only if the operator $d_H = d + H \wedge \cdot$ satisfies*

$$d_H(C^\infty(U_0)) \subset C^\infty(U_1),$$

i.e. L is H-involutive if and only if, for any local trivialization ρ of U, there exists a section $X + \xi \in C^\infty((T \oplus T^*) \otimes \mathbb{C})$ such that

$$d_H \rho = i_X \rho + \xi \wedge \rho. \tag{3.12}$$

Proof. The key result of the proof of Theorem 3.38 was that

$$[A, B] \cdot \rho = A \cdot B \cdot d\rho.$$

From this we see immediately that

$$[A, B]_H \cdot \rho = A \cdot B \cdot d\rho + i_Y i_X H \wedge \rho,$$

where $A = X + \xi$ and $B = Y + \eta$. Using the fact that $i_X \rho + \xi \wedge \rho = i_Y \rho + \eta \wedge \rho = 0$, we obtain

$$[A, B]_H \cdot \rho = A \cdot B \cdot (d\rho + H \wedge \rho),$$

which is what we require: using the same reasoning as the untwisted case, ρ determines a H-Dirac structure if and only if $d_H \rho = i_X \rho + \xi \wedge \rho$ for some section $X + \xi \in C^\infty(T \oplus T^*)$. □

3.8 Relation to gerbes

At this point we would like to give some indication of how gerbes are related to Courant algebroids; in particular how the H-twisted Courant bracket can be viewed as a "twist by a gerbe" when $[H]$ is an integral cohomology class. This section is independent of the rest of the thesis and is mostly a collection of remarks intended to demonstrate that the algebraic structures introduced in this chapter have geometric underpinnings. The gerbe interpretation is particularly relevant for physicists working with sigma models, for whom H is known as the Neveu-Schwarz 3-form flux: in current theories $[H]$ is required to be integral for the Lagrangian to be well-defined. Among other things, the gerbe interpretation provides a reason for the fact that B-field transformations with $[B] \in H^2(M, \mathbb{Z})$ should be interpreted as gauge transformations, e.g. for questions of moduli they should be quotiented out.

The Courant bracket on $T \oplus T^*$ is just "level 1" of a hierarchy of brackets on the bundles $T \oplus \wedge^p T^*$, $p = 0, 1, \ldots$, defined by the same formula

$$[X + \sigma, Y + \tau] = [X, Y] + \mathcal{L}_X \tau - \mathcal{L}_Y \sigma - \tfrac{1}{2} d(i_X \tau - i_Y \sigma),$$

where now $\sigma, \tau \in C^\infty(\wedge^p T^*)$. In his thesis [41], Roytenberg showed that for $p = 1$, the Courant bracket defines an L_∞ algebra of level 2. The exact nature of the full algebraic structure for all p remains a work in progress. Using identical arguments to those used for $T \oplus T^*$, one sees that symmetries of the bracket are given by closed $p + 1$ forms, and that the bracket may be twisted by a closed $p + 2$ form.

Level 0: It will be fruitful to begin our investigation with the case $p = 0$, i.e. the bracket on $T \oplus 1$ given by

$$[X + f, Y + g] = [X, Y] + Xg - Yf,$$

where $X, Y \in C^\infty(T)$ and $f, g \in C^\infty(M)$. Unlike the bracket on $T \oplus T^*$, this one actually satisfies the Jacobi identity, and makes $T \oplus 1$ into a Lie algebroid, with anchor the obvious projection $\pi : T \oplus 1 \longrightarrow T$.

This bracket operation should be familiar: it is the Lie bracket structure on the Atiyah sequence associated to the trivial S^1 principal bundle $P = S^1 \times M$. In particular, if $X + f\partial_t, Y + g\partial_t$ are S^1-invariant vector fields on P (here t is the coordinate on S^1 and f, g have no dependence on t), then their Lie bracket is

$$[X + f\partial_t, Y + g\partial_t] = [X, Y] + (Xg - Yf)\partial_t.$$

In this way we achieve a geometrical interpretation of the untwisted Courant bracket for $p = 0$. The bracket, however, may be twisted by a closed 2-form F:

$$[X + f, Y + g]_F = [X + f, Y + g] + i_Y i_X F,$$

and we would like to interpret this twisting in some geometrical way.

The interpretation using $S^1 \times M$ can be generalized since there is an Atiyah sequence associated to any S^1 principal bundle P:

$$0 \longrightarrow 1 \xrightarrow{j} TP/S^1 \xrightarrow{\pi_*} T \longrightarrow 0\ .$$

By choosing a splitting of this extension, we choose an isomorphism $TP/S^1 \cong T \oplus 1$, and we may then transport the Lie algebroid structure on TP/S^1 to $T \oplus 1$. As is well known, choosing a splitting ∇ for this sequence is equivalent to choosing a connection on P:

$$\begin{tikzcd} 0 \arrow[r] & 1 \arrow[r, shift left, "j"] & TP/S^1 \arrow[l, shift left, "s"] \arrow[r, shift left, "\pi_*"] & T \arrow[l, shift left, "\nabla"] \arrow[r] & 0\ , \end{tikzcd}$$

and the curvature of the connection measures the failure of ∇ to be a Lie algebra morphism:

$$F^\nabla(X, Y) = s(\nabla(X), \nabla(Y)),$$

where $X, Y \in C^\infty(T)$. Now let us calculate the Lie bracket on $T \oplus 1$ induced by this choice of splitting:

$$\begin{aligned} [X + f, Y + g] &= \pi_*[\nabla(X), \nabla(Y)] + s([\nabla(X), j(g)] - [\nabla(Y), j(f)]) + s([\nabla(X), \nabla(Y)]) \\ &= [X, Y] + Xg - Yf + F^\nabla(X, Y), \end{aligned}$$

where we have used the Lie algebroid axioms (satisfied by TP/S^1) to establish the final equality. Hence we see that from a line bundle with connection we obtain a twisted Lie algebroid bracket on $T \oplus 1$; in this case, the twisting 2-form satisfies $[F^\nabla/2\pi] \in H^2(M, \mathbb{Z})$. Note that choosing a different connection $\nabla' = \nabla + A$, for $A \in C^\infty(T^*)$, modifies the curvature, and therefore the twisting 2-form, by an exact form:

$$F^{\nabla'} = F^\nabla + dA$$

So, we see that while we do obtain a geometrical interpretation of certain twisted Lie algebroid brackets on $T \oplus 1$, it is only those for which the twisting 2-form is *integral* which can be interpreted in terms of line bundles with connection.

Proposition 3.45. *The twisted Courant bracket $[,]_F$ on $T \oplus 1$ can be obtained from a principal S^1 bundle with connection when $[F/2\pi] \in H^2(M, \mathbb{Z})$.*

From this point of view, the symmetries of the bracket also become clear. Changing the splitting to $\nabla' = \nabla + A$ corresponds to mapping $X + f \mapsto X + f - i_X A$, and this preserves the bracket as long as the curvature is unchanged, i.e. A is a closed 1-form. This could be thought of as taking the tensor product with the trivial bundle equipped with a flat connection $d + A$, $dA = 0$.

If $[F/2\pi]$ is integral, then some of these symmetries actually come from gauge transformations of the underlying S^1 bundle, i.e. those for which $A = f^{-1}df$, for f an S^1-valued function. In other words, those symmetries for which $[A] \in H^1(M, \mathbb{Z})$ are gauge transformations modulo constant gauge transformations.

Proposition 3.46. *Let $[,]_F$ be a Courant bracket on $T \oplus 1$ derived from a principal S^1 bundle with connection. Its symmetries correspond to tensoring with the trivial S^1 bundle with a flat connection $d + A$, of which those with $[A] \in H^1(M, \mathbb{Z})$ derive from gauge transformations.*

In the case that $[F/2\pi]$ is not integral, it is of course still possible to interpret the twisted bracket as deriving from an exact Lie algebroid

$$0 \longrightarrow 1 \xrightarrow{j} E \xrightarrow{a} T \longrightarrow 0$$

equipped with a splitting ∇. Then the symmetries of the bracket are simply automorphisms of the exact sequence preserving the Lie bracket structure.

However, it is possible to go further, and interpret this non-integral case not as a line bundle, but as the following generalization of a line bundle: a trivialization, or section, of an S^1 gerbe. In particular we need a trivialization (with connection) of an S^1 gerbe with flat connection. This is directly analogous to the fact that we can interpret a closed 1-form A as given by a trivialization (nonvanishing section) s of a trivial line bundle with connection (L, ∇) in the manner

$$\nabla(s) = A \otimes s.$$

We will follow an argument similar to that used by Hitchin in [18], and we use the Čech description of gerbes he espouses, which was developed in the thesis of Chatterjee [12].

In the same way that an S^1 principal bundle can be specified by a cocycle $g_{\alpha\beta} : U_\alpha \cap U_\beta \to S^1$ in $\check{C}^1(M, C^\infty(S^1))$, an S^1 gerbe can be specified by a cocycle $g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to S^1$ in $\check{C}^2(M, C^\infty(S^1))$. A connection on a gerbe is specified by 1-forms $A_{\alpha\beta}$ and 2-forms B_α satisfying the following deRham-Čech conditions:

$$iA_{\alpha\beta} + iA_{\beta\gamma} + iA_{\gamma\alpha} = g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma},$$
$$B_\beta - B_\alpha = dA_{\alpha\beta}$$

and we see immediately that there is a globally-defined 3-form H (with $[H/2\pi]$ integral), known as the *curvature* of the gerbe connection and defined by

$$H\big|_{U_\alpha} = dB_\alpha.$$

A trivialization of a gerbe is given by $h_{\alpha\beta} \in \check{C}^1(M, C^\infty(S^1))$ satisfying

$$h_{\alpha\beta} h_{\beta\gamma} h_{\gamma\alpha} = g_{\alpha\beta\gamma},$$

and the existence of such a section implies that the gerbe is trivial, i.e. $[g_{\alpha\beta\gamma}] = 0$.

If the connection on the gerbe is flat, then since $dB_\alpha = 0$ we can write (for a suitable cover refinement) $B_\alpha = da_\alpha$, and then since $dA_{\alpha\beta} = B_\beta - B_\alpha = d(a_\beta - a_\alpha)$, we obtain functions $f_{\alpha\beta}$ such that $A_{\alpha\beta} - a_\beta + a_\alpha = df_{\alpha\beta}$. This then implies that

$$idf_{\alpha\beta} + idf_{\beta\gamma} + idf_{\gamma\alpha} = g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma}.$$

A trivialization $h_{\alpha\beta}$ of such a flat gerbe produces immediately a cocycle

$$\mathfrak{a}_{\alpha\beta} = -i\left(\frac{dh_{\alpha\beta} + idf_{\alpha\beta}h_{\alpha\beta}}{h_{\alpha\beta}}\right),$$

(analogous to A in $\nabla(s) = iA \otimes s$) which can be used to manufacture an extension

$$0 \longrightarrow 1 \longrightarrow E \longrightarrow T \longrightarrow 0\ , \tag{3.13}$$

by gluing $(T \oplus 1)|_{U_\alpha}$ to $(T \oplus 1)|_{U_\beta}$ using the automorphism

$$\begin{pmatrix} 1 & 0 \\ \mathfrak{a}_{\alpha\beta} & 1 \end{pmatrix},$$

where $\mathfrak{a}_{\alpha\beta}$ acts on T by contraction $X \mapsto i_X \mathfrak{a}_{\alpha\beta}$. This extension inherits a Lie bracket since the one forms $\mathfrak{a}_{\alpha\beta}$ are closed.

The trivialization $h_{\alpha\beta}$ is *with connection* relative to the gerbe connection if we have 1-forms A_α such that

$$iA_\beta - iA_\alpha = h^{-1}_{\alpha\beta} dh_{\alpha\beta} + idf_{\alpha\beta}.$$

This is precisely the data determining a splitting ∇ of the extension (3.13), and dA_α determines the global closed 2-form F. Hence we see that the twisted Courant bracket on $T \oplus 1$ may be viewed as a trivialization with connection of a flat gerbe. Chatterjee studies this situation (but calls a trivialization an "object") and concludes ([12], Prop. 3.2.5) that any two such trivializations with connection are equivalent if and only if they differ by a trivial line bundle with flat connection, corresponding to the fact that the symmetries of $[,]_F$ are given by closed 1-forms.

Level 1: In the same way that $[,]_F$ on $T \oplus 1$ can be understood in terms of the Atiyah sequence of a line bundle when $[F/2\pi]$ is integral, the twisted Courant bracket $[,]_H$ on $T \oplus T^*$ can be understood in terms of gerbes when $[H/2\pi]$ is integral. In the non-integral case, one would need to pass to 2-gerbes, which we will not address.

Any twisted Courant bracket on $T \oplus T^*$ can be obtained by choosing a splitting of an exact Courant algebroid, of the form

$$0 \longrightarrow T^* \xrightarrow{j} E \xrightarrow{\pi} T \longrightarrow 0\ .$$

In [19], Hitchin demonstrates how, in the integral case, such a split algebroid can be naturally obtained from a gerbe with connection. The first condition on the connection data, namely

$$A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma},$$

implies that $dA_{\alpha\beta}$ is a cocycle, which can be used to produce an extension by gluing $(T \oplus T^*)|_{U_\alpha}$ to $(T \oplus T^*)|_{U_\beta}$ using the automorphism

$$\begin{pmatrix} 1 & 0 \\ dA_{\alpha\beta} & 1 \end{pmatrix},$$

where $dA_{\alpha\beta}$ acts on T by contraction $X \mapsto i_X dA_{\alpha\beta}$. This extension inherits a bracket structure since the 2-forms $dA_{\alpha\beta}$ are closed. The second condition on the connection data, namely

$$B_\beta - B_\alpha = dA_{\alpha\beta},$$

defines a splitting (∇, s) of the extension as in the Lie algebroid case, with curvature

$$s([\nabla(X), \nabla(Y)]) = i_Y i_X H$$

given by H, the gerbe curvature. In this way, we see that a gerbe connection gives rise to a generalized Atiyah sequence together with a splitting.

Proposition 3.47. *The twisted Courant bracket $[,]_H$ on $T \oplus T^*$ can be obtained from a S^1 gerbe with connection when $[H/2\pi] \in H^3(M, \mathbb{Z})$.*

The advantage of this point of view, as in the “level 0” case, is that we gain an understanding of the symmetries of the Courant bracket. In the “level 0” case, when $[F/2\pi]$ is integral, trivializations of a flat line bundle act as symmetries of $[,]_F$, and the difference of two such trivializations is a gauge transformation. In the “level 1” case, again when $[H/2\pi]$ is integral, trivializations (with connection) of a flat gerbe act as symmetries of $[,]_H$ (B-field transforms), and the difference of two such trivializations, a line bundle with connection, acts as a gauge transformation (integral B-fields).

In this way, we see that trivializations (with connection) of a flat gerbe not only give rise to twisted Lie algebroid structures on $T \oplus 1$ but also to symmetries of $T \oplus T^*$.

Chapter 4

Generalized complex structures

In this chapter we introduce our main object of study: generalized complex structures. The idea originates with Nigel Hitchin [19], and is an extension of the use of Dirac structures and Lie algebroids to incorporate complex geometry, and, as we shall see, many other new forms of geometry. In this way, symplectic and complex geometry can be viewed as extremal cases of a more general structure.

In view of recent work in mirror symmetry, it is clear that there are deep connections between the holomorphic and symplectic categories. This makes the explicit unification of both structures all the more intriguing; indeed generalized complex structures provide a natural framework in which to discuss mirror symmetry. In particular, the concepts of the *B-field*, the *extended deformation space* of Kontsevich [25], as well as the newly-discovered *coisotropic D-branes* of Kapustin [22], not to mention the target space bi-Hermitian geometry discovered by Gates, Hull and Roček [42], all have natural interpretations in terms of generalized complex geometry.

In the first section we will introduce the algebraic nature of generalized complex structures. In section 4.2 we will transport the algebraic structure to a manifold and investigate the topological implications of having a generalized complex structure. In section 4.3, we impose the integrability condition, and describe how it unifies complex and symplectic geometry. In section 4.4, we describe how a generalized complex structure affects the differential forms on a manifold. In section 4.5, we provide some examples of manifolds which are generalized complex, and yet have no known complex or symplectic structures. In section 4.6, we describe a family of generalized complex structures interpolating between a complex and a symplectic structure, demonstrating that in some cases, the moduli space of complex and symplectic structures may be connected through generalized complex structures. In section 4.7 we prove a Darboux-type theorem describing the local form of a generalized complex structure in a regular neighbourhood. In the final section of the chapter, we discuss the jumping phenomenon, where generalized complex structures may change algebraic type along loci in the manifold.

4.1 Linear generalized complex structures

We begin by defining the notion of generalized complex structure on a real vector space. We will use the well known structures of complex and symplectic geometry to guide us.

Let V be a real, finite dimensional vector space. A *complex* structure on V is an endomorphism $J : V \longrightarrow V$ satisfying $J^2 = -1$. By comparison, a *symplectic* structure on V is a nondegenerate skew form $\omega \in \wedge^2 V$. We may, however, view ω as a map $V \longrightarrow V^*$ via interior product:

$$\omega : v \mapsto i_v \omega, \quad v \in V.$$

With this in mind, a symplectic structure on V can be defined as an isomorphism $\omega : V \longrightarrow V^*$ satisfying $\omega^* = -\omega$. Note that we are using an asterisk to denote the linear dual of a space or mapping, so that ω^* maps $(V^*)^* = V$ to V^*.

In attempting to include both these structures in a higher algebraic structure, we will consider endomorphisms of the direct sum $V \oplus V^*$. Recall from section 2 that $V \oplus V^*$ may be identified with its dual space using the natural inner product $\langle , \rangle$.

Definition 4.1. *A generalized complex structure on V is an endomorphism $\mathcal{J}$ of the direct sum $V \oplus V^*$ which satisfies two conditions. First, it is complex, i.e. $\mathcal{J}^2 = -1$; and second, it is symplectic, i.e. $\mathcal{J}^* = -\mathcal{J}$.*

Proposition 4.2. *Equivalently, we could define a generalized complex structure on V as a complex structure on $V \oplus V^*$ which is orthogonal in the natural inner product.*

Proof. If $\mathcal{J}^2 = -1$ and $\mathcal{J}^* = -\mathcal{J}$, then $\mathcal{J}^*\mathcal{J} = 1$, i.e. $\mathcal{J}$ is orthogonal. Conversely, if $\mathcal{J}^2 = -1$ and $\mathcal{J}^*\mathcal{J} = 1$, then $\mathcal{J}^* = -\mathcal{J}$. □

The usual complex and symplectic structures are embedded in the notion of a generalized complex structure in the following way. Consider the endomorphism

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix},$$

where J is a usual complex structure on V, and the matrix is written with respect to the direct sum $V \oplus V^*$. Then we see that $\mathcal{J}_J^2 = -1$ and $\mathcal{J}_J^* = -\mathcal{J}_J$, i.e. $\mathcal{J}_J$ is a generalized complex structure. Similarly, consider the endomorphism

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix},$$

where ω is a usual symplectic structure. Again, we observe that $\mathcal{J}_\omega$ is a generalized complex structure. Therefore we see that diagonal and anti-diagonal generalized complex structures correspond to complex and symplectic structures, respectively. Now we wish to understand any intermediate structures which mix V and V^*; indeed we wish to understand the space of all generalized complex structures for V.

The important observation is that specifying $\mathcal{J}$ is equivalent to specifying a maximal isotropic subspace of $(V \oplus V^*) \otimes \mathbb{C}$:

Proposition 4.3. *A generalized complex structure on V is equivalent to the specification of a maximal isotropic complex subspace $L < (V \oplus V^*) \otimes \mathbb{C}$ of real index zero, i.e. such that $L \cap \overline{L} = \{0\}$.*

Proof. If $\mathcal{J}$ is a generalized complex structure, then let L be its $+i$-eigenspace in $(V \oplus V^*) \otimes \mathbb{C}$. Then if $x, y \in L$, $\langle x, y \rangle = \langle \mathcal{J}x, \mathcal{J}y \rangle$ by orthogonality and $\langle \mathcal{J}x, \mathcal{J}y \rangle = \langle ix, iy \rangle = -\langle x, y \rangle$, implying that $\langle x, y \rangle = 0$. Therefore L is isotropic and half-dimensional, i.e. maximally isotropic. Also, $\overline{L}$ is the $-i$-eigenspace of $\mathcal{J}$ and thus $L \cap \overline{L} = \{0\}$. Conversely, given such an L, simply define $\mathcal{J}$ to be multiplication by i on L and by $-i$ on $\overline{L}$. This real transformation then defines a generalized complex structure on $V \oplus V^*$. □

This means that studying generalized complex structures is equivalent to studying complex maximal isotropics with real index zero, which is the most generic possible real index. The real index zero condition may also be expressed in terms of the data (E, ε) (see section 2.2 for notation):

Proposition 4.4. *The maximal isotropic $L(E,\varepsilon)$ has real index zero if and only if $E+\overline{E}=V\otimes\mathbb{C}$ and ε is such that the real skew 2-form $\omega_\Delta=\mathrm{Im}(\varepsilon\big|_{E\cap\overline{E}})$ is nondegenerate on $E\cap\overline{E}=\Delta\otimes\mathbb{C}$.*

Proof. Let L have real index zero. Then since $(V\oplus V^*)\otimes\mathbb{C}=L\oplus\overline{L}$, we see that $E+\overline{E}=V\otimes\mathbb{C}$. Also, if $0\neq X\in\Delta$ such that $(\varepsilon-\overline{\varepsilon})(X)=0$ then there exists $\xi\in V\otimes\mathbb{C}$ such that $X+\xi\in L\cap\overline{L}$, which is a contradiction. Hence ω_Δ is nondegenerate.

Conversely, assume $E+\overline{E}=V\otimes\mathbb{C}$ and that ω_Δ is nondegenerate. Suppose $0\neq X+\xi\in L\cap\overline{L}$; then $\xi|_E=\varepsilon(X)$ and $\xi|_{\overline{E}}=\overline{\varepsilon}(X)$, so that $(\varepsilon-\overline{\varepsilon})(X)=0$, which implies $X=0$. But then $\xi|_E=\xi|_{\overline{E}}=0$, hence $\xi=0$ as well, a contradiction. Hence $L\cap\overline{L}=\{0\}$. This completes the proof. $\square$

The result (2.19) about the parity of the real index indicates that generalized complex structures may only exist on even-dimensional spaces:

Proposition 4.5. *The vector space V admits a generalized complex structure if and only if it is even dimensional.*

Proof. An even-dimensional real vector space always admits complex and symplectic structures, and these are generalized complex structures as we have just shown. For the converse, equation (2.19) implies that the real index must be congruent to $\dim(V)$ modulo 2, showing that generalized complex structures exist only on vector spaces of even dimension. We also include the following alternative argument:

Let $\mathcal{J}$ be a generalized complex structure on V. Since the natural inner product on $V\oplus V^*$ is indefinite, we can find a null vector $x\in V\oplus V^*$, i.e. $\langle x,x\rangle=0$. Since $\mathcal{J}$ is an orthogonal complex structure, $\mathcal{J}x$ is also null and is orthogonal to x. Therefore $x,\mathcal{J}x$ span an isotropic subspace $N<V\oplus V^*$. We can iteratively enlarge the isotropic subspace N by adding a pair of vectors, consisting of a null vector x' orthogonal to N together with $\mathcal{J}x'$, to the spanning set until N becomes maximally isotropic. Since the inner product has split signature, N will finally have dimension $\dim V$. Thus V must be even dimensional. $\square$

In view of this, let $2n$ be the dimension of V, and let $\mathcal{J}$ be a generalized complex structure on $V\oplus V^*$. We may now properly describe the G-structure determined by a generalized complex structure $\mathcal{J}$:

Proposition 4.6. *A generalized complex structure on $V\oplus V^*$, for $\dim V=2n$, is equivalent to a reduction of structure from $O(2n,2n)$ to $U(n,n)=O(2n,2n)\cap GL(2n,\mathbb{C})$.*

Any other generalized complex structure can be obtained by conjugating $\mathcal{J}$ by an element of the orthogonal group $O(2n,2n)$. The stabilizer of $\mathcal{J}$ for this action is $U(n,n)=O(2n,2n)\cap GL(2n,\mathbb{C})$. Therefore the space of generalized complex structures on V is given by the coset space

$$S_{\mathcal{J}}\cong\frac{O(2n,2n)}{U(n,n)}.$$

The Lie group $O(2n,2n)$ is homotopic to $O(2n)\times O(2n)$ and so has four connected components, while $U(n,n)\sim U(n)\times U(n)$ has only one. Therefore the space $S_{\mathcal{J}}$ has four connected components. We can distinguish between these by noting that a generalized complex structure induces an orientation on maximal positive-definite complex subspaces of $(V\oplus V^*,\mathcal{J})$, and another on maximal negative-definite subspaces, yielding four possibilities, one for each component of $S_{\mathcal{J}}$. These orientations pair to give an orientation on the total space $V\oplus V^*$, which may agree or disagree with the canonical orientation on $V\oplus V^*$. In this way we have identified a simple $\mathbb{Z}/2\mathbb{Z}$ invariant of a generalized complex structure, which may be calculated by taking the top power of $\mathcal{J}$, thinking of it as a 2-form on $V\oplus V^*$.

Proposition 4.7. *A generalized complex structure $\mathcal{J}$ is said to have* even *or* odd *parity depending on whether its induced orientation on $V \oplus V^*$ is ± 1, i.e. $\mathcal{J}$ is even if $\frac{1}{(2n)!}\mathcal{J}^{2n} = 1$ and odd if $\frac{1}{(2n)!}\mathcal{J}^{2n} = -1$. This is equivalent to whether the parity of its $+i$-eigenbundle L (as a maximal isotropic subspace) is even or odd.*

Proof. Suppose L is even, i.e. it is in the same component of the space of maximal isotropics as $V \otimes \mathbb{C}$. Since $O(4n, \mathbb{C})$ acts transitively on the space of maximal isotropics, there exists $R \in SO(4n, \mathbb{C})$ such that $R(L) = V \otimes \mathbb{C}$. Then $R(\overline{L})$ intersects trivially with $V \otimes \mathbb{C}$, and so is the graph of a bivector $\beta : V^* \otimes \mathbb{C} \to V \otimes \mathbb{C}$ so that $e^{-\beta}R(\overline{L}) = V^* \otimes \mathbb{C}$ and $e^{-\beta}R(L) = V \otimes \mathbb{C}$. Hence, conjugating $\mathcal{J}$ by $e^{-\beta}R \in SO(4n, \mathbb{C})$ produces the transformation

$$(e^{-\beta}R)\mathcal{J}(e^{-\beta}R)^{-1} = \begin{pmatrix} i\mathbf{1} & \\ & -i\mathbf{1} \end{pmatrix},$$

written in the $V \oplus V^*$ splitting. This transformation then can be thought of as a 2-form $\kappa = i\sum_{k=1}^{2n} e^k \wedge e_k$ for e_k, e^k dual bases for V, V^* respectively. Computing, we obtain $\frac{1}{(2n)!}\kappa^{2n} = 1$, showing that κ, and therefore $\mathcal{J}$, induces the canonical orientation on $V \oplus V^*$.

On the other hand, if L is odd, then it is obtained from $V \otimes \mathbb{C}$ by applying an orientation-reversing element of $O(4n, \mathbb{C})$; therefore the orientation induced by $\mathcal{J}$ is opposite to the canonical orientation. $\square$

Since a generalized complex structure is given by the maximal isotropic L, then by Proposition 2.25, it can equally be specified by the spinor line determining it, $U_L < \wedge^\bullet T^* \otimes \mathbb{C}$, generated by

$$\varphi_L = \exp(B + i\omega)\theta_1 \wedge \cdots \wedge \theta_k, \tag{4.1}$$

where $(\theta_1, \ldots, \theta_k)$ are linearly independent complex 1-forms in $V^* \otimes \mathbb{C}$ spanning $\mathrm{Ann}(\pi_{V\otimes\mathbb{C}}(L))$, and B, ω are the real and imaginary parts of a complex 2-form in $\wedge^2(V^* \otimes \mathbb{C})$. As we showed, the parity of φ_L as a differential form is the same as the parity of L as a maximal isotropic. Thus, expressing the generalized complex structure as a form renders its parity manifest. We call the complex line U_L the *canonical line* of the generalized complex structure.

The additional constraint that L is of real index zero imposes an additional constraint on the line U_L, namely:

Theorem 4.8. *Every maximal isotropic in $V \oplus V^*$ corresponds to a pure spinor line generated by*

$$\varphi_L = \exp(B + i\omega)\Omega,$$

where B, ω are real 2-forms and $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$ for some linearly independent complex 1-forms $(\theta_1, \ldots, \theta_k)$. The integer k is called the type *of the maximal isotropic, as in section 2.2.*

The maximal isotropic is of real index zero if and only if

$$\omega^{n-k} \wedge \Omega \wedge \overline{\Omega} \neq 0, \tag{4.2}$$

or in other words

- *$(\theta_1, \ldots, \theta_k, \overline{\theta}_1, \ldots, \overline{\theta}_k)$ are linearly independent, and*
- *ω is nondegenerate when restricted to the real $(2n - 2k)$-dimensional subspace $\Delta \leq V$ defined by $\Delta = \mathrm{Ker}(\Omega \wedge \overline{\Omega})$.*

Proof. By Corollary 2.21, we see that $\dim L\cap\overline{L}=0$ if and only if $(\varphi_L,\overline{\varphi}_L)\neq 0$, yielding

$$\begin{aligned}0\neq(e^{B+i\omega}\Omega,e^{B-i\omega}\overline{\Omega})&=(e^{2i\omega}\Omega,\overline{\Omega})\\&=\tfrac{(-1)^{2n-k}(2i)^{n-k}}{(n-k)!}\omega^{n-k}\wedge\Omega\wedge\overline{\Omega},\end{aligned}$$

as required. Alternatively, note that this result is a direct consequence of Proposition 4.4, since $B+i\omega$ has been chosen so that $i^*(B+i\omega)=\varepsilon$, where $i:E\hookrightarrow V\otimes\mathbb{C}$ is the inclusion. □

Remark 4.9. *While $O(4n,\mathbb{C})$ acts transitively on the space of maximal isotropics of $V\otimes\mathbb{C}$, the real group $O(2n,2n)$ preserves the real index $r=\dim L\cap\overline{L}$, and this integer labels the orbits of $O(2n,2n)$, as shown in [26]. In particular, note that we may apply B-field transforms to generalized complex structures to obtain new ones: the condition $r=0$ is preserved since B is chosen to be real.*

We will now describe in detail certain examples of generalized complex structures.

Example 4.10 (Symplectic type $(k=0)$). The generalized complex structure determined by a symplectic structure

$$\mathcal{J}_\omega=\begin{pmatrix}0&-\omega^{-1}\\ \omega&0\end{pmatrix}$$

determines a maximal isotropic

$$L=\{X-i\omega(X)\ :\ X\in V\otimes\mathbb{C}\}$$

and a spinor line generated by

$$\varphi_L=e^{i\omega},$$

showing that symplectic structures are always of even parity. This generalized complex structure has type $k=0$, where we recall that k is the codimension of the projection of L to $V\otimes\mathbb{C}$. With respect to the stratification on maximal isotropics according to type, the case $k=0$ is the generic stratum. Since a B-field transformation does not affect projections to $V\otimes\mathbb{C}$, it preserves type. So we may transform this example by a B-field and obtain another generalized complex structure of type 0:

$$e^{-B}\mathcal{J}_\omega e^{B}=\begin{pmatrix}-\omega^{-1}B&-\omega^{-1}\\ \omega+B\omega^{-1}B&B\omega^{-1}\end{pmatrix},$$

$$e^{-B}(L)=\{X-(B+i\omega)(X)\ :\ X\in V\otimes\mathbb{C}\}$$

$$\varphi_{e^{-B}L}=e^{B+i\omega}.$$

We will call this a B-symplectic structure; by (4.1) we see that any generalized complex structure of type $k=0$ is a B-field transform of a symplectic structure.

It is an important observation that for a generalized complex structure, $(V\oplus V^*)\otimes\mathbb{C}=L\oplus\overline{L}$, and projecting to $V\otimes\mathbb{C}$, it follows that $V\otimes\mathbb{C}=E+\overline{E}$, where $E=\pi_{V\otimes\mathbb{C}}(L)$. For this reason, the type of the generalized complex structure cannot exceed n. Our next example is of this extremal type.

Example 4.11 (Complex type $(k=n)$). The generalized complex structure determined by a complex structure

$$\mathcal{J}_J=\begin{pmatrix}-J&0\\ 0&J^*\end{pmatrix}$$

determines a maximal isotropic

$$L=V_{0,1}\oplus V^*_{1,0}$$

(where $V_{1,0} = \overline{V_{0,1}}$ is the $+i$-eigenspace of J), as well a spinor line generated by

$$\varphi_L = \Omega^{n,0},$$

where $\Omega^{n,0}$ is any generator of the $(n,0)$-forms for the complex n-dimensional space (V,J). Hence we see that a complex structure is of even parity when n is even and of odd parity when n is odd. This example may be transformed by a B-field, yielding the equivalent representations

$$e^{-B}\mathcal{J}_J e^B = \begin{pmatrix} -J & 0 \\ BJ + J^*B & J^* \end{pmatrix}$$

$$e^{-B}(L) = \{X + \xi - i_X B \ : \ X + \xi \in V_{0,1} \oplus V^*_{1,0}\},$$
$$\varphi_{e^{-B}L} = e^B \Omega^{n,0}.$$

Notice that only the $(0,2)$ component of the real 2-form B has any effect in this transformation. According to (4.1), the most general form of a type n generalized complex structure is given by

$$\varphi = e^{B+i\omega}\Omega,$$

but since $\Omega \wedge \overline{\Omega} \neq 0$, the n-form Ω determines a complex structure for which it is of type $(n,0)$. Using this complex structure, we see that only the $(0,2)$ component $c = (B + i\omega)^{0,2}$ is effective. Hence, defining the real 2-form $B' = c + \overline{c}$, we obtain $\varphi = e^{B'}\Omega$, showing that any generalized complex structure of type n is the B-field transform of a complex structure.

Example 4.12 (Products). There is a natural notion of *direct sum* of generalized complex structures; that is, if $\mathcal{J}_1$ and $\mathcal{J}_2$ are generalized complex structures on V_1, V_2 respectively, then $\mathcal{J}_1 \oplus \mathcal{J}_2$ is a generalized complex structure on $V_1 \oplus V_2$. This corresponds to taking direct sums $\pi_1^* L_1 \oplus \pi_2^* L_2$ of maximal isotropics (where π_i are the natural projections onto V_i), or forming the wedge product of the defining spinor lines, i.e. $\pi_1^*\varphi_1 \wedge \pi_2^*\varphi_2$. Clearly the nondegeneracy condition (4.2) is preserved, and by the properties of wedge product it is clear that parity is additive with respect to the product of generalized complex structures. An example of such a product structure is the product of a complex structure on V and a symplectic structure on W: the product structure can be described by the line in $\wedge^\bullet(V \oplus W)^*$ generated by

$$\varphi = e^{i\omega}\Omega,$$

where $\omega \in \wedge^2 V^*$ is the symplectic structure on V and $\Omega \in \wedge^{n,0} W^*$ defines the complex structure on W.

Theorem 4.13. *Any linear generalized complex structure of type k can be (noncanonically) expressed as a B-field transform of the direct sum of a complex structure of complex dimension k and a symplectic structure of real dimension $2n - 2k$.*

Proof. The general form of a generalized complex structure is

$$\varphi_L = e^{B+i\omega}\Omega,$$

where the data satisfy the nondegeneracy condition (4.2). If we choose a subspace $N \leq V$ transverse to Δ, then $V = N \oplus \Delta$ and while Δ carries a symplectic structure $\omega_0 = \omega|_\Delta$, N inherits a complex structure determined by $\Omega|_N$. The space $\wedge^2 V^*$ then decomposes as

$$\wedge^2 V^* = \bigoplus_{p+q+r=2} \wedge^p \Delta^* \otimes \wedge^q N^*_{1,0} \otimes \wedge^r N^*_{0,1},$$

so that forms have tri-degree (p,q,r). While Ω is purely of type $(0,k,0)$, the complex 2-form $A = B + i\omega$ decomposes into six components:

$$\begin{array}{lll} A^{200} & & \\ A^{110} & A^{101} & \\ A^{020} & A^{011} & A^{002} \end{array}.$$

Only the components $A^{200}, A^{101}, A^{002}$ act nontrivially on Ω in the expression $e^A\Omega$. Hence we are free to modify the other three components at will. Note that $\omega_0 = -\frac{i}{2}(A^{200} - \overline{A^{200}})$, i.e. the imaginary part of A^{200} is precisely the symplectic structure on Δ. Therefore, define the real 2-form

$$\tilde{B} = \tfrac{1}{2}(A^{200} + \overline{A^{200}}) + A^{101} + \overline{A^{101}} + A^{002} + \overline{A^{002}},$$

and observe that $e^{\tilde{B}+i\omega_0}\Omega = e^{B+i\omega}\Omega$, demonstrating that $\varphi_L = e^{\tilde{B}+i\omega_0}\Omega$, i.e. φ_L is a B-field transform of $e^{i\omega_0}\Omega$, which is a direct sum of a symplectic and complex structure. □

4.2 Almost structures and topological obstructions

We now wish to transport generalized complex structures onto manifolds. In the case of complex or symplectic manifolds, this involves two steps: the specification of an algebraic or 'almost' structure on the tangent bundle, as well as an integrability condition imposed on this structure. In the case of generalized complex structures, our algebraic structure exists on the sum $T \oplus T^*$ of the tangent and cotangent bundles, and our integrability condition involves the Courant bracket. In this section we will describe the algebraic consequences of having a generalized almost complex structure, as well as the topological obstruction to its existence.

Definition 4.14. *A generalized almost complex structure on a real $2n$-dimensional manifold M is given by the following equivalent data:*

- *an almost complex structure $\mathcal{J}$ on $T \oplus T^*$ which is orthogonal with respect to the natural inner product $\langle , \rangle$, i.e. a reduction of structure for the $O(2n,2n)$-bundle $T \oplus T^*$ to the group $U(n,n)$,*
- *a maximal isotropic sub-bundle $L < (T \oplus T^*) \otimes \mathbb{C}$ of real index zero, i.e. $L \cap \overline{L} = 0$,*
- *A pure spinor line sub-bundle $U < \wedge^\bullet T^* \otimes \mathbb{C}$, called the canonical line bundle, satisfying $(\varphi, \overline{\varphi}) \neq 0$ at each point $x \in M$ for any generator $\varphi \in U_x$*

The fact that L is of real index zero leads to the important fact that we have the decomposition

$$(T \oplus T^*) \otimes \mathbb{C} = L \oplus \overline{L},$$

hence we may use the inner product $\langle , \rangle$ to identify $\overline{L} = L^*$. In this way, we obtain an alternative splitting into the sum of dual spaces

$$(T \oplus T^*) \otimes \mathbb{C} = L \oplus L^*.$$

This is particularly significant as it converts the filtration of $\wedge^\bullet T^* \otimes \mathbb{C}$ discussed in section 3.6 into an actual grading. That is, if $U = U_0$ is the canonical bundle, then let $U_k = \wedge^k \overline{L} \cdot U_0$ for $k = 1, \dots, 2n$. Then we obtain an alternative grading for the differential forms:

$$\wedge^\bullet T^* \otimes \mathbb{C} = U_0 \oplus U_1 \oplus \cdots \oplus U_{2n},$$

where, depending on the parity of $\mathcal{J}$,

$$U_0 \oplus U_2 \oplus \cdots \oplus U_{2n} = \wedge^{ev/od} T^* \otimes \mathbb{C}. \tag{4.3}$$

Note also that there is a conjugation symmetry $\overline{U_k} = U_{2n-k}$. Clifford multiplication by elements in $L, \overline{L}$ is of degree $-1, +1$ respectively, in this grading. The bundle U_k could alternatively be defined as the eigenbundle of $\mathcal{J}$ (acting via the Spin representation) with eigenvalue $i(n-k)$.

From a topological point of view, a generalized complex structure is a reduction to $U(n,n)$, but this group is homotopic to its maximal compact subgroup $U(n)\times U(n)$, and so the $U(n,n)$ structure may be further reduced to $U(n)\times U(n)$. This corresponds geometrically to the choice of a positive definite sub-bundle $C_+ < T \oplus T^*$ which is complex with respect to $\mathcal{J}$. The orthogonal complement $C_- = C_+^\perp$ is negative-definite and also complex, and so we obtain the orthogonal decomposition

$$T \oplus T^* = C_+ \oplus C_-.$$

Note that since $C_\pm$ are definite and T is null, the projection $\pi_T : C_\pm \to T$ is an isomorphism. Hence we can transport the complex structure on $C_\pm$ to T, obtaining two almost complex structures J_+, J_- on T. Thus we see that a generalized almost complex structure exists on a manifold if and only if an almost complex structure does.

Proposition 4.15. *The obstruction to the existence of a generalized almost complex structure is the same as that for an almost complex structure, which itself is the same as that for a nondegenerate 2-form (almost symplectic structure).*

The choice of an almost complex structure on a manifold is a reduction of the structure group from $Gl(2n,\mathbb{R})$ to $GL(n,\mathbb{C})$. Noone has yet described the sufficient cohomological conditions for such a reduction to exist, (this has only been done in dimensions ≤ 10, see [45]), however the known necessary conditions in general dimension can be easily obtained.

Proposition 4.16. *The following are necessary conditions for the existence of a (generalized) almost complex structure on a $2n$-manifold M:*

- *The odd Stiefel-Whitney classes of T must be zero.*
- *There must exist classes $c_i \in H^{2i}(M,\mathbb{Z})$, $i = 0,\ldots,n$ whose mod 2 reductions are the even Stiefel-Whitney classes of T. Also, c_n must be the Euler class of T, and*

$$\sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i p_i = \sum_{j=0}^{n} c_j \cup \sum_{k=0}^{n} (-1)^k c_k,$$

where p_i are the Pontrjagin classes of T.

Proof. The Stiefel-Whitney classes of a complex bundle are the mod 2 reductions of the Chern classes, and so vanish in odd degree. Therefore if T is to admit a complex structure, it must have vanishing odd Stiefel-Whitney classes. Furthermore, the Pontrjagin class $p_i(T)$ is equal to $(-1)^i c_{2i}(T\otimes\mathbb{C})$, but if T admits a complex structure J, then $T\otimes\mathbb{C} = T_{1,0}\oplus T_{0,1}$, where $T_{1,0}, T_{0,1}$ are respectively the $+i,-i$-eigenbundles of J. Therefore, $c(T\otimes\mathbb{C}) = c(T_{1,0})\cup c(T_{0,1})$, and since $c_i(T_{0,1}) = (-1)^i c_i(T_{1,0})$ we obtain finally that

$$\sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i p_i = c(T\otimes\mathbb{C}) = c(T_{1,0}) \cup c(T_{0,1}).$$

Also, by definition of the Chern classes, c_n is the Euler class. Hence we have the required result. □

Remark 4.17 (Characteristic classes). *Because of the decomposition $T \oplus T^* = C_+ \oplus C_-$, a generalized almost complex structure has two sets of Chern classes; $\{c_k^+ = c_k(C_+)\}$ and $\{c_k^- = c_k(C_-)\}$. While $T \oplus T^*$ itself is a complex bundle and so has Chern classes, they can be expressed in terms of $c_k^\pm$, as follows:*

$$c(T \oplus T^*, \mathcal{J}) = c(C_+) \cup c(C_-).$$

The canonical bundle U also has a characteristic class $c_1(U)$, and since by squaring the spinor (see Proposition (2.22) we know that

$$U \otimes U \cong \det L,$$

and since $L = (T \oplus T^*, \mathcal{J})$ as complex bundles, we obtain the fact that

$$c_1(U) = \tfrac{1}{2}c_1(L) = \tfrac{1}{2}(c_1^+ + c_1^-).$$

Note here that $c_1^\pm$ is congruent to $w_2(C_\pm) = w_2(T)$ modulo 2, so that $c_1^+ + c_1^-$ is even.

We will explore the $U(n) \times U(n)$ reduction in greater detail in chapter 6, where we define an integrability condition generalizing the Kähler condition.

4.3 The Courant integrability condition

We now introduce the integrability condition on generalized almost complex structures which interpolates between the symplectic condition $d\omega = 0$ and the complex condition that $[T_{1,0}, T_{1,0}] \subset T_{1,0}$.

Definition 4.18. *The generalized almost complex structure $\mathcal{J}$ is said to be integrable to a generalized complex structure when its $+i$-eigenbundle $L < (T \oplus T^*) \otimes \mathbb{C}$ is Courant involutive. In other words, a generalized complex structure is a complex Dirac structure of real index zero.*

The requirement that L be a complex Dirac structure and that $L \cap \overline{L} = \{0\}$ leads us directly into the situation described in Theorem 3.36. That is, $L, \overline{L}$ are transverse Dirac structures of a Courant algebroid and therefore form a *Lie bialgebroid*. Again, this means that the exterior derivative d_L on $\wedge^\bullet L^* = \wedge^\bullet \overline{L}$ is a derivation of the Courant bracket $[,]$ on $\wedge^\bullet \overline{L}$.

Furthermore, because of the decomposition $(T \oplus T^*) \otimes \mathbb{C} = L \oplus \overline{L}$, the projection $E = \pi_T(L)$ satisfies $E + \overline{E} = T \otimes \mathbb{C}$, and we are in the situation of Proposition 3.11, which concludes that E gives rise to a smooth integrable distribution Δ, defined by $\Delta \otimes \mathbb{C} = E \cap \overline{E}$. Recall that a point at which $\dim \Delta$ is locally constant is called a regular point, and from Proposition 3.12 we conclude that near a regular point of a generalized complex structure we obtain a transverse complex structure to the foliation defined by Δ. The *type*, $k \in \{0, \ldots, n\}$, of the generalized complex structure at $x \in M$ is defined as the codimension of $E_x < T_x \otimes \mathbb{C}$, and therefore the leaves of the induced foliation have dimension $\dim_{\mathbb{R}} \Delta = 2n - 2k$. These leaves inherit symplectic structure, as follows: in the regular neighbourhood, the complex Dirac structure L may be expressed, as in Proposition 4.4, as $L(E, \varepsilon)$, where $E < T \otimes \mathbb{C}$ is a sub-bundle and $\varepsilon \in C^\infty(\wedge^2 E^*)$, such that $E + \overline{E} = T \otimes \mathbb{C}$ and $\omega_\Delta = \mathrm{Im}(\varepsilon|_{E \cap \overline{E}})$ is a nondegenerate real 2-form on Δ. The integrability of ω_Δ follows from the Courant involutivity of $L(E, \varepsilon)$:

Proposition 4.19. *Let $E < T \otimes \mathbb{C}$ be a sub-bundle and $\varepsilon \in C^\infty(\wedge^2 E^*)$. Then the maximal isotropic $L(E, \varepsilon)$ defines an integrable generalized complex structure if and only if E is involutive and $d_E \varepsilon = 0$.*

Proof. Let $i : E \hookrightarrow T \otimes \mathbb{C}$ be the inclusion. Then $d_E : C^\infty(\wedge^k E^*) \to C^\infty(\wedge^{k+1}E^*)$ is defined by $i^* \circ d = d_E \circ i^*$. Now let $\sigma \in C^\infty(\wedge^2 T^* \otimes \mathbb{C})$ be a smooth extension of ε, i.e. $i^*\sigma = \varepsilon$. Suppose that $X + \xi, Y + \eta \in C^\infty(L)$, which means that $\xi|_E = i_X\varepsilon$ and $\eta|_E = i_Y\varepsilon$. Consider the bracket $Z + \zeta = [X + \xi, Y + \eta]$; if L is Courant involutive, then $Z \in C^\infty(E)$, showing E is involutive, and the difference

$$\begin{aligned}\zeta|_E - i_Z\varepsilon &= i^*(\mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d(i_X\eta - i_Y\xi)) - i_{[X,Y]}i^*\sigma \\ &= i_Xd_Ei^*\eta - i_Yd_Ei^*\xi + \tfrac{1}{2}d_E(i_Xi_Y\varepsilon - i_Yi_X\varepsilon) - i^*[\mathcal{L}_X, i_Y]\sigma \\ &= i_Xd_Ei^*\eta - i_Yd_Ei^*\xi + d_Ei_Xi_Y\varepsilon - i^*(i_Xdi_Y\sigma + di_Xi_Y\sigma - i_Ydi_X\sigma - i_Yi_Xd\sigma) \\ &= i_Yi_Xd_E\varepsilon\end{aligned}$$

must vanish for all $X + \xi, Y + \eta \in C^\infty(L)$, showing that $d_E\varepsilon = 0$. Reversing the argument we see that the converse holds as well. □

A corollary to this result is that the nondegenerate 2-form $\omega_\Delta \in C^\infty(\wedge^2\Delta^*)$ is closed along the leaves, showing that in a regular neighbourhood, a generalized complex structure gives rise to a foliation with symplectic leaves and a transverse complex structure.

We may now verify that the integrability condition on generalized almost complex structures yields the classical conditions on symplectic and complex structures. Recall that by *type* we mean the codimension of $E = \pi_T(L)$ in $T \otimes \mathbb{C}$. As we are projecting a bundle, this codimension may not be constant throughout the manifold, a fact we investigate in section 4.8. In this section we consider only generalized complex structures of constant type.

Example 4.20 (Symplectic type ($k = 0$)). The generalized almost complex structure determined by a symplectic structure

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

has $+i$-eigenbundle

$$L = \{X - i\omega(X) \ : \ X \in T \otimes \mathbb{C}\},$$

which is Courant involutive if and only if $d\omega = 0$. Of course we may apply a B-field transform (B a real closed 2-form) to $\mathcal{J}_\omega$, obtaining what we will call a B-symplectic generalized complex structure. In fact, any generalized almost complex structure which is everywhere of type $k = 0$ must be of this form: its $+i$-eigenbundle can be expressed as $L(E, \varepsilon)$, where $E = T \otimes \mathbb{C}$ and $\varepsilon = -B - i\omega$ is a complex 2-form with ω non-degenerate. The maximal isotropic is Courant involutive if and only if $d(B + i\omega) = 0$. Hence every generalized complex structure of type zero is B-symplectic.

Example 4.21 (Complex type ($k = n$)). The generalized complex structure determined by a complex structure

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$$

has maximal isotropic eigenbundle

$$L = T_{0,1} \oplus T^*_{1,0},$$

which, as we saw in Example 3.34, is Courant involutive if and only if J is integrable as a complex structure.

The general form for the $+i$-eigenbundle of a generalized almost complex structure $\mathcal{J}$ of constant type $k = n$ is $L(E, \varepsilon)$, where $E \cap \overline{E} = \{0\}$ and $\varepsilon \in \wedge^2 E^*$. In this case E determines an almost complex structure J on T (i.e. $E = T_{0,1}$) and $\varepsilon \in C^\infty(\wedge^{2,0}T^*)$. The involutivity of $L(E, \varepsilon)$ is equivalent to the condition that $E = T_{0,1}$ is involutive and $\partial\varepsilon = 0$. Hence we see that a generalized complex structure of constant type $k = n$ must be the conjugation of a bare complex structure by a ∂-closed $(2,0)$-form, i.e.

$$\mathcal{J} = e^{-\varepsilon}\mathcal{J}_J e^{\varepsilon} = \begin{pmatrix} -J & 0 \\ \varepsilon J + J^*\varepsilon & J^* \end{pmatrix}.$$

Note that ε is not necessarily closed and is not real, and so $\mathcal{J}$ is not, in general, a B-field transform of a complex structure. However, if $[\varepsilon] = 0$ in $H^{2,0}_{\partial}(M)$, then we can find a $(1,0)$-form φ such that $\varepsilon = \partial\varphi$ and then $\mathcal{J}$ is the B-field transform of $\mathcal{J}_J$, where

$$B = \varepsilon + \overline{\varepsilon} + \overline{\partial}\varphi + \partial\overline{\varphi}.$$

Alternatively, if the complex structure J satisfies the $\partial\overline{\partial}$-lemma, we may again express ε as the $(2,0)$-part of a real, closed 2-form B.

Summarizing, we obtain the following result:

Proposition 4.22. *On a $2n$-dimensional manifold, a generalized complex structure of type zero is a B-symplectic structure, while a generalized complex structure of type n is the transform of a complex structure by a ∂-closed $(2,0)$-form.*

4.4 Integrability and differential forms

The presence of a generalized complex structure on a manifold has implications for complex differential forms, analogous to the fact that a complex structure induces a (p,q)-decomposition of forms and a splitting $d = \partial + \overline{\partial}$.

Let $\mathcal{J}$ be a generalized complex structure, and let $U < \wedge^\bullet T^*$ be the canonical line bundle of the generalized complex structure. We have already observed that $\mathcal{J}$ determines an alternative grading for the differential forms

$$\wedge^\bullet T^* \otimes \mathbb{C} = U_0 \oplus U_1 \oplus \cdots \oplus U_{2n},$$

where $U_0 = U$ and $U_k = \wedge^k \overline{L} \cdot U_0$ for $k = 1, \ldots, 2n$. We now demonstrate that the integrability of $\mathcal{J}$ is equivalent to the fact that the exterior derivative d splits into the sum $d = \partial + \overline{\partial}$ where for each $k = 0, \ldots, 2n-1$,

$$C^\infty(U_k) \underset{\partial}{\overset{\overline{\partial}}{\rightleftarrows}} C^\infty(U_{k+1}) \ .$$

Theorem 4.23. *Let $\mathcal{J}$ be a generalized almost complex structure, and define*

$$\begin{aligned} \overline{\partial} &= \pi_{k+1} \circ d : C^\infty(U_k) \longrightarrow C^\infty(U_{k+1}) \\ \partial &= \pi_{k-1} \circ d : C^\infty(U_k) \longrightarrow C^\infty(U_{k-1}), \end{aligned}$$

where π_k is the projection onto U_k, and $U_k = \{0\}$ for $k < 0$ and $k > 2n$. Then $\mathcal{J}$ is integrable if and only if $d = \partial + \overline{\partial}$.

To prove this theorem we need a generalization of formula (3.11):

Lemma 4.24. *For any differential form ρ and any sections $A, B \in C^\infty(T \oplus T^*)$, we have the following identity*

$$A \cdot B \cdot d\rho = d(B \cdot A \cdot \rho) + B \cdot d(A \cdot \rho) - A \cdot d(B \cdot \rho) + [A, B] \cdot \rho - d\langle A, B\rangle \wedge \rho. \tag{4.4}$$

Proof. Let $A = X + \xi, B = Y + \eta$. then

$$\begin{aligned}
A \cdot B \cdot d\rho &= (i_X + \xi\wedge)(i_Y + \eta\wedge)d\rho \\
&= i_X i_Y d\rho + i_X \eta \wedge d\rho - \eta \wedge i_X d\rho + \xi \wedge i_Y d\rho + \xi \wedge \eta \wedge d\rho \\
&= d i_Y i_X \rho + i_Y d i_X \rho - i_X d i_Y \rho + i_{[X,Y]}\rho + i_X \eta \wedge d\rho - \eta \wedge i_X d\rho + \xi \wedge i_Y d\rho + \xi \wedge \eta \wedge d\rho \\
&= d((i_Y + \eta\wedge)(i_X + \xi\wedge)\rho) + (i_Y + \eta\wedge)d((i_X + \xi\wedge)\rho) - (i_X + \xi\wedge)d((i_Y + \eta\wedge)\rho) \\
&\quad + [X + \xi, Y + \eta]\rho - \tfrac{1}{2}d(i_Y \xi + i_X \eta) \\
&= d(B \cdot A \cdot \rho) + B \cdot d(A \cdot \rho) - A \cdot d(B \cdot \rho) + [A, B] \cdot \rho - d\langle A, B\rangle \wedge \rho.
\end{aligned}$$

□

Proof of Theorem 4.23. By induction; recall that Clifford multiplication by $L, \overline{L}$ is of degree $-1, +1$ respectively in the alternative grading. First let $\rho \in C^\infty(U_0)$. Then for any $A, B \in C^\infty(L)$, equation (4.4) implies that $A \cdot B \cdot d\rho = [A, B] \cdot \rho$. This shows that $d(C^\infty(U_0)) \subset C^\infty(U_1)$ if and only if L is Courant involutive, since d is degree 1 in the usual grading of forms and hence $d\rho$ can have no U_0 component (see equation (4.3)). Now assume that $d = \partial + \overline{\partial}$ for all U_i such that $0 \leq i < k$. Then take $\rho \in C^\infty(U_k)$ and for any $A, B \in C^\infty(L)$ we apply (4.4) to obtain

$$A \cdot B \cdot d\rho = d(B \cdot A \cdot \rho) + B \cdot d(A \cdot \rho) - A \cdot d(B \cdot \rho) + [A, B] \cdot \rho.$$

On the right hand side, by induction each term is in $C^\infty(U_{k-3} \oplus U_{k-1})$; therefore $d\rho$ is in $C^\infty(U_{k-1} \oplus U_{k+1})$, again using the fact that d is of degree 1 in the usual grading of forms, ensuring that $d\rho$ has no U_k component. This completes the proof. □

Example 4.25. In the complex case, $U_0 = \wedge^{n,0}T^*$ and

$$U_k = \bigoplus_p \wedge^{n-p,k-p}T^*,$$

so that ∂ and $\overline{\partial}$ are the usual operators on differential forms for a complex manifold.

Just as in the complex case, once we have the decomposition $d = \partial + \overline{\partial}$ according to the grading, the fact that $d^2 = 0$ implies that $\partial^2 = \overline{\partial}^2 = 0$, and $\partial\overline{\partial} = -\overline{\partial}\partial$. Furthermore we can form another real operator $d^{\mathcal{J}} = i(\overline{\partial} - \partial)$, which can also be written $d^{\mathcal{J}} = [d, \mathcal{J}]$, and which satisfies $(d^{\mathcal{J}})^2 = 0$.

Remark 4.26. *It is interesting to note that while in the complex case $d^{\mathcal{J}}$ is just the usual d^c-operator $d^c = i(\overline{\partial} - \partial)$, in the symplectic case $d^{\mathcal{J}}$ is equal to the symplectic adjoint of d defined by Koszul [28] and studied by Brylinski [8] in the context of symplectic harmonic forms.*

The natural differential operator $\overline{\partial}$ can be viewed as an operator

$$\overline{\partial} : C^\infty(\wedge^k L^* \otimes U) \to C^\infty(\wedge^{k+1} L^* \otimes U),$$

extended from $d : C^\infty(U) \to C^\infty(L^* \otimes U)$ via the rule

$$\overline{\partial}(\mu \otimes s) = d_L \mu \otimes s + (-1)^{|\mu|} \mu \wedge ds,$$

for $\mu \in C^\infty(\wedge^k L^*)$ and $s \in C^\infty(U)$. As such, $\overline{\partial}$ is an example of a Lie algebroid connection on U, in the sense of Definition 3.40. Since $\overline{\partial}^2 = 0$, this Lie algebroid connection has vanishing curvature, and because of this, we say that $(U, \overline{\partial})$ is a *generalized holomorphic bundle.*

Definition 4.27. *Let E be a complex vector bundle on a generalized complex manifold with $+i$-eigenbundle L. Then the data $(E, \overline{\partial}_E)$, where $\overline{\partial}_E$ is a Lie algebroid connection on E with respect to L, is said to be a generalized holomorphic bundle if and only if $\overline{\partial}_E^2 = 0$.*

Of course, the trivial complex line bundle is always generalized holomorphic, using the operator $d_L : C^\infty(\wedge^k L^*) \to C^\infty(\wedge^{k+1} L^*)$ as the Lie algebroid connection.

In special cases, the canonical line bundle may be *holomorphically trivial*, in the sense that $(U, \overline{\partial})$ is isomorphic to the trivial bundle together with its Lie algebroid connection d_L. This is equivalent to the existence of a nowhere-vanishing section $\rho \in C^\infty(U)$ satisfying $d\rho = 0$. In [19], Hitchin calls these generalized Calabi-Yau structures:

Definition 4.28. *A generalized Calabi-Yau structure is a generalized complex structure with holomorphically trivial canonical bundle, i.e. there exists a nowhere-vanishing closed section $\rho \in C^\infty(U)$.*

4.5 Exotic examples of generalized complex structures

In this section we describe examples of six-dimensional manifolds which admit no known complex (type 0) or symplectic (type 3) structures, and yet do admit generalized complex structures of types 1 or 2. These manifolds are *nilmanifolds*; a more extensive survey of generalized complex structures on 6-nilmanifolds has been carried out in collaboration with Cavalcanti [10].

A nilmanifold is a homogeneous space $M = G/\Gamma$, where G is a simply-connected nilpotent real Lie group and Γ is a lattice of maximal rank in G. The simplest nilmanifold is the torus $\mathbb{R}^k/\mathbb{Z}^k$. The differential graded algebra of left-invariant forms on G is quasi-isomorphic to the de Rham complex of M, and serves as a rational minimal model for the nilmanifold. Hence, up to equivalence by common finite covers, the nilmanifolds can be fully distinguished simply by giving the differentials of a set $\{e_1, \ldots, e_6\}$ of linearly independent left-invariant 1-forms. In the nilmanifold literature this information is usually presented as in the following example: the array $(0, 0, 0, 12, 13, 14 + 35)$ describes a nilmanifold with de Rham complex generated by 1-forms $e_1, \ldots e_6$ and such that $de_1 = de_2 = de_3 = 0$, while $de_4 = e_1 \wedge e_2, de_5 = e_1 \wedge e_3$, and $de_6 = e_1 \wedge e_4 + e_3 \wedge e_5$.

In six dimensions, there are 34 isomorphism classes of real nilpotent Lie algebras (see [43] for a detailed list and literature review). For each of the 34 algebras there may be many nilmanifolds, and these are distinguished by their fundamental group.

Salamon studied the problem in [43] of which 6-nilmanifolds carry left-invariant complex or symplectic structures. Of course this reduces to determining which of the 34 nilpotent lie algebras admit invariant structures. His results are as follows: exactly 18 of the 34 admit complex structures, exactly 26 admit symplectic structures, and 15 admit both, of which the 6-torus is the only Kähler example. This leaves 5 classes of nilmanifold which admit no known complex or symplectic structure. They are (as listed in [43]):

- (0,0,12,13,14+23, 34+52)
- (0,0,12,13,14,34+52)
- (0,0,0,12,13,14+35)
- (0,0,0,12,23,14+35)
- (0,0,0,0,12,15+34)

In fact these 5 families admit generalized complex structures, which we will exhibit by listing the pure spinor lines defining them. This is greatly simplified since we have an explicit handle on the de Rham complex. The situation is even more special since in each case, the canonical bundle is holomorphically trivial: the following examples are all generalized Calabi-Yau.

Example 4.29 (0,0,12,13,14+23, 34+52). Let $\rho = e^{B+i\omega}\Omega$, where

$$\begin{aligned}\Omega &= e_1 + ie_2\\ B &= e_2 \wedge e_6 - e_3 \wedge e_5 + e_3 \wedge e_6 - e_4 \wedge e_5\\ \omega &= e_3 \wedge e_6 + e_4 \wedge e_5.\end{aligned}$$

As defined, ρ is a closed pure spinor of real index zero, and defines a generalized complex structure of type $k = 1$.

Example 4.30 (0,0,12,13,14,34+52). Let $\rho = e^{B+i\omega}\Omega$, where

$$\begin{aligned}\Omega &= e_1 + ie_2\\ B &= e_3 \wedge e_6 - e_4 \wedge e_5\\ \omega &= e_3 \wedge e_6 + e_4 \wedge e_5.\end{aligned}$$

As defined, ρ is a closed pure spinor of real index zero, and defines a generalized complex structure of type $k = 1$.

Example 4.31 (0,0,0,12,13,14+35). Let $\rho = e^{B+i\omega}\Omega$, where

$$\begin{aligned}\Omega &= e_1 + ie_2\\ B &= 0\\ \omega &= e_3 \wedge e_6 + e_4 \wedge e_5.\end{aligned}$$

As defined, ρ is a closed pure spinor of real index zero, and defines a generalized complex structure of type $k = 1$.

Example 4.32 (0,0,0,12,23,14+35). Let $\rho = e^{B+i\omega}\Omega$, where

$$\begin{aligned}\Omega &= e_1 + ie_2\\ B &= -e_3 \wedge e_6 + e_4 \wedge e_5\\ \omega &= e_3 \wedge e_6 + e_4 \wedge e_5.\end{aligned}$$

As defined, ρ is a closed pure spinor of real index zero, and defines a generalized complex structure of type $k = 1$.

Example 4.33 (0,0,0,0,12,15+34). Let $\rho = e^{B+i\omega}\Omega$, where

$$\begin{aligned}\Omega &= (e_1 + ie_2) \wedge (e_3 + ie_4)\\ B &= 0\\ \omega &= e_5 \wedge e_6.\end{aligned}$$

As defined, ρ is a closed pure spinor of real index zero, and defines a generalized complex structure of type $k = 2$.

Our purpose here is simply to demonstrate that there are examples of generalized complex structures which are not simply B-field transforms of products of complex and symplectic manifolds. A fuller exploration of these structures and their moduli on nilmanifolds will appear in our work with Cavalcanti [10].

4.6 Interpolation between complex and symplectic structures

We learned from the case of linear generalized complex structures that complex and symplectic structures have opposite parity in dimensions $4k+2$ and the same parity in dimension $4k$. We now show that it is possible to interpolate smoothly between a complex structure and a symplectic structure *through* integrable generalized complex structures when M is a hyperkähler manifold, e.g. a K3 surface.

If M is a Kähler manifold then it is equipped with an integrable complex structure J and a symplectic form ω which is of type $(1,1)$, which means that $\omega J = -J^*\omega$, which implies that

$$\begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}\begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}\begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix},$$

or in other words, the generalized complex structures commute.

On the other hand, suppose that M is hyperkähler, which means it has a triple of Kähler complex structures I, J, K with coincident Kähler metric satisfying the quaternionic relations $IJ = K = -JI$. Then we have the Kähler forms $\omega_I, \omega_J, \omega_K$ and as before $\omega_J J = -J^*\omega_J$. However, since $IJ = -JI$, we see that $\omega_J I = I^*\omega_J$, which implies that the generalized complex structures $\mathcal{J}_{\omega_J}$ and $\mathcal{J}_I$ *anticommute*. Hence form the one-parameter family of generalized almost complex structures

$$\mathcal{J}_t = \sin t \mathcal{J}_I + \cos t \mathcal{J}_{\omega_J}, \quad t \in [0, \tfrac{\pi}{2}].$$

Clearly $\mathcal{J}_t$ is a generalized almost complex structure; we now check that it is integrable.

Proposition 4.34. *Let M be a hyperkähler manifold as above. Then the generalized complex structure $\mathcal{J}_t = \sin t \mathcal{J}_I + \cos t \mathcal{J}_{\omega_J}$ is integrable $\forall t \in [0, \frac{\pi}{2}]$. Therefore it is a family of generalized complex structures interpolating between a symplectic structure and a complex structure.*

Proof. Let $B = \tan t\ \omega_K$, a closed 2-form which is well defined $\forall\ t \in [0, \frac{\pi}{2})$. Noting that $\omega_K I = I^*\omega_K = \omega_J$, we obtain the following expression:

$$e^B \mathcal{J}_t e^{-B} = \begin{pmatrix} 0 & -(\sec t\ \omega_J)^{-1} \\ \sec t\ \omega_J & 0 \end{pmatrix}.$$

We conclude from this that for all $t \in [0, \frac{\pi}{2})$, $\mathcal{J}_t$ is in fact the B-field transform of the symplectic structure determined by $\sec t\ \omega_J$, with $B = \tan t\ \omega_K$, and is therefore integrable as a generalized complex structure; at $t = \frac{\pi}{2}$, $\mathcal{J}_t$ is purely complex, and is integrable as well, completing the proof. Note that this interpolation argument applies to holomorphic symplectic manifolds as well. $\square$

4.7 Local structure: The generalized Darboux theorem

The Newlander-Nirenberg theorem tells us that an integrable complex structure on a $2n$-manifold is locally equivalent, via a diffeomorphism, to $\mathbb{C}^n$. Similarly, the Darboux theorem states that a symplectic structure on a $2n$-manifold is locally equivalent, via a diffeomorphism, to the standard symplectic structure $(\mathbb{R}^{2n}, \omega)$, where

$$\omega = dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_n.$$

In this section we prove the analogous theorem in the generalized context, which states that at a regular point of type k, a generalized complex structure on a $2n$-manifold is locally equivalent, via a diffeomorphism and a B-field, to the standard product generalized complex structure $\mathbb{C}^k \times (\mathbb{R}^{2n-2k}, \omega)$.

We saw in Proposition 4.19 that in a regular neighbourhood, a generalized complex structure may be expressed as $L(E, \varepsilon)$ where $E < T \otimes \mathbb{C}$ is an involutive sub-bundle and $\varepsilon \in C^\infty(\wedge^2 E^*)$ satisfies $d_E\varepsilon = 0$. By Proposition 3.12, the distribution E determines a foliation of the neighbourhood with transverse complex structure isomorphic to an open set in $\mathbb{R}^{2n-2k} \times \mathbb{C}^k$, where E is spanned by $\{\partial/\partial x_1, \ldots, \partial/\partial x_{2n-2k}, \partial/\partial z_1, \ldots, \partial/\partial z_k\}$, where $\{x_i\}$ are coordinates for the leaves $\mathbb{R}^{2n-2k}$ and $\{z_i\}$ are transverse complex coordinates. Therefore, by choosing $B + i\omega \in C^\infty(\wedge^2 T^* \otimes \mathbb{C})$ such that $i^*(B + i\omega) = \varepsilon$, we may write a generator for the canonical bundle defining $L(E, \varepsilon)$ as follows:

$$\rho = e^{B+i\omega}\Omega,$$

where $\Omega = dz_1 \wedge \cdots \wedge dz_k$; note ρ is independent of the choice of extension for ε. Furthermore, we see that

$$i^*d(B + i\omega) = d_E i^*(B + i\omega) = d_E\varepsilon = 0,$$

which means that $d(B + i\omega) \in \mathrm{Ann}^\bullet E$, implying finally that

$$d\rho = e^{B+i\omega} d(B + i\omega) \wedge \Omega = 0.$$

We have shown that in a regular neighbourhood, any generalized complex structure on a $2n$-dimensional manifold may be expressed as a closed complex differential form $\rho = e^{B+i\omega}\Omega$, where Ω is decomposable of degree $0 \le k \le n$ and such that

$$\omega^{n-k} \wedge \Omega \wedge \overline{\Omega} \neq 0.$$

Weinstein's proof of the Darboux normal coordinate theorem for a family of symplectic structures (see [33]) can be used to find a leaf-preserving local diffeomorphism φ taking ω to a 2-form whose pullback to each leaf is the standard Darboux symplectic form on $\mathbb{R}^{2n-2k}$, i.e.

$$\varphi^*\omega\big|_{\mathbb{R}^{2n-2k}\times\{pt\}} = \omega_0 = dx_1 \wedge dx_2 + \cdots + dx_{2n-2k-1} \wedge dx_{2n-2k}.$$

Let us apply this diffeomorphism, obtaining new 2-forms $\varphi^*B + i\varphi^*\omega$. Note that Ω is unaffected by this diffeomorphism, since $\{z_i\}$ are constant along the leaves.

For convenience, let $K = \mathbb{R}^{2n-2k}$ and $N = \mathbb{C}^k$, so that differential forms now have tri-degree (p, q, r) for components in $\wedge^p K^* \otimes \wedge^q N^*_{1,0} \otimes \wedge^r N^*_{0,1}$. Furthermore, the exterior derivative decomposes into a sum of three operators

$$d = d_f + \partial + \overline{\partial},$$

each of degree 1 in the respective component of the tri-grading. Note that d_f is the leafwise exterior derivative. While Ω is purely of type $(0, k, 0)$, the complex 2-form $A = \varphi^*B + i\varphi^*\omega$ decomposes into six components:

$$\begin{array}{lll} A^{200} & & \\ A^{110} & A^{101} & \\ A^{020} & A^{011} & A^{002} \end{array}$$

Note that only the components $A^{200}, A^{101}, A^{002}$ act nontrivially on Ω in the expression $e^A\Omega$. Hence we are free to modify the other three components at will. Also, note that the imaginary part of A^{200} is simply ω_0, so that $d(A^{200} - \overline{A^{200}}) = 0$, since ω_0 is in constant Darboux form.

From the condition $d(B+i\omega)\wedge\Omega = 0$, we obtain the following four equations:

$$\overline{\partial} A^{002} = 0 \tag{4.5}$$
$$\overline{\partial} A^{101} + d_f A^{002} = 0 \tag{4.6}$$
$$\overline{\partial} A^{200} + d_f A^{101} = 0 \tag{4.7}$$
$$d_f A^{200} = 0. \tag{4.8}$$

The last equation simply states that the pull-back of $B + i\omega$ to any leaf is closed in the leaf, as we know already.

We will now endeavour to modify A so that $\varphi^*\rho = e^A\Omega$ is unchanged but A is replaced with $\tilde{A} = \tilde{B} + \frac{1}{2}(A^{200} - \overline{A^{200}})$, where $\tilde{B}$ is a real closed 2-form. This would demonstrate that

$$\varphi^*\rho = e^{\tilde{B}+i\omega_0}\Omega,$$

i.e. ρ is equivalent, via the composition of a B-field transformation and a diffeomorphism, to the product of a symplectic with a complex structure.

In order to preserve $\varphi^*\rho$, the most general form for $\tilde{B}$ is

$$\tilde{B} = \tfrac{1}{2}(A^{200} + \overline{A^{200}}) + A^{101} + \overline{A^{101}} + A^{002} + \overline{A^{002}} + C,$$

where C is a real 2-form of type (011). Then clearly $\varphi^*\rho = e^{\tilde{B}+i\omega_0}\Omega$. Requiring that $d\tilde{B} = 0$ imposes two constraint equations:

$$(d\tilde{B})^{012} = \partial A^{002} + \overline{\partial} C = 0. \tag{4.9}$$
$$(d\tilde{B})^{111} = \partial A^{101} + \overline{\partial A^{101}} + d_f C = 0 \tag{4.10}$$

The question then becomes whether we can find a real (011)-form C such that these equations are satisfied. The following are all local arguments, making repeated use of the Dolbeault lemma.

- From equation (4.5) we obtain that $A^{002} = \overline{\partial}\alpha$ for some (001)-form α. Then condition (4.9) is equivalent to $\overline{\partial}(C - \partial\alpha) = 0$, whose general solution is

  $$C = \partial\alpha + \overline{\partial\alpha} + i\partial\overline{\partial}\chi$$

  for any real function χ. We must now check that it is possible to choose χ so that the second condition (4.10) is satisfied by this C.

- From equation (4.6) we obtain that $\overline{\partial}(A^{101} - d_f\alpha) = 0$, implying that $A^{101} = d_f\alpha + \overline{\partial}\beta$ for some (100)-form β. Condition (4.10) then is equivalent to the fact that

  $$-id_f\partial\overline{\partial}\chi = \partial\overline{\partial}(\beta - \overline{\beta}),$$

  which can be solved (for the unknown χ) if and only if the right hand side is d_f-closed. From equation (4.7) we see that $\overline{\partial}(A^{200} - d_f\beta) = 0$, showing that $A^{200} = d_f\beta + \delta$, where δ is a $\overline{\partial}$-closed (200)-form. Hence

  $$d_f\partial\overline{\partial}(\beta - \overline{\beta}) = \partial\overline{\partial}(A^{200} - \overline{A^{200}}),$$

  and the right hand side vanishes precisely because $A^{200} - \overline{A^{200}} = 2\omega_0$, which is closed. Hence χ may be chosen to satisfy condition (4.10), and so we obtain a closed 2-form $\tilde{B}$.

Finally, we have proven the normal coordinate theorem for regular neighbourhoods of generalized complex manifolds:

Theorem 4.35 (Generalized Darboux theorem). *Any regular point in a generalized complex manifold has a neighbourhood which is equivalent, via a diffeomorphism and a B-field transformation, to the product of an open set in $\mathbb{C}^k$ with an open set in the standard symplectic space $(\mathbb{R}^{2n-2k}, \omega_0)$.*

4.8 The jumping phenomenon

While we have fully characterized generalized complex structures in regular neighbourhoods, it remains an essential feature of generalized complex geometry that the *type* of the structure may vary throughout the manifold. In view of the normal form theorem, the type can be thought of as the number of transverse complex directions, and this is an upper semi-continuous function on the manifold, i.e. each point has a neighbourhood in which it does not increase. The most generic type is zero, when there are only symplectic directions, and the most special type is n, when all directions are complex. Note that the type may jump up, but always by an even number, since types of the same parity are in the same connected component of the space of linear generalized complex structures (see section 2.2 for details).

In this section we present a simple example of a generalized complex structure on $\mathbb{R}^4$ which is of symplectic type ($k = 0$) outside a codimension 2 hypersurface and jumps up to complex type ($k = 2$) along the hypersurface.

Consider the differential form

$$\rho = z_1 + dz_1 \wedge dz_2,$$

where z_1, z_2 are the standard coordinates on $\mathbb{C}^2 \cong \mathbb{R}^4$. When $z_1 = 0$, $\rho = dz_1 \wedge dz_2$ and so defines the standard complex structure, whereas when $z_1 \neq 0$, ρ defines a B-symplectic structure since

$$\rho = z_1 e^{\frac{dz_1 \wedge dz_2}{z_1}}.$$

Hence, algebraically the form ρ defines a generalized almost complex structure whose type jumps along $z_1 = 0$.

To check the integrability of this structure, we take the exterior derivative:

$$d\rho = dz_1 = i_{-\partial_{z_2}}(z_1 + dz_1 \wedge dz_2) = (-\partial_{z_2}) \cdot \rho,$$

showing that ρ indeed satisfies the integrability condition of Theorem 4.23, and defines a generalized complex structure on all of $\mathbb{R}^4$. In this case it is easy to see that although the canonical line bundle is topologically trivial, it is not holomorphically trivial, i.e. there is no nowhere-vanishing closed section of U.

In the next chapter we will produce more general examples of the jumping phenomenon, and on compact manifolds as well.

4.9 Twisted generalized complex structures

Following on from section 3.7, where we described the twisted Courant bracket, we now define the notion of *twisted* generalized complex structure. The underlying algebraic structure is the same as the generalized complex case:

Definition 4.36. *A generalized almost complex structure $\mathcal{J}$ is said to be twisted generalized complex with respect to the closed 3-form H when its $+i$-eigenbundle L is involutive with respect to the H-twisted Courant bracket.*

Remark 4.37. *It is important to note that given any H-twisted generalized complex structure $\mathcal{J}$, the conjugate $e^b \mathcal{J} e^{-b}$, for b any smooth 2-form, is integrable with respect to the $H + db$-twisted Courant bracket. This means that the space of twisted generalized complex structures depends only on the cohomology class $[H] \in H^3(M, \mathbb{R})$.*

As we described in Proposition 3.44, the integrability condition on the differential forms defining the generalized almost complex structure is simply that $d_H = d + H \wedge \cdot$ maps $C^\infty(U_0)$ to $C^\infty(U_1)$. A Darboux theorem for H-twisted structures follows from the methods developed in section 4.7: in a regular neighbourhood an H-twisted generalized complex structure can be expressed as the B-transform of a product of a symplectic by a complex structure, except that in the twisted case B is not a closed form. Instead, $dB = H$ in the neighbourhood.

We will provide interesting examples of twisted generalized complex structures in section 6.6.

Chapter 5

Deformations of generalized complex structures

In the deformation theory of complex manifolds developed by Kodaira, Spencer, and Kuranishi, one begins with a compact complex manifold (M, J) with holomorphic tangent bundle $\mathcal{T}$, and constructs an analytic subvariety $\mathcal{Z} \subset H^1(M, \mathcal{T})$ (containing 0) which is the base space of a family of deformations $\mathcal{M} = \{\varepsilon_z \ : \ z \in \mathcal{Z}, \ \varepsilon_0 = 0\}$ of the original complex structure J. This family is *locally complete* (also called *miniversal*), in the sense that any family of deformations of J can be obtained, up to equivalence, by pulling $\mathcal{M}$ back by a map f to $\mathcal{Z}$, as long as the family is restricted to a sufficiently small open set in its base. The subvariety $\mathcal{Z} \subset H^1(M, \mathcal{T})$ is defined as the zero set of a holomorphic map $\Phi : H^1(M, \mathcal{T}) \to H^2(M, \mathcal{T})$, and so the base of the miniversal family is certainly smooth when this obstruction map vanishes.

In this section we extend these results to the generalized complex setting, following the method of Kuranishi [29]. In particular, we construct, for any generalized complex manifold, a locally complete family of deformations. We then proceed to produce new examples of generalized complex structures by deforming known ones.

5.1 The deformation complex

The generalized complex structure $\mathcal{J}$ is determined by its $+i$-eigenbundle $L < (T \oplus T^*) \otimes \mathbb{C}$ which is isotropic, satisfies $L \cap \overline{L} = \{0\}$, and is closed under the Courant bracket. Recall that since $(T \oplus T^*) \otimes \mathbb{C} = L \oplus \overline{L}$, we use the natural metric $\langle , \rangle$ to identify $\overline{L}$ with L^*.

To deform $\mathcal{J}$ we will vary L in the Grassmannian of maximal isotropics. Any maximal isotropic having zero intersection with $\overline{L}$ (this is an open set containing L) can be uniquely described as the graph of a homomorphism $\epsilon : L \longrightarrow \overline{L}$ satisfying $\langle \epsilon X, Y \rangle + \langle X, \epsilon Y \rangle = 0 \ \ \forall X, Y \in C^\infty(L)$, or equivalently $\epsilon \in C^\infty(\wedge^2 L^*)$. Therefore the new isotropic is given by $L_\epsilon = (1 + \epsilon)L$. As the deformed $\mathcal{J}$ is to remain real, we must have $\overline{L}_\epsilon = (1 + \overline{\epsilon})\overline{L}$. Now we observe that L_ϵ has zero intersection with its conjugate if and only if the endomorphism we have described on $L \oplus L^*$, namely

$$A_\epsilon = \begin{pmatrix} 1 & \overline{\epsilon} \\ \epsilon & 1 \end{pmatrix}, \tag{5.1}$$

is invertible; this is the case for ϵ in an open set around zero.

So, providing ϵ is small enough, $\mathcal{J}_\epsilon = A_\epsilon \mathcal{J} A_\epsilon^{-1}$ is a new generalized almost complex structure, and all nearby almost structures are obtained in this way. Note that while A_ϵ itself is not an orthogonal transformation, of course $\mathcal{J}_\epsilon$ is.

To describe the integrability condition for $\epsilon \in C^\infty(\wedge^2 L^*)$ which guarantees that $\mathcal{J}_\epsilon$ is integrable, we notice that we are precisely in the situation of Theorem 3.37, which states that $\mathcal{J}_\epsilon$ is integrable if and only if $\epsilon \in C^\infty(\wedge^2 L^*)$ satisfies the equation

$$d_L\epsilon + \frac{1}{2}[\epsilon, \epsilon] = 0. \tag{5.2}$$

In this way, we can interpret sufficiently small solutions of equation (5.2) as deformations of a genuine geometrical structure, thereby solving the open problem stated in [30] of interpreting the equation as deriving from a deformation theory.

Infinitesimally, this means that nearby generalized complex structures are in the kernel of the $d_L : C^\infty(\wedge^2 L^*) \to C^\infty(\wedge^3 L^*)$, and so we need to know the nature of this linear operator.

Proposition 5.1. *If L is the Lie algebroid deriving from a generalized complex structure, then the Lie algebroid differential complex*

$$d_L : C^\infty(\wedge^p L^*) \to C^\infty(\wedge^{p+1} L^*)$$

is elliptic. Hence its cohomology groups, which we denote by $H^p_L(M)$, are finite dimensional complex vector spaces.

Proof. The principal symbol $s(d_L) : (T^* \otimes \mathbb{C}) \otimes \wedge^p L^* \to \wedge^{p+1} L^*$ is given by $\pi^* : T^* \otimes \mathbb{C} \to L^*$ (where $\pi : L \to T \otimes \mathbb{C}$ is the projection) composed with wedge product, i.e.

$$s_\xi(d_L) = \pi^*(\xi) \wedge \cdot \quad ,$$

where $\xi \in T^* \otimes \mathbb{C}$. If ξ is a nonzero real 1-form, then since $(T \oplus T^*) \otimes \mathbb{C} = L \oplus \overline{L}$, we have the decomposition $\xi = x + \overline{x}$ for a nonzero element $x \in L$, and then $s_\xi(d_L) = \overline{x} \wedge \cdot$ defines an elliptic Koszul complex. Hence $(C^\infty(\wedge^\bullet L^*), d_L)$ is an elliptic differential complex. See [50] for a proof that the cohomology groups of an elliptic differential complex on a compact manifold are finite dimensional. □

Now that we have described which tensors $\varepsilon \in C^\infty(\wedge^2 L^*)$ are integrable deformations, we must explain when two such deformations are considered equivalent. In the case of complex or symplectic geometry, two deformations are considered equivalent if they are related by a small diffeomorphism. In the case of generalized complex geometry, however, the Courant bracket on $T \oplus T^*$ has a larger group of symmetries, and so we will consider two deformations to be equivalent if they are related by a diffeomorphism (connected to the identity) and an *exact* B-field transformation. A special case of such a transformation is one generated by a vector field X and a 1-form ξ:

$$F_{X+\xi} = e^{d\xi} \circ e^X$$

for $X + \xi \in C^\infty(T \oplus T^*)$, where by e^{tX} we mean the one-parameter group of diffeomorphisms generated by the vector field X. Note that there is redundancy in expressing a symmetry as a section of $T \oplus T^*$, since the 1-form ξ could be exact, in which case it has no action whatsoever.

Remark 5.2. *In the presence of a generalized complex structure $\mathcal{J}$, certain infinitesimal symmetries $X + \xi \in C^\infty(T \oplus T^*)$ preserve the tensor $\mathcal{J}$; these* generalized holomorphic symmetries *are given by sections of $T \oplus T^*$ whose L^* component lies in the kernel of d_L. In the complex case these are holomorphic vector fields together with $\overline{\partial}$-closed $(0,1)$-forms, while in the symplectic case they arise from symplectic vector fields, i.e. sections $X \in C^\infty(T)$ such that $\mathcal{L}_X\omega = 0$. Note that any complex-valued function $f \in C^\infty(M, \mathbb{C})$ generates a symmetry $X + \xi = d_L f + \overline{d_L f}$; such holomorphic* *symmetries could be called* Hamiltonian *symmetries of the generalized complex structure, and do coincide with the notion of Hamiltonian vector field in the symplectic case. In the complex case they are given by 1-forms $\xi = \overline{\partial} f + \partial \overline{f}$, and generate B-field transformations with $B = \partial\overline{\partial}(f - \overline{f})$.*

By differentiating, we now show the relationship between sections in the image of $d_L : C^\infty(L^*) \to C^\infty(\wedge^2 L^*)$ and equivalent deformations.

Proposition 5.3. *Let $\mathcal{J}$ be a generalized complex structure with $+i$-eigenbundle L, and let $\varepsilon_0 \in C^\infty(\wedge^2 L^*)$ be a deformation of this structure. If $X+\xi \in C^\infty(T \oplus T^*)$, and if $t \in \mathbb{R}$ is in a sufficiently small neighbourhood of 0, then we have the following expression for the equivalent deformation induced by $F_{t(X+\xi)} = e^{td\xi} \circ e^{tX}$ by its action on the graph of ε_0:*

$$F_{t(X+\xi)}(\varepsilon_0) = \varepsilon_0 + t d_L((X+\xi)_{L^*}) + R(\varepsilon_0, t(X+\xi)), \tag{5.3}$$

where $(X+\xi)_{L^*}$ is the component in L^* according to the splitting $L \oplus L^*$, and R satisfies

$$R(t\varepsilon_0, t(X+\xi)) = t^2 \widetilde{R}(\varepsilon_0, X+\xi, t),$$

where $\widetilde{R}(\varepsilon_0, X+\xi, t)$ is smooth. In this sense, $R(\varepsilon_0, X+\xi)$ is $O(t^2)$ in the deformations.

Proof. Let $\{s\varepsilon_0 \ : \ s \in \mathbb{R}\}$ be a straight line in the space of smooth sections of $\wedge^2 L^*$ passing through 0 and the deformation ε_0. Let $F_{t(X+\xi)}$, for t in some neighbourhood of $0 \in \mathbb{R}$, be the family of automorphisms of $T \oplus T^*$ defined by $F_{t(X+\xi)} = e^{td\xi} \circ e^{tX}$. The combined action of the section $s\varepsilon_0$ and the automorphism $F_{t(X+\xi)}$ on the $+i$-eigenbundle L is given by the composition

$$F_{t(X+\xi)} A_{s\varepsilon_0} = \begin{pmatrix} \sigma & \overline{\tau} \\ \tau & \overline{\sigma} \end{pmatrix},$$

where $A_{s\varepsilon_0}$ is as in (5.1) and the right hand side is written in the splitting $L \oplus \overline{L}$. Assuming t is small enough, σ is invertible, and we may factorise

$$F_{t(X+\xi)} A_{s\varepsilon_0} = \begin{pmatrix} \sigma & \\ & \overline{\sigma} \end{pmatrix} \begin{pmatrix} 1 & \overline{\varepsilon(s,t)} \\ \varepsilon(s,t) & 1 \end{pmatrix} = C_\sigma A_{\varepsilon(s,t)},$$

where $\varepsilon(s,t) \in C^\infty(\wedge^2 L^*)$ is the new section of $\wedge^2 L^*$ given by the action of $F_{t(X+\xi)}$ on $s\varepsilon_0$, i.e.

$$F_{t(X+\xi)}(s\varepsilon_0) = \varepsilon(s,t).$$

Differentiating $A_{\varepsilon(s,t)} = C_\sigma^{-1} F_{t(X+\xi)} A_{s\varepsilon_0}$ and evaluating at $(s,t) = (0,0)$, we obtain

$$\dot{A}_{\varepsilon(s,t)}\big|_{(0,0)} = -\dot{C}_\sigma\big|_{(0,0)} + \dot{F}_{t(X+\xi)}\big|_{(0,0)} + \dot{A}_{s\varepsilon_0}\big|_{(0,0)}.$$

Now let $A+a, B+b \in C^\infty(L)$, so that

$$\varepsilon(s,t)(A+a, B+b) = \langle A_{\varepsilon(s,t)}(A+a), B+b \rangle.$$

Differentiating, we see that since C_σ is simply an automorphism of L, the $\dot{C}_\sigma$ term has no effect, leaving two terms:

$$\dot{\varepsilon}(s,t)(A+a, B+b) = \langle \dot{F}_{t(X+\xi)}\big|_{(0,0)} A+a, B+b \rangle + \langle \dot{A}_{s\varepsilon_0}\big|_{(0,0)} A+a, B+b \rangle.$$

We compute each term separately:

$$\begin{aligned} \tfrac{\partial}{\partial s}\varepsilon(s,t)\big|_{t=s=0}(A+a, B+b) &= \tfrac{d}{ds}\langle A_{s\varepsilon_0}(A+a), B+b \rangle\big|_{s=0} \\ &= \varepsilon_0(A+a, B+b), \end{aligned}$$

while

$$
\begin{aligned}
\tfrac{\partial}{\partial t}\varepsilon(s,t)\big|_{t=s=0}(A+a,B+b) &= \tfrac{d}{dt}\langle F(t(X+\xi))(A+a),B+b\rangle\big|_{t=0} \\
&= \tfrac{d}{dt}\langle e_*^{tX}A+(e^{tX})^{*-1}a+i_{e_*^{tX}A}(td\xi),B+b\rangle\big|_{t=0} \\
&= -\langle [X,A]+\mathcal{L}_X a+i_A d\xi, B+b\rangle.
\end{aligned}
$$

Now note that

$$
\begin{aligned}
d_L(X+\xi)_{L^*}(A+a,B+b) &= i_A d\langle X+\xi,B+b\rangle - i_B d\langle X+\xi,A+a\rangle - \langle X+\xi,[A+a,B+b]\rangle \\
&= -\tfrac{1}{2}(i_B\mathcal{L}_X a + i_B i_A d\xi + [\mathcal{L}_X,i_A]b) \\
&= -\langle [X,A]+\mathcal{L}_X a+i_A d\xi, B+b\rangle \\
&= \tfrac{\partial}{\partial t}\varepsilon(s,t)\big|_{t=s=0}(A+a,B+b).
\end{aligned}
$$

Therefore, by Taylor's theorem,

$$
\varepsilon(s,t) = s\varepsilon_0 + t d_L(X+\xi)_{L^*} + R(s\varepsilon_0, t(X+\xi)), \tag{5.4}
$$

where $R(s\varepsilon_0, t(X+\xi))$ is of order s^2, st, and t^2, so that, taking $s=t$,

$$
R(t\varepsilon_0, t(X+\xi)) = t^2\widetilde{R}(\varepsilon_0, X+\xi, t)
$$

with $\widetilde{R}(\varepsilon_0, X+\xi, t)$ smooth. Setting $s=1$ in (5.4), we obtain the required result. □

This proposition shows us that, infinitesimally, deformations which differ by sections which lie in the image of d_L in $C^\infty(\wedge^2 L^*)$ are equivalent by transformations generated by vector fields and 1-forms. Hence we expect the tangent space to the moduli space to lie in $H_L^2(M)$. In the remainder of this section we develop the Hodge theory for the elliptic complex $(C^\infty(\wedge^\bullet L^*), d_L)$ so that we may prove this assertion rigorously.

We follow the usual treatment of Hodge theory as described in [50]. Choose a Hermitian metric on the complex Lie algebroid L and let $(\varphi,\psi)_k$ be the L_k^2 Sobolev inner product on sections $\varphi,\psi\in C^\infty(\wedge^p L^*)$ induced by the metric. Recall that L_0^2 is the usual L^2 inner product. We will use $|\ |_k$ to denote the Sobolev norm, defined by

$$
|u|_k = ((u,u)_k)^{1/2}.
$$

We define the L^2 adjoint d_L^* of d_L via the formula

$$
(d_L\varphi,\phi) = (\varphi, d_L^*\phi),
$$

and obtain the elliptic, self-adjoint Laplacian

$$
\Delta_L = d_L d_L^* + d_L^* d_L.
$$

Let $\mathcal{H}^p$ be the space of Δ_L-harmonic forms, which is isomorphic to $H_L^p(M)$ by the standard argument, and let H be the orthogonal projection of $C^\infty(\wedge^p L)$ onto the closed subspace $\mathcal{H}^p$. Also, let G be the Green smoothing operator quasi-inverse to Δ_L, i.e.

$$
G : L_k^2 \to L_{k+2}^2,
$$

so that we have

$$
\mathrm{Id} = H + \Delta G = H + G\Delta.
$$

Also, G satisfies $[G, d_L] = [G, d_L^*] = 0$. We will find it useful, as Kuranishi did, to define the once-smoothing operator

$$Q = d_L^* G : L_k^2 \to L_{k+1}^2,$$

which then satisfies

$$\mathrm{Id} = H + d_L Q + Q d_L, \tag{5.5}$$

$$Q^2 = d_L^* Q = Q d_L^* = HQ = QH = 0. \tag{5.6}$$

We now have the algebraic and analytical tools we need to describe the deformation theory of generalized complex manifolds.

5.2 The deformation theorem

Let $\mathcal{J}$ be a generalized complex structure on the compact manifold M with $+i$-eigenbundle L, and let $\varepsilon \in C^\infty(\wedge^2 L^*)$. Then recall that since $(L, \overline{L})$ is a Lie bialgebroid, we have compatibility between the Courant bracket on L^* and the differential d_L.

Theorem 5.4 (Deformation theorem for generalized complex structures). *There exists an open neighbourhood $U \subset H_L^2(M)$ containing zero, a smooth family $\widetilde{\mathcal{M}} = \{\varepsilon_u \ : \ u \in U, \ \varepsilon_0 = 0\}$ of generalized almost complex deformations of $\mathcal{J}$, and an analytic obstruction map $\Phi : U \to H_L^3(M)$ with $\Phi(0) = 0$ and $d\Phi(0) = 0$, such that the deformations in the sub-family $\mathcal{M} = \{\varepsilon_z \ : \ z \in \mathcal{Z} = \Phi^{-1}(0)\}$ are precisely the integrable ones. Furthermore, any sufficiently small deformation ε of J is equivalent to at least one member of the family $\mathcal{M}$. Finally, in the case that the obstruction map vanishes, we show that $\mathcal{M}$ is a smooth locally complete family.*

Proof. The proof is divided into two parts: first, we construct a smooth family $\widetilde{\mathcal{M}}$, and show it contains the family of integrable deformations $\mathcal{M}$ defined by the map Φ; second, we describe its miniversality property. We follow the paper of Kuranishi [29] closely.

Part I: For sufficiently large k, $L_k^2(M, \mathbb{R})$ is a Banach algebra (see [37]), and the map $F : \varepsilon \mapsto \varepsilon + \frac{1}{2}Q[\varepsilon, \varepsilon]$ extends to a map

$$F : L_k^2(\wedge^2 L^*) \longrightarrow L_k^2(\wedge^2 L^*),$$

which is a smooth map of the Hilbert space into itself whose derivative at the origin is clearly the identity mapping. Hence by the Banach space inverse function theorem, F^{-1} maps a neighbourhood of the origin in $L_k^2(\wedge^2 L^*)$ smoothly and bijectively to another neighbourhood of the origin. Hence, if we choose a sufficiently small $\delta > 0$ then the following finite-dimensional subset of harmonic sections

$$U = \{u \in \mathcal{H}^2 < L_k^2(\wedge^2 L^*) \ : \ |u|_k < \delta\}$$

is taken by F^{-1} into another smooth finite-dimensional set, i.e. defines a family of sections

$$\widetilde{M} = \{\varepsilon(u) = F^{-1}(u) \ : \ u \in U\}$$

where $\varepsilon(u)$ depends smoothly (in fact, holomorphically) on u, and satisfies $F(\varepsilon(u)) = u$. Applying the Laplacian to this equation, we obtain

$$\Delta_L \varepsilon(u) + \tfrac{1}{2}\Delta_L Q[\varepsilon(u), \varepsilon(u)] = 0,$$

and using $\Delta_L Q = \Delta_L d_L^* G = d_L^* \Delta_L G = d_L^*$, we see that

$$\Delta_L \varepsilon(u) + \tfrac{1}{2} d_L^*[\varepsilon(u), \varepsilon(u)] = 0.$$

This is a quasi-linear elliptic PDE, and by the standard result of Morrey [34], we conclude that the solutions $\varepsilon(u)$ of this equation are actually smooth sections, i.e.

$$\varepsilon(u) \in C^\infty(\wedge^2 L^*).$$

Hence we have constructed a smooth family of generalized almost complex deformations of $\mathcal{J}$, over an open set $U \subset H^2_L(M)$. We emphasize that we have exhibited this family as a genuine finite-dimensional submanifold of a Hilbert space whose tangent space at the origin is $\mathcal{H}^2 = H^2_L(M)$.

We now ask which of these deformations are integrable, i.e. satisfy the equation $d_L\varepsilon + \frac{1}{2}[\varepsilon, \varepsilon] = 0$. Since $\varepsilon(u) + \frac{1}{2}Q[\varepsilon(u), \varepsilon(u)] = u$ and $d_L u = d_L^* u = 0$, we see that $d_L\varepsilon(u) = -\frac{1}{2}d_L Q[\varepsilon(u), \varepsilon(u)]$, and using (5.5) we obtain

$$\begin{aligned} d_L\varepsilon(u) + \tfrac{1}{2}[\varepsilon(u), \varepsilon(u)] &= -\tfrac{1}{2}d_L Q[\varepsilon(u), \varepsilon(u)] + \tfrac{1}{2}[\varepsilon(u), \varepsilon(u)] \\ &= \tfrac{1}{2}(Qd_L + H)[\varepsilon(u), \varepsilon(u)]. \end{aligned}$$

Since the images of Q and H are orthogonal in the L_2 inner product, we see that $\varepsilon(u)$ is integrable if and only if $H[\varepsilon(u), \varepsilon(u)] = Qd_L[\varepsilon(u), \varepsilon(u)] = 0$.

Now we claim that $H[\varepsilon(u), \varepsilon(u)] = 0$ implies that $Qd_L[\varepsilon(u), \varepsilon(u)] = 0$: using the compatibility of $[,]$ and d_L we obtain

$$\begin{aligned} Qd_L[\varepsilon(u), \varepsilon(u)] &= 2Q[d_L\varepsilon(u), \varepsilon(u)] \\ &= -Q[d_L Q[\varepsilon(u), \varepsilon(u)], \varepsilon(u)] \\ &= -Q[(\mathrm{Id} - Qd_L - H)[\varepsilon(u), \varepsilon(u)], \varepsilon(u)]. \end{aligned}$$

So, assuming that $H[\varepsilon(u), \varepsilon(u)] = 0$, we obtain

$$\begin{aligned} Qd_L[\varepsilon(u), \varepsilon(u)] &= -Q[(\mathrm{Id} - Qd_L)[\varepsilon(u), \varepsilon(u)], \varepsilon(u)] \\ &= Q[Qd_L[\varepsilon(u), \varepsilon(u)], \varepsilon(u)]. \end{aligned}$$

Letting $\zeta(u) = Qd_L[\varepsilon(u), \varepsilon(u)]$, we have that

$$\zeta(u) = Q[\zeta(u), \varepsilon(u)],$$

and since for sufficiently large k the map $(\alpha, \beta) \mapsto Q[\alpha, \beta]$ satisfies $|Q[\alpha, \beta]|_k \leq c|\alpha|_k|\beta|_k$ for some $c > 0$, we have

$$|\zeta(u)|_k \leq c|\zeta(u)|_k|\varepsilon(u)|_k$$

for some $c > 0$. Therefore, if we take δ to be so small that $|\varepsilon(u)|_k < \frac{1}{c}$ for all $|u|_k < \delta$, we obtain that $\zeta(u) = 0$.

Hence, we have shown that $\varepsilon(u)$ is integrable precisely when u lies in the vanishing set of the analytic mapping $\Phi : U \to H^3_L(M)$ defined by $\Phi(u) = H[\varepsilon(u), \varepsilon(u)]$. Note that $\Phi(0) = d\Phi(0) = 0$. Before we proceed to the second part of the proof, we wish to give an alternative characterisation of the family $\mathcal{M} = \{\varepsilon(z) \ : \ z \in \mathcal{Z} = \Phi^{-1}(0)\}$. We claim that $\mathcal{M}$ is actually a neighbourhood around zero in the set

$$\mathcal{M}' = \left\{\varepsilon \in C^\infty(\wedge^2 L^*) \ : \ d_L\varepsilon + \tfrac{1}{2}[\varepsilon, \varepsilon] = d_L^*\varepsilon = 0\right\}.$$

To show this, let $\varepsilon(u) \in \mathcal{M}$. Then since $\varepsilon(u) = u - \frac{1}{2}Q[\varepsilon(u), \varepsilon(u)]$ and $d_L^* Q = 0$, we see that $d_L^*\varepsilon(u) = 0$, showing that $\mathcal{M} \subset \mathcal{M}'$. Conversely, let $\varepsilon \in \mathcal{M}'$. Then since $d_L^*\varepsilon = 0$, applying d_L^* to the equation $d_L\varepsilon + \frac{1}{2}[\varepsilon, \varepsilon] = 0$ we obtain $\Delta_L\varepsilon + \frac{1}{2}d_L^*[\varepsilon, \varepsilon] = 0$, and applying Green's operator we see that $\varepsilon + \frac{1}{2}Q[\varepsilon,\varepsilon] = H\varepsilon$, i.e. $F(\varepsilon) = H\varepsilon \in \mathcal{H}^2$, proving that a small open set in $\mathcal{M}'$ is contained in $\mathcal{M}$, completing the argument.

Part II: Let $P < C^\infty(L^*)$ be the L^2 orthogonal complement of the d_L-closed sections $\ker d_L < C^\infty(L^*)$, or in other words, sections in the image of d_L^*. We show that there exist neighbourhoods of the origin $V \subset C^\infty(\wedge^2 L^*)$ and $W \subset P$ such that for any $\varepsilon \in V$ there is a unique $X+\xi \in C^\infty(T\oplus T^*)$ such that $(X+\xi)_{L^*} \in W$ and

$$d_L^*(e^{d\xi}e^X(\varepsilon)) = 0. \tag{5.7}$$

This would imply that any sufficiently small solution to $d_L\varepsilon + \frac{1}{2}[\varepsilon,\varepsilon] = 0$ is equivalent to another solution ε' such that $d_L^*\varepsilon' = 0$, i.e. a solution in $\mathcal{M}$. Extended to smooth families, this result would prove local completeness.

Restricting to a sufficiently small neighbourhood in $C^\infty(T\oplus T^*)$ so that we may take $t=1$ in Equation (5.3), we see that $d_L^*(e^{d\xi}e^X(\varepsilon)) = 0$ if and only if

$$d_L^*\varepsilon + d_L^* d_L(X+\xi)_{L^*} + d_L^* R(\varepsilon, X+\xi) = 0.$$

Assuming $(X+\xi)_{L^*} \in P$, we see that $d_L^*(X+\xi)_{L^*} = H(X+\xi)_{L^*} = 0$, so that

$$d_L^*\varepsilon + \Delta_L(X+\xi)_{L^*} + d_L^* R(\varepsilon, X+\xi) = 0,$$

and applying G,

$$(X+\xi)_{L^*} + Qd_L^*\varepsilon + QR(\varepsilon, X+\xi) = 0.$$

Since $R(\varepsilon, X+\xi)$ involves one derivative of $X+\xi$, the map

$$F : (\varepsilon, X+\xi) \mapsto (X+\xi)_{L^*} + Qd_L^*\varepsilon + QR(\varepsilon, X+\xi)$$

is continuous from a neighbourhood of the origin $V_0 \times W_0$ in $C^\infty(\wedge^2 L^*) \times P$ (where $R(\varepsilon, X+\xi)$ is defined) to P, where all spaces are endowed with the L^2_k norm, k sufficiently large. F can therefore be extended to a continuous map from the completion of the domain, $\widehat{V_0} \times \widehat{W_0}$, to the completion of P. The derivative of F with respect to $X+\xi$ is the identity map, and so by the implicit function theorem there are neighbourhoods $V \subset V_0, W_1 \subset \widehat{W_0}$ such that given $\varepsilon \in V$, $F(\varepsilon, X+\xi) = 0$ is satisfied for a unique $(X+\xi)_{L^*} \in W_1$, and which depends smoothly on $\varepsilon \in V$. Furthermore, since $\varepsilon \in V$ is itself smooth, the unique solution $X+\xi$ satisfies the quasi-linear elliptic PDE $\Delta_L(X+\xi)_{L^*} + d_L^*\varepsilon + d_L^*R(\varepsilon, X+\xi) = 0$, implying that $X+\xi$ is smooth as well, hence $(X+\xi)_{L^*}$ lies in the neighbourhood $W = W_1 \cap P$. Therefore we have shown that every sufficiently small deformation of the generalized complex structure is equivalent to one in our finite-dimensional family $\mathcal{M}$.

If the obstruction map Φ vanishes, so that $\mathcal{M}$ is a smooth family, then given any other smooth family $\mathcal{M}_S = \{\varepsilon_s \ : \ s \in S, \ \varepsilon_{s_0} = 0\}$ with basepoint $s_0 \in S$, the above argument provides a smooth family of equivalences $(X+\xi)_s$ taking each ε_s for s in some neighbourhood T of s_0 to $\varepsilon_{f(s)}$, $f(s) \in U \subset H^2_L(M)$, defining a smooth map $f : T \to U$, $f(s_0) = 0$, so that $f^*\mathcal{M} = \mathcal{M}_S$. Thus we establish that $\mathcal{M}$ is a locally complete family of deformations. □

Remark 5.5. *The natural complex structure on $H^2_L(M)$ and on the vanishing set of the holomorphic obstruction map Φ raises the question of whether there is a notion of holomorphic family of generalized complex structures. There is: if S is a complex manifold then a holomorphic family of generalized complex structures on M is a generalized complex structure on $M \times S$ which can be pushed down via the projection to yield the complex structure on S. The family $\mathcal{M}$ is actually such a holomorphic family.*

5.3 Examples of deformed structures

Consider deforming a complex manifold (M, J) as a generalized complex manifold. Then since

$$L = T_{0,1} \oplus T^*_{1,0},$$

the deformation complex is actually

$$(\oplus_{p+q=k}\Omega^{0,q}(\wedge^p T_{1,0}), \quad \overline{\partial}),$$

so that the base of the Kuranishi family lies in

$$H^2_L(M) = \oplus_{p+q=2} H^q(M, \wedge^p T_{1,0}).$$

The image of the obstruction map lies in

$$H^3_L(M) = \oplus_{p+q=3} H^q(M, \wedge^p T_{1,0})$$

Therefore we see immediately that generalized complex manifolds provide a solution to the problem of finding a geometrical interpretation of the "extended complex deformation space" defined by Kontsevich and Barannikov [3]. Any deformation ε has three components

$$\beta \in H^0(M, \wedge^2 T_{1,0}), \quad \varphi \in H^1(M, T_{1,0}), \quad B \in H^2(M, \mathcal{O}).$$

The component φ is a usual deformation of the complex structure, as discovered by Kodaira and Spencer. The component B is a complex B-field action as we have discussed. The component β, however, is a completely new type of deformation for complex manifolds. The integrability condition on such a deformation $\beta \in C^\infty(\wedge^2 T_{1,0})$ is simply that

$$\overline{\partial}\beta + \tfrac{1}{2}[\beta, \beta] = 0,$$

which is satisfied if and only if the bivector β is holomorphic and Poisson. As we saw in section 2.1, a β-transform acts by shearing $T \oplus T^*$ in the T direction, and hence may change the type of the generalized complex structure. As an example, let us explore such deformations on $\mathbb{C}P^2$.

Example 5.6 (Deformed generalized complex structure on $\mathbb{C}P^2$). On $\mathbb{C}P^2$, $\wedge^2 T_{1,0}$ is simply the anticanonical bundle $\mathcal{O}(3)$, whose nonzero holomorphic sections vanish along cubics. Any holomorphic bivector $\beta \in H^0(M, \mathcal{O}(3))$ is automatically Poisson since we are in complex dimension 2, and hence any sufficiently small holomorphic section of $\mathcal{O}(3)$ defines an integrable deformation of the complex structure into a generalized complex structure.

β takes $X + \xi \in T_{0,1} \oplus T^*_{1,0}$ to $X + \xi + i_\xi \beta$ and so whenever β is nonzero, the deformed Lie algebroid projects surjectively onto $T \otimes \mathbb{C}$. Hence the deformed structure is of B-symplectic type (type 0) outside the cubic vanishing set and of complex type (type 2) along the cubic. The B-field and symplectic form go to infinity as one approaches the cubic curve. Therefore we obtain a generalized complex structure on $\mathbb{C}P^2$ which is clearly inequivalent to either a complex or a symplectic structure. This compact generalized complex manifold exhibits a jumping phenomenon along a codimension 2 subvariety.

To check the smoothness of the locally complete family, note that since we are in complex dimension 2, $H^0(\wedge^3 T_{1,0}) = H^3(\mathcal{O}) = 0$. Also, by Serre duality, $H^2(T_{1,0}) \cong H^0(T^*_{1,0} \otimes \mathcal{O}(-3)) = 0$, since $T^*_{1,0}$ has no holomorphic sections. Furthermore, $H^1(\wedge^2 T_{1,0}) = H^1(\mathcal{O}(3)) = 0$ by the Bott formulae. Hence we see that the obstruction space vanishes for $\mathbb{C}P^2$, and so we conclude that there is a smooth locally complete family of deformations for $\mathbb{C}P^2$ as a generalized complex manifold.

Example 5.7. One can of course deform $\mathbb{C}^2$ in the same way that we have deformed $\mathbb{C}P^2$; we choose the holomorphic bivector

$$\beta = z_1 \partial_{z_1} \wedge \partial_{z_2},$$

where z_1, z_2 are the usual complex coordinates. Then applying a β-transform to the usual complex structure defined by the spinor $\Omega = dz_1 \wedge dz_2$, we obtain

$$e^{\beta}\Omega = dz_1 \wedge dz_2 + z_1,$$

which is precisely the example we provided in section 4.8 of a jumping generalized complex structure on $\mathbb{C}^2$. We see now that it is actually a deformation of the usual complex structure by a holomorphic Poisson structure. To see the behaviour of the B-symplectic form as we approach the vanishing set $z_1 = 0$ of the bivector, express the differential form for $z_1 \neq 0$ as

$$z_1 e^{\frac{dz_1 \wedge dz_2}{z_1}},$$

showing that as z_1 approaches zero, $B + i\omega = \frac{1}{z_1} dz_1 \wedge dz_2$ approaches infinity.

These deformations by holomorphic Poisson bivectors can be thought of as non-commutative deformations of the complex manifold, in the sense of quantization of Poisson structures. The connection between non-commutative geometry and these deformations of generalized complex structure is explored by Kapustin in [21], where the relations to topological string theory are described as well.

Chapter 6

Generalized Kähler geometry

As we have seen, a generalized complex structure is an integrable reduction of the structure group of $T \oplus T^*$ to $U(n,n)$. This structure group may always be further reduced to its maximal compact subgroup $U(n)\times U(n)$ by the choice of an appropriate metric G on $T\oplus T^*$. In this section we show that there is an integrability condition which applies to such $U(n) \times U(n)$ structures, which generalizes the usual Kähler condition. We then show that this generalized Kähler geometry is equivalent to a geometry first discovered by physicists (see [42]) investigating supersymmetric nonlinear sigma-models. Aspects of this geometry, in particular the fact that it involves a bi-Hermitian structure, were later studied (in the four-dimensional case) by mathematicians (see [1],[38],[24]). The main open problem in this field, as stated in [1], is to determine whether or not there exist bi-Hermitian structures on complex surfaces not admitting any anti-self-dual metric, for example, $\mathbb{C}P^2$. Using generalized Kähler structures, we are able to provide an affirmative solution to this problem. Finally, we define twisted generalized Kähler structures, and describe an interesting class of examples: the even-dimensional semi-simple Lie groups.

6.1 Definition

Since the bundle $T\oplus T^*$ has a natural inner product $\langle , \rangle$, it has structure group $O(2n, 2n)$ in a natural way. A reduction from $O(2n, 2n)$ to its maximal compact subgroup $O(2n) \times O(2n)$ is equivalent to the choice of a $2n$-dimensional subbundle C_+ which is positive definite with respect to the inner product. Let C_- be the (negative definite) orthogonal complement to C_+. Note that the splitting $T \oplus T^* = C_+ \oplus C_-$ defines a positive definite metric on $T \oplus T^*$ via

$$G = \langle , \rangle\big|_{C_+} - \langle , \rangle\big|_{C_-}.$$

Using the inner product to identify $T \oplus T^*$ with its dual, the metric G may be viewed as an automorphism of $T \oplus T^*$ which is symmetric, i.e. $G^* = G$, and which squares to the identity, i.e. $G^2 = 1$. Note that $C_\pm$ are the ± 1 eigenspaces of G. Hence we have the following:

Proposition 6.1. *A reduction to $O(2n)\times O(2n)$ is equivalent to specifying a positive definite metric on $T \oplus T^*$ which is compatible with the pre-existing inner product, i.e. $G^2 = 1$.*

Now suppose that we have a generalized almost complex structure $\mathcal{J}$ defining a further reduction to $U(n,n) \subset O(n,n)$. To now reduce to $U(n) \times U(n)$ is equivalent to choosing a metric G as above, which commutes with the generalized complex structure $\mathcal{J}$. This is the same as choosing the space

C_+ to be stable under $\mathcal{J}$. Since C_+ is stable under $\mathcal{J}$ we see that $\mathcal{J}$ is orthogonal with respect to G and so we obtain a Hermitian structure on $T \oplus T^*$ compatible with the pre-existing inner product.

Note that since $G^2 = 1$ and $G\mathcal{J} = \mathcal{J}G$, the map $G\mathcal{J}$ squares to -1, and since G is symmetric while $\mathcal{J}$ is skew, $G\mathcal{J}$ is also skew, and therefore defines a generalized almost complex structure. Hence we have the following:

Proposition 6.2. *A reduction to $U(n)\times U(n)$ is equivalent to the existence of two generalized almost complex structures $\mathcal{J}_1, \mathcal{J}_2$ as well as a positive definite metric G satisfying $G^2 = 1$, which are related by the following commuting diagram:*

$$\begin{tikzcd} T \oplus T^* \arrow[rr, "G"] & & T \oplus T^* \\ & T \oplus T^* \arrow[ul, "\mathcal{J}_1"] \arrow[ur, "\mathcal{J}_2"'] & \end{tikzcd}$$

Note that the conditions on G are equivalent to requiring that $\mathcal{J}_1$ and $\mathcal{J}_2$ commute and that $-\mathcal{J}_1\mathcal{J}_2$ is positive definite.

We are now in a position to impose the integrability condition on the $U(n) \times U(n)$ structure which defines generalized Kähler structure. We simply require that both $\mathcal{J}_1$ and $\mathcal{J}_2$ are integrable.

Definition 6.3. *A generalized Kähler structure is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of commuting generalized complex structures such that $G = -\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric on $T \oplus T^*$.*

Our first example of a generalized Kähler structure justifies the nomenclature.

Example 6.4. Let (g, J, ω) be a usual Kähler structure on a manifold, i.e. a Riemannian metric g, a complex structure J, and a symplectic structure ω such that the following diagram commutes.

$$\begin{tikzcd} T \arrow[rr, "g"] & & T^* \\ & T \arrow[ul, "J"] \arrow[ur, "\omega"'] & \end{tikzcd}$$

Then forming the generalized complex structures

$$\mathcal{J}_J = \begin{pmatrix} J & \\ & -J^* \end{pmatrix}, \quad \mathcal{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix},$$

we see immediately that $\mathcal{J}_J, \mathcal{J}_\omega$ commute and

$$G = -\mathcal{J}_J\mathcal{J}_\omega = \begin{pmatrix} & g^{-1} \\ g & \end{pmatrix}$$

is a positive definite metric on $T \oplus T^*$. Hence $(\mathcal{J}_J, \mathcal{J}_\omega)$ defines a generalized Kähler structure.

Example 6.5. Given any generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$, we may transform it by a B-field, for B any closed 2-form: $(\mathcal{J}_1^B, \mathcal{J}_2^B) = (\mathcal{B}\mathcal{J}_1\mathcal{B}^{-1}, \mathcal{B}\mathcal{J}_2\mathcal{B}^{-1})$ is also generalized Kähler. Applying such a transformation to the first example $(\mathcal{J}_J, \mathcal{J}_\omega)$, we obtain the following generalized complex structures

$$\mathcal{J}_J^B = \begin{pmatrix} J & \\ BJ + J^*B & -J^* \end{pmatrix}, \quad \mathcal{J}_\omega^B = \begin{pmatrix} \omega^{-1}B & -\omega^{-1} \\ \omega + B\omega^{-1}B & -B\omega^{-1} \end{pmatrix}. \tag{6.1}$$

Similarly, the metric G becomes

$$G^B = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix},$$

showing that the metric G in a generalized Kähler structure need not be diagonal. Note also that the restriction of G^B to the tangent bundle is the component $g - Bg^{-1}B$, which is indeed a Riemannian metric for any two-form B.

6.2 Torsion and the generalized Kähler metric

The last example gives an indication of the general form of a generalized Kähler metric. Let $(\mathcal{J}_1, \mathcal{J}_2)$ be any generalized Kähler structure and write the metric $G = -\mathcal{J}_1\mathcal{J}_2$ as follows.

$$G = \begin{pmatrix} A & g^{-1} \\ \sigma & A^* \end{pmatrix},$$

where g, σ are genuine Riemannian metrics on the manifold and A is an endomorphism of T. The condition $G^2 = 1$ implies that A is skew-symmetric with respect to both metrics g and σ, and that if we define the 2-form $b = -gA$, we can write

$$G = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \\ b & 1 \end{pmatrix} \begin{pmatrix} & g^{-1} \\ g & \end{pmatrix} \begin{pmatrix} 1 & \\ -b & 1 \end{pmatrix}.$$

We see from this argument that any generalized Kähler metric is uniquely determined by a Riemannian metric g together with a 2-form b. We may also see g and b as follows: The $+1$-eigenbundle C_+ of G is positive definite in the natural inner product, and since the tangent bundle is isotropic, C_+ can be expressed as the graph of a positive definite linear map from T to T^*.

Proposition 6.6. *$C_\pm$ is the graph of $b \pm g : T \longrightarrow T^*$*

Proof. Let $X + \xi \in C_+$, so that

$$\begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = \begin{pmatrix} X \\ \xi \end{pmatrix}.$$

The equation in vector fields states that $-g^{-1}bX + g^{-1}\xi = X$, i.e. $\xi = (b + g)X$, as required. The equation in 1-forms is automatically satisfied. Similarly for C_-. □

It may seem from the discussion above that any generalized Kähler metric is the B-field transform of a bare Riemannian metric, but this is not the case, as the 2-form b need not be closed. We will present an example of this in a later section. The derivative of b actually plays an important role in generalized Kähler geometry.

Definition 6.7. *The* torsion *of a generalized Kähler structure is the 3-form $h = db$.*

6.3 Courant integrability

We wish to describe the meaning of the generalized Kähler condition in terms of subbundles of $(T \oplus T^*) \otimes \mathbb{C}$. As usual, $\mathcal{J}_1$ and $\mathcal{J}_2$ engender a decomposition into $\pm i$ eigenbundles:

$$\begin{aligned} (T \oplus T^*) \otimes \mathbb{C} &= L_1 \oplus \overline{L_1} \\ &= L_2 \oplus \overline{L_2}. \end{aligned}$$

Since $\mathcal{J}_1$ and $\mathcal{J}_2$ commute, L_1 must decompose into $\pm i$ eigenbundles of $\mathcal{J}_2$, which we denote $L_1^\pm$. Then we have the following decomposition into four isotropic subbundles:

$$(T \oplus T^*) \otimes \mathbb{C} = L_1^+ \oplus L_1^- \oplus \overline{L_1^+} \oplus \overline{L_1^-}.$$

Note that $L_2 = L_1^+ \oplus \overline{L_1^-}$. Since $C_\pm$ is the ± 1 eigenbundle of $G = -\mathcal{J}_1\mathcal{J}_2$, we see that

$$C_\pm \otimes \mathbb{C} = L_1^\pm \oplus \overline{L_1^\pm},$$

which, incidentally, proves that $\operatorname{rk} L_1^+ = \operatorname{rk} L_1^- = n$, yielding the following result.

Proposition 6.8. *The generalized complex structures $\mathcal{J}_1, \mathcal{J}_2$ in a generalized Kähler pair must have the same parity if n is even and must have opposite parity if n is odd. For example, on a 4-dimensional manifold, the two generalized complex structures comprising the $U(2) \times U(2)$ structure must have the same parity.*

Proof. The rank n bundle L_1^+ is the intersection of L_1 and L_2, hence if n is even, the maximal isotropics L_1 and L_2 must have the same parity, and similarly for n odd they must have opposite parity. □

The positive-definiteness condition, in terms of $L_1^\pm$, becomes

$$\langle p, \bar{p} \rangle - \langle q, \bar{q} \rangle \geq 0,$$

for all $p \in L_1^+$, $q \in L_1^-$, with equality if and only if $p = q = 0$. In other words, $\pm\langle x, \bar{x} \rangle > 0$ for all nonzero $x \in L_1^\pm$.

Furthermore, since $L_1^+ = L_1 \cap L_2$ and $L_1^- = L_1 \cap \overline{L_2}$, each of $L_1^\pm$ is closed under the Courant bracket. Using this information, we obtain a useful characterization of a generalized Kähler structure. First we need a useful lemma which generalizes the decomposition $d = \partial + \overline{\partial}$ for complex manifolds:

Definition 6.9. *Let L be a complex Lie algebroid with bracket $[,]$ and differential $d_L : C^\infty(\wedge^k L^*) \longrightarrow C^\infty(\wedge^{k+1} L^*)$. If $L = L^+ \oplus L^-$ then we can define $\wedge^{p,q}(L^*) = \wedge^p(L^+)^* \otimes \wedge^q(L^-)^*$, as well as the operators*

$$\begin{aligned}
\partial_L^+ &= \pi_{p+1,q} \circ d_L : C^\infty(\wedge^{p,q} L^*) \longrightarrow C^\infty(\wedge^{p+1,q} L^*) \\
\partial_L^- &= \pi_{p,q+1} \circ d_L : C^\infty(\wedge^{p,q} L^*) \longrightarrow C^\infty(\wedge^{p,q+1} L^*),
\end{aligned}$$

where $\pi_{p,q}$ is the projection $\wedge^{p+q} L^* \longrightarrow \wedge^{p,q} L^*$. If $L^\pm$ are closed under the Lie bracket, then we have the equality

$$d_L = \partial_L^+ + \partial_L^-.$$

Proposition 6.10. *A generalized Kähler structure on a real $2n$-dimensional manifold is equivalent to the specification of two complex rank n subbundles L_1^+, L_1^- of $(T \oplus T^*) \otimes \mathbb{C}$ satisfying*

- *$L_1^\pm$ are isotropic, and*
- *$L_1^+ \perp L_1^-$ and $L_1^+ \perp \overline{L_1^-}$, where $\perp$ indicates orthogonality with respect to the inner product, and*
- *$L_1^\pm$ are $\pm$-definite, in the sense that*

$$\pm\langle x, \bar{x} \rangle > 0 \quad \forall x \in L_1^\pm,\ x \neq 0, \tag{6.2}$$

with the integrability condition that both $L_1^\pm$ are closed under the Courant bracket, and also that $L_1^+ \oplus L_1^-$ is closed under the Courant bracket.

Proof. As we saw above, a generalized Kähler structure certainly provides two subbundles with the required properties. We show the converse. Given subbundles as above, condition (6.2) implies that $L_1^{\pm}, \overline{L_1^{\pm}}$ are all mutually transverse, and that $L_1 = L_1^+ \oplus L_1^-$ and $L_2 = L_1^+ \oplus \overline{L_1^-}$ are maximally isotropic, defining two commuting almost generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ respectively. The condition that $L_1^+ \oplus L_1^-$ is integrable implies that $\mathcal{J}_1$ is integrable, and we must show that $\mathcal{J}_2$ is integrable as well. For this we use the expression for the Courant bracket in the presence of the dual Lie algebroid splitting $L_1 \oplus \overline{L_1} = L_1 \oplus L_1^*$ (we identify $\overline{L_1} = L_1^*$ using the inner product):

$$\begin{aligned}[A+\alpha, B+\beta] = [A,B] + \mathcal{L}_\alpha B - \mathcal{L}_\beta A - d_{\overline{L_1}}(\langle A,\beta\rangle - \langle B,\alpha\rangle)\\ + [\alpha,\beta] + \mathcal{L}_A\beta - \mathcal{L}_B\alpha + d_{L_1}(\langle A,\beta\rangle - \langle B,\alpha\rangle),\end{aligned}$$

where $A, B \in C^\infty(L_1)$ and $\alpha, \beta \in C^\infty(\overline{L_1})$. Now, as a special case, take $A \in C^\infty(L_1^+)$ and $\beta \in C^\infty(\overline{L_1^-})$, setting $B = \alpha = 0$. Then, using the fact that L_1^+ and $\overline{L_1^-}$ are orthogonal,

$$\begin{aligned}[A,\beta] &= \mathcal{L}_A\beta - \mathcal{L}_\beta A\\ &= i_A d_{L_1}\beta - i_\beta d_{L_1^*}A.\end{aligned}$$

where we recall that $\mathcal{L}_A$ is defined by the Cartan formula $\mathcal{L}_A = i_A d_{L_1} + d_{L_1} i_A$. Since $L_1^{\pm}$ are closed under the bracket, we can use the decomposition of d_{L_1} described in the previous lemma, obtaining

$$\begin{aligned}[A,\beta] &= i_A(\partial^+_{L_1} + \partial^-_{L_1})\beta + i_\beta(\partial^+_{L_1^*} + \partial^-_{L_1^*})A\\ &= i_A\partial^+_{L_1}\beta + i_\beta\partial^-_{L_1^*}A\end{aligned}$$

showing that $[A,\beta] \in C^\infty(L_1^+ \oplus \overline{L_1^-})$, and therefore that $L_2 = L_1^+ \oplus \overline{L_1^-}$ is also closed under the Courant bracket, i.e. $\mathcal{J}_2$ is integrable. □

6.4 Relation to bi-Hermitian geometry

In this section we begin with a generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ and 'project' it to the tangent bundle, obtaining more familiar types of structures, but with interesting integrability conditions.

6.4.1 The algebraic structure

Since the bundle C_+ is positive definite while T is null, the projection $\pi : T \oplus T^* \longrightarrow T$ induces isomorphisms

$$\pi : C_\pm \xrightarrow{\cong} T\ .$$

This means that any algebraic structure existing on $C_\pm$ may be transported to the tangent bundle.

- (Metric and 2-form) As $C_\pm$ are definite subspaces of $T \oplus T^*$, and as there are natural symmetric and skew-symmetric inner products on $T \oplus T^*$, namely

$$\begin{aligned}\langle X+\xi, Y+\eta\rangle_+ &= \frac{1}{2}(\xi(Y) + \eta(X))\\ \langle X+\xi, Y+\eta\rangle_- &= \frac{1}{2}(\xi(Y) - \eta(X)),\end{aligned}$$

  we obtain natural Riemannian metrics and 2-forms on both of $C_\pm$ by restriction. If we transport these structures via π to the tangent bundle, we obtain precisely the metric and 2-form discussed in proposition 6.6, that is, projecting from $C_\pm$ yields $b \pm g$, with g a Riemannian metric on T and b a 2-form.

- (Compatible almost complex structures) Since $C_\pm$ are stable under both $\mathcal{J}_1$ and $\mathcal{J}_2$, they carry complex structures which are each compatible with the metric $\langle,\rangle_+$, in the sense that each is an orthogonal transformation. Note that $\mathcal{J}_1 = \mathcal{J}_2$ on C_+ and $\mathcal{J}_1 = -\mathcal{J}_2$ on C_-, so we only need to project one of them, say $\mathcal{J}_1$. By projection from $C_\pm$, $\mathcal{J}_1$ induces two almost complex structures on T, which we denote $J_\pm$, and these are compatible with the induced Riemannian metric g, so we will call them Hermitian almost complex structures.

In fact, as we now show, algebraically a $U(n)\times U(n)$ structure is equivalent to the specification of the quadruple (g,b,J_+,J_-), that is, a Riemannian metric g, a 2-form b, and two Hermitian almost complex structures $J_\pm$. One could call this structure an 'almost bi-Hermitian structure with b-field'.

Definition 6.11. *Let $\omega_\pm$ be the 2-forms associated to the Hermitian almost complex structures $J_\pm$, i.e.*

$$\omega_\pm = gJ_\pm.$$

Proposition 6.12. *The generalized Kähler structure $(\mathcal{J}_1,\mathcal{J}_2)$ can be reconstructed from the data (g,b,J_+,J_-).*

Proof. The maps $b\pm g$ determine the metric G by the formula

$$G = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g-bg^{-1}b & bg^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \\ b & 1 \end{pmatrix}\begin{pmatrix} & g^{-1} \\ g & \end{pmatrix}\begin{pmatrix} 1 & \\ -b & 1 \end{pmatrix},$$

and therefore they determine $C_\pm$. Using π we can reconstruct $\mathcal{J}_1$ by defining it to be the transport of J_+ on C_+ and the transport of J_- on C_-. To reconstruct $\mathcal{J}_2$ we use J_+ on C_+ and $-J_-$ on C_-. In formulae:

$$\begin{aligned} \mathcal{J}_1 &= \pi|_{C_+}^{-1} J_+\pi P_+ + \pi|_{C_-}^{-1} J_-\pi P_- \\ \mathcal{J}_2 &= \pi|_{C_+}^{-1} J_+\pi P_+ - \pi|_{C_-}^{-1} J_-\pi P_-, \end{aligned}$$

where $P_\pm$ are the projections from $T\oplus T^*$ to $C_\pm$, namely $P_\pm = \frac{1}{2}(1\pm G)$. Using these formulae we are able to write $\mathcal{J}_1$ and $\mathcal{J}_2$ explicitly:

$$\mathcal{J}_{1/2} = \frac{1}{2}\begin{pmatrix} 1 & \\ b & 1 \end{pmatrix}\begin{pmatrix} J_+\pm J_- & -(\omega_+^{-1}\mp\omega_-^{-1}) \\ \omega_+\mp\omega_- & -(J_+^*\pm J_-^*) \end{pmatrix}\begin{pmatrix} 1 & \\ -b & 1 \end{pmatrix}. \tag{6.3}$$

□

Remark 6.13. *From equation (6.3) we see that the degenerate case where the almost complex structures J_+,J_- are equal or conjugate ($J_+ = \pm J_-$) corresponds to $(\mathcal{J}_1,\mathcal{J}_2)$ being the B-field transform of a genuine Kähler structure.*

Before proceeding to the integrability conditions, we mention here a fact about the relationship between the parity of the generalized Kähler pair and the orientations of the almost complex structures $J_\pm$.

Remark 6.14. *As we have seen, in dimension $4k$ the generalized Kähler pair must have the same parity, i.e. must have parity (even,even) or (odd,odd). In the former case, the complex structures $J_\pm$ induce the same orientation on the manifold. In the latter case, $J_\pm$ induce opposite orientations.*

In dimension $4k+2$, the parities of the generalized complex structures must be different, and this places no restriction on the orientations of $J_\pm$; indeed changing the sign of either $J_\pm$ reverses its orientation in this dimension.

6.4.2 The integrability condition: part I

Now that we can express the generalized Kähler structure in terms of an almost bi-Hermitian structure with b-field, we must describe what the Courant integrability condition implies for the quadruple $(g, b, J_\pm)$.

The first observation is that since the map $\pi : T \oplus T^* \longrightarrow T$ is the Courant algebroid anchor, it satisfies the condition

$$\pi[X + \xi, Y + \eta] = [\pi(X + \xi), \pi(Y + \eta)] = [X, Y].$$

Therefore we see that if a subbundle of $T \oplus T^*$ is closed under the Courant bracket, then its image under π will be closed under the Lie bracket. Immediately we have the following result.

Proposition 6.15. *The complex structures J_+, J_- coming from a generalized Kähler structure are integrable, and therefore (g, J_+, J_-) is a bi-Hermitian structure.*

Proof. The integrability condition for the complex structure J_+ is that the $+i$ eigenbundle in the complexified tangent bundle is closed under the Lie bracket. We denote this subbundle by $T_+^{1,0} < T \otimes \mathbb{C}$. Since J_+ is obtained from the complex structure $\mathcal{J}_1\big|_{C_+}$ via the isomorphism $\pi|_{C_+}$, we see that the preimage of $T_+^{1,0}$ is the bundle $L_1^+ < (T \oplus T^*) \otimes \mathbb{C}$:

$$\pi : L_1^+ \xrightarrow{\cong} T_+^{1,0} .$$

But we know that the bundle L_1^+ is closed under the Courant bracket. Hence $T_+^{1,0}$ is closed under the Lie bracket, as required. Similarly for J_-. $\square$

To obtain the full integrability conditions it will be useful to explicitly describe the data entering into proposition 6.10 in terms of the bi-Hermitian structure with b-field. To this end, we observe from proposition 6.6 that the bundle L_1^+ may be described as the graph of $g + b$ thought of as a map from $T_+^{1,0}$ to $T^* \otimes \mathbb{C}$:

$$\begin{aligned} L_1^+ &= \{X + (b + g)X \mid X \in C^\infty(T_+^{1,0})\} \\ &= \{X + (b - i\omega_+)X \mid X \in C^\infty(T_+^{1,0})\}, \end{aligned} \tag{6.4}$$

where in the second line we use the fact that $g = -\omega_+ J_+$. We can use the same argument for L_1^-, obtaining the expression

$$\begin{aligned} L_1^- &= \{X + (b - g)X \mid X \in C^\infty(T_-^{1,0})\} \\ &= \{X + (b + i\omega_-)X \mid X \in C^\infty(T_-^{1,0})\}. \end{aligned} \tag{6.5}$$

Now that we have expressed the bundles $L_1^\pm$ in terms of the bi-Hermitian data, we must discover the meaning of the three integrability conditions required by proposition 6.10, namely that L_1^+, L_1^-, and $L_1^+ \oplus L_1^-$ are all Courant integrable. The first two conditions are easily understood, since each bundle $L_1^\pm$ has been expressed as the graph of a complex two-form, and we may use the following result about Courant integrability:

Proposition 6.16. *The subbundle of $(T \oplus T^*) \otimes \mathbb{C}$ defined by*

$$F = \{X + cX \mid X \in C^\infty(E)\},$$

for some complex 2-form c and subbundle E of $T \otimes \mathbb{C}$, is Courant integrable if and only if E is Lie integrable and the form c satisfies

$$i_Y i_X dc = 0 \ \ \forall \ X, Y \in C^\infty(E).$$

Based on this, we obtain equivalent conditions to the Courant integrability of $L_1^{\pm}$:

Proposition 6.17. *Given an almost generalized Kähler structure* $(\mathcal{J}_1, \mathcal{J}_2)$ *and its related almost bi-Hermitian structure* $(g, b, J_\pm)$*, the bundles* $L_1^{\pm}$ *are Courant integrable if and only if*

- $T_{\pm}^{1,0}$ *are Lie integrable, i.e.* $J_\pm$ *are integrable complex structures, and*
- *The three 2-forms* $\omega_\pm, b$ *satisfy the conditions*

$$d_-^c \omega_- = -d_+^c \omega_+ = db, \tag{6.6}$$

where $d_\pm^c = i(\bar{\partial}_\pm - \partial_\pm)$*, and* $\partial_\pm$ *is the* ∂*-operator for the complex structure* $J_\pm$*. Condition (6.6) may also be written as follows:*

$$db(X, Y, Z) = d\omega_+(J_+X, J_+Y, J_+Z) = -d\omega_-(J_-X, J_-Y, J_-Z) \tag{6.7}$$

for all vector fields X, Y, Z*.*

Proof. Proposition 6.16 implies that $L_1^{\pm}$ is Courant integrable if and only if $J_\pm$ is integrable and the following condition holds:

$$i_Y i_X d(b \mp i\omega_\pm) = 0, \quad \forall X, Y \in C^\infty(T_\pm^{1,0}).$$

Using the (p, q) decomposition of forms determined by $J_\pm$, we obtain the equivalent condition

$$db \pm d_\pm^c \omega_\pm = 0,$$

as required. To obtain the final equation (6.7), note that since $\omega_\pm$ is of type $(1, 1)$, $d\omega_\pm$ is of type $(2, 1) + (1, 2)$. The $(2, 1)$ forms have eigenvalue i under $\wedge^3 J_\pm^*$ while the $(1, 2)$ forms have eigenvalue $-i$. Hence when acting on a $(1, 1)$ form, $d_\pm^c = i(\bar{\partial}_\pm - \partial_\pm)$ is the same as $-\wedge^3 J_\pm^* \circ d$, that is,

$$d_\pm^c \omega_\pm(X, Y, Z) = -d\omega_\pm(J_\pm X, J_\pm Y, J_\pm Z),$$

as required. □

An immediate result of these relations is a constraint on the *torsion* $h = db$ of a generalized Kähler structure, namely:

Corollary 6.18. *The torsion* h *of a generalized Kähler structure is of type* $(2, 1) + (1, 2)$ *with respect to both complex structures* $J_\pm$*; equivalently, it satisfies the condition*

$$h(X, Y, Z) = h(X, JY, JZ) + h(JX, Y, JZ) + h(JX, JY, Z) \tag{6.8}$$

for all vector fields X, Y, Z*, and for both complex structures* $J = J_\pm$*. Note that this is equivalent to*

$$h(JX, JY, JZ) = h(JX, Y, Z) + h(X, JY, Z) + h(X, Y, JZ). \tag{6.9}$$

Also, we obtain a description of the degenerate case when $h = 0$:

Corollary 6.19. *For a generalized Kähler structure with data* $(g, b, J_\pm)$*, the following are equivalent:*

- $h = db = 0$
- (J_+, g) *is Kähler*
- (J_-, g) *is Kähler.*

Now that we have proven that the integrability of $L_1^{\pm}$ implies the existence of a bi-Hermitian structure with the additional constraint (6.6), we ask what additional constraint arises from the final integrability condition in Proposition 6.10, namely, that $L_1^+ \oplus L_1^-$ must be Courant integrable. In fact we intend to show that no additional constraint arises; indeed, this last condition is redundant. However, to do this, we must develop some of the Riemannian geometry on the generalized Kähler manifold.

6.4.3 The Bismut connection versus the Levi-Civita connection

Unless a Hermitian manifold is Kähler, the Levi-Civita connection is not a $U(n)$ connection. However there are several canonically defined $U(n)$ connections on a Hermitian manifold, including the Chern connection (the unique $U(n)$ connection with torsion τ such that $\tau(JX, JY) = -\tau(X, Y)$), and more importantly for us, the Bismut connection, defined as follows.

Definition 6.20. *The Bismut connection associated to a Hermitian structure is the unique $U(n)$ connection with totally skew-symmetric torsion. See [6] for details.*

In the case of a Kähler structure, all these connections coincide, as we now recall.

Proposition 6.21. *Let (g, J, ω) be an almost Hermitian structure. Then the following are equivalent:*

- $\nabla J = 0$*, where* ∇ *is the Levi-Civita connection*
- $N_J = 0$ *and* $d\omega = 0$*, i.e.* (g, J, ω) *is Kähler.*

It will be useful to precisely describe the difference between the Levi-Civita and the Bismut connection for a general Hermitian structure. To do this, we generalize Proposition 6.21. We need two lemmas:

Lemma 6.22. *Let (g, J, ω) be an almost Hermitian structure. Then if N_J is the Nijenhuis tensor of J and ∇ is the Levi-Civita connection,*

$$N_J(X, Y) = (\nabla_{JX} J)Y - (\nabla_{JY} J)X + (\nabla_X J)JY - (\nabla_Y J)JX, \tag{6.10}$$

for any vector fields X, Y.

Proof. By definition, the Nijenhuis tensor is

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y].$$

Using the fact that ∇ is torsion-free, we obtain

$$\begin{aligned} N_J(X, Y) &= \nabla_{JX} JY - \nabla_{JY} JX - J(\nabla_{JX} Y - \nabla_Y JX) - J(\nabla_X JY - \nabla_{JY} X) - (\nabla_X Y - \nabla_Y X) \\ &= (\nabla_{JX} J)Y - (\nabla_{JY} J)X - J(\nabla_X J)Y + J(\nabla_Y J)X. \end{aligned}$$

Differentiating $J^2 = -1$ we obtain $(\nabla_X J)J + J(\nabla_X J) = 0$, which provides the result. ∎

Lemma 6.23. *Let (g, J, ω) be an almost Hermitian structure. Then if ∇ is the Levi-Civita connection,*

$$d\omega(X, Y, Z) = g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y), \tag{6.11}$$

for any vector fields X, Y, Z.

Proof. The exterior derivative is defined such that

$$d\omega(X,Y,Z) = \nabla_X\omega(Y,Z) - \omega([X,Y],Z) \ + \ \text{c.p.},$$

where 'c.p.' stands for cyclic permutations. Using the facts that $gJ = \omega$, that ∇ is metric, and finally that ∇ is torsion free, we obtain

$$\begin{aligned} d\omega(X,Y,Z) &= g(\nabla_X(JY),Z) + g(JY,\nabla_X Z) - \omega([X,Y],Z) \ + \ \text{c.p.} \\ &= g((\nabla_X J)Y,Z) + g(J\nabla_X Y,Z) + g(JY,\nabla_X Z) - \omega([X,Y],Z) \ + \ \text{c.p.} \\ &= g((\nabla_X J)Y,Z) + g(J\nabla_X Y,Z) + g(JZ,\nabla_Y X) - \omega([X,Y],Z) \ + \ \text{c.p.} \\ &= g((\nabla_X J)Y,Z) + g(J[X,Y],Z) - \omega([X,Y],Z) \ + \ \text{c.p.} \\ &= g((\nabla_X J)Y,Z) \ + \ \text{c.p.}, \end{aligned}$$

as required. □

Proposition 6.24. *Let h be any 3-form and let (g,J,ω) be an almost Hermitian structure. Consider the connection*

$$\nabla^h = \nabla + \tfrac{1}{2}g^{-1}h. \tag{6.12}$$

This is a metric connection with torsion $g^{-1}h$ and the following are equivalent:

- $\nabla^h J = 0$ *(or $\nabla^h\omega = 0$).*
- $N_J = 4g^{-1}h^{(3,0)+(0,3)}$ *and* $d\omega^{(2,1)+(1,2)} = -ih^{(2,1)} + ih^{(1,2)}$.

Proof. Substituting the definition of ∇^h into equation (6.10), we obtain the expression

$$\begin{aligned} g(N_J(X,Y),Z) &= g((\nabla^h_{JX}J)Y - (\nabla^h_{JY}J)X + (\nabla^h_X J)JY - (\nabla^h_Y J)JX, Z) \\ &\quad - h(JX,JY,Z) - h(JX,Y,JZ) - h(X,JY,JZ) + h(X,Y,Z) \\ &= g((\nabla^h_{JX}J)Y - (\nabla^h_{JY}J)X + (\nabla^h_X J)JY - (\nabla^h_Y J)JX, Z) + 4h^{(3,0)+(0,3)}(X,Y,Z), \end{aligned}$$

showing that if $\nabla^h J = 0$, then $N_J = 4g^{-1}h^{(3,0)+(0,3)}$. Substituting the definition of ∇^h into equation (6.11), we obtain the expression

$$\begin{aligned} d\omega(X,Y,Z) &= g((\nabla^h_X J)Y,Z) + g((\nabla^h_Y J)Z,X) + g((\nabla^h_Z J)X,Y) \\ &\quad - h(JX,Y,Z) - h(X,JY,Z) - h(X,Y,JZ), \end{aligned}$$

showing that if $\nabla^h J = 0$, then $d\omega^{(2,1)+(1,2)} = -ih^{(2,1)} + ih^{(1,2)}$, as required.

To show the converse, we combine equations (6.10) and (6.11) to express ∇J in terms of N_J and $d\omega$:

$$\begin{aligned} g(N_J(X,Y),Z) &= g((\nabla_{JX}J)Y - (\nabla_{JY}J)X + (\nabla_X J)JY - (\nabla_Y J)JX, Z) \\ &= d\omega(JX,Y,Z) + d\omega(X,JY,Z) - 2g((\nabla_Z J)X,JY). \end{aligned}$$

Now, using the definition of ∇^h:

$$\begin{aligned} 2g((\nabla^h_Z J)X,JY) &= 2g(((\nabla_Z - \tfrac{1}{2}g^{-1}h_Z)J)X,JY) \\ &= d\omega(JX,Y,Z) + d\omega(X,JY,Z) - g(N_J(X,Y),Z) - h(JX,JY,Z) + h(X,Y,Z), \end{aligned}$$

which vanishes if we use the expressions for N_J in terms of $h^{(3,0)+(0,3)}$ and $d\omega^{(2,1)+(1,2)}$ in terms of $h^{(2,1)+(1,2)}$, as well as the standard fact that for any almost Hermitian structure, the skew part of the Nijenhuis tensor is related to $d\omega^{(3,0)}$, i.e.

$$(i(d\omega)^{(3,0)} - i(d\omega)^{(0,3)})(X,Y,Z) = \tfrac{1}{4}(g(N_J(X,Y),Z) + \ \text{c.p.}) \tag{6.13}$$

This completes the proof. □

This proposition finally allows us to identify the Bismut connection of any Hermitian structure:

Corollary 6.25. *Let (g, J, ω) be a Hermitian structure. Then the Bismut connection is given by $\nabla^h = \nabla + \frac{1}{2}g^{-1}h$, where $h = -d^c\omega = -i(\bar{\partial} - \partial)\omega$, and ∇ is the Levi-Civita connection.*

On a generalized Kähler manifold there are two Hermitian structures (g, J_+, J_-) and therefore two Bismut connections, which we denote by $\nabla^{\pm}$. By our work on Hermitian structures, we see that the torsion 3-form of the generalized Kähler structure serves also as the torsion of the Bismut connections:

Theorem 6.26. *Let $(g, b, J_\pm)$ derive from an almost generalized Kähler structure. Define $h = db$ and connections as follows*

$$\nabla^{\pm} = \nabla \pm \tfrac{1}{2}g^{-1}h.$$

These are metric connections with torsion

$$\tau(\nabla^{\pm}) = \pm g^{-1}h.$$

The following are equivalent:

- *$J_\pm$ are integrable and $d^c_-\omega_- = -d^c_+\omega_+ = h$,*
- *$\nabla^{\pm}J_\pm = 0$ and h is of type $(2,1) + (1,2)$ with respect to both $J_\pm$.*

Finally, if these conditions hold, $\nabla^{\pm}$ become the Bismut connections.

Proof. Let $J_\pm$ be integrable. Then we have two Hermitian structures, with Bismut connections $\nabla^{\pm} = \nabla - \frac{1}{2}g^{-1}d^c\omega_\pm$ such that $\nabla^{\pm}J_\pm = 0$. But the condition $d^c_-\omega_- = -d^c_+\omega_+ = h$ implies that $\nabla^{\pm} = \nabla \pm g^{-1}h$, and also that h is of type $(2,1) + (1,2)$, as required. For the converse, $\nabla^{\pm}J_\pm = 0$ implies that $N_{J_\pm} = \pm 4g^{-1}h^{(3,0)+(0,3)}$ and $d\omega_\pm^{(2,1)+(1,2)} = \mp ih^{(2,1)} \pm ih^{(1,2)}$. If h is of type $(2,1) + (1,2)$, then $N_J = 0$ and $d^c_\mp\omega_\mp = \pm h$, as required. □

Remark 6.27. *We now see that generalized Kähler geometry naturally induces a bi-Hermitian structure whose integrability can be characterized by two connections with totally skew-symmetric torsion which average to the Levi-Civita connection:*

$$\frac{\nabla^+ + \nabla^-}{2} = \nabla.$$

This kind of geometry was first introduced in [42] and arises in physics as the natural geometry present on the target of a N=(2,2) non-linear sigma model with torsion.

6.4.4 The integrability condition: part II

Having described the relevance of torsion connections to the generalized Kähler geometry, we may now dispense with the remaining integrability condition, that $L_1^+ \oplus L_1^-$ be Courant integrable, obtaining the full set of conditions for generalized Kähler structures.

Theorem 6.28. *Let $(\mathcal{J}_1, \mathcal{J}_2)$ be an almost generalized Kähler structure, with associated data $(g, b, J_\pm)$, and let $h = db$. Any one of the following conditions is equivalent to the integrability of the generalized Kähler structure:*

i) The bundles $L_1^{\pm}$ are Courant integrable.

ii) $J_\pm$ are integrable and $d^c_-\omega_- = -d^c_+\omega_+ = h$.

iii) $\nabla^{\pm} J_{\pm} = 0$ *and h is of type $(2,1)+(1,2)$ with respect to both $J_{\pm}$ (the latter condition is implied by $J_{\pm}$ integrable).*

Proof. We have shown that integrability implies condition $i)$, and we have shown that $i), ii), iii)$ are equivalent. All that remains is to prove that one of $i), ii), iii)$ implies that $L_1^+ \oplus L_1^-$ is Courant integrable.

Let $X+\xi \in C^\infty(L_1^+)$ and $Y+\eta \in C^\infty(L_1^-)$. We now show that $iii)$ implies that $[X+\xi, Y+\eta] \in C^\infty(L_1^+ \oplus L_1^-)$, which would complete the proof. Recall that by expressions (6.4) and (6.5), $X+\xi = X+(b+g)X$ and $Y+\eta = Y+(b-g)Y$, and so

$$\begin{aligned}
[X+\xi, Y+\eta] &= [X+gX+bX, Y-gY+bX] \\
&= [X+gX, Y-gY] + b([X,Y]) + i_Y i_X db \\
&= [X,Y] - \mathcal{L}_X(gY) - \mathcal{L}_Y(gX) + d(g(X,Y)) + b([X,Y]) + i_Y i_X h.
\end{aligned}$$

First we consider the tangent vector component. Since $X+\xi \in C^\infty(L_1^+)$, by projection $X \in C^\infty(T_+^{1,0})$. Similarly, $Y \in C^\infty(T_-^{1,0})$. Since the Levi-Civita connection is torsion free, we have

$$\begin{aligned}
[X,Y] &= \nabla_X Y - \nabla_Y X \\
&= (\nabla_X^- + \tfrac{1}{2} g^{-1} h_X)Y - (\nabla_Y^+ - \tfrac{1}{2} g^{-1} h_Y)X \\
&= \nabla_X^- Y - \nabla_Y^+ X + \tfrac{1}{2} g^{-1}(h(X,Y) + h(Y,X)) \\
&= \nabla_X^- Y - \nabla_Y^+ X.
\end{aligned}$$

If we assume $\nabla^{\pm} J_{\pm} = 0$, then $\nabla_X^- Y \in C^\infty(T_-^{1,0})$ and $\nabla_Y^+ X \in C^\infty(T_+^{1,0})$.

Now, consider the 1-form component

$$\begin{aligned}
i_Z(-\mathcal{L}_X(gY) - \mathcal{L}_Y(gX) + d(g(X,Y))) &= -i_Z\mathcal{L}_X(gY) - i_Z\mathcal{L}_Y(gX) + g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \\
&= -i_Z\mathcal{L}_X(gY) - i_Z\mathcal{L}_Y(gX) + g([Z,X],Y) + g(X,[Z,Y]) + g(\nabla_X Z, Y) + g(X, \nabla_Y Z) \\
&= -\mathcal{L}_X i_Z(gY) - \mathcal{L}_Y i_Z(gX) + i_X d(i_Z(gY)) + i_Y d(i_Z(gX)) - g(Z, \nabla_X Y + \nabla_Y X) \\
&= -i_Z g(\nabla_X Y + \nabla_Y X).
\end{aligned}$$

Therefore, we obtain the expression

$$\begin{aligned}
[X+\xi, Y+\eta] &= [X,Y] - g(\nabla_X Y + \nabla_Y X) + b([X,Y]) + i_Y i_X h \\
&= \nabla_X^- Y - \nabla_Y^+ X + b(\nabla_X^- Y - \nabla_Y^+ X) - g(\nabla_X^- Y + \nabla_Y^+ X) \\
&= (-\nabla_Y^+ X + (b+g)(-\nabla_Y^+ X)) \;+\; ((\nabla_X^- Y + (b-g)(\nabla_X^- Y)),
\end{aligned}$$

which is clearly in $C^\infty(L_1^+ \oplus L_1^-)$, as required. This completes the proof. □

6.5 Examples of generalized Kähler structures

We will now provide some examples of generalized Kähler four-manifolds. Because of the equivalence between generalized Kähler structures and bi-Hermitian structures, this places us within the study of complex structures on Riemannian 4-manifolds. In four dimensions there are two kinds of generalized Kähler structure, as the pair of generalized complex structures may be of parity (even,even) or (odd,odd), as we showed in Proposition 6.8. We will concentrate on the case (even,even), which corresponds to the situation that both induced complex structures $J_{\pm}$ in the bi-Hermitian structure have the same orientation. A remarkable result of Salamon and Pontecorvo [38] is that once a Riemannian 4-manifold admits three distinct Hermitian complex structures with the same orientation, it must admit infinitely many. Note that by "distinct" we mean in the sense that $\{J_i\}_{i=1}^n$ are distinct when for each $i \neq j$ there exists at least one point $p \in M$ such that $J_i(p) \neq \pm J_j(p)$.

Theorem 6.29 ([38]). *Let (M, g) be a Riemannian 4-manifold. Then it may admit either 0, 1, 2, or infinitely many distinct integrable Hermitian complex structures with the same orientation. Furthermore, if it admits infinitely many, then it must be hyperhermitian and therefore admit an S^2 family of orthogonal complex structures: it must be either a flat torus, a $K3$ surface with its Ricci-flat Kähler metric, or a hyperhermitian Hopf surface.*

Of course any Kähler surface provides us with an example of a generalized Kähler 4-manifold; in this case the generalized Kähler pair is simply $(\mathcal{J}_J, \mathcal{J}_\omega)$, where J, ω are the complex and symplectic structures. It is not difficult to see that any Hyperkähler structure provides us with another example:

Example 6.30 (Hyperkähler). Let (M, g, I, J, K) be a hyperkähler structure. Then clearly (g, I, J) is a bi-Hermitian structure, and since $d\omega_I = d\omega_J = 0$, we see that (g, I, J) defines a generalized Kähler structure with $b = 0$. From formula (6.3), we reconstruct the generalized complex structures:

$$\mathcal{J}_{1/2} = \frac{1}{2}\begin{pmatrix} I \pm J & -(\omega_I^{-1} \mp \omega_J^{-1}) \\ \omega_I \mp \omega_J & -(I^* \pm J^*) \end{pmatrix}. \tag{6.14}$$

From the expression (6.1) for B-field transforms, we notice that (6.14) describes two generalized complex structures, each a B-field transform of a symplectic structure:

$$\mathcal{J}_1 = \begin{pmatrix} 1 & \\ \omega_K & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{2}(\omega_I^{-1} - \omega_J^{-1}) \\ \omega_I - \omega_J & 0 \end{pmatrix} \begin{pmatrix} 1 & \\ -\omega_K & 1 \end{pmatrix}$$
$$\mathcal{J}_2 = \begin{pmatrix} 1 & \\ -\omega_K & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{2}(\omega_I^{-1} + \omega_J^{-1}) \\ \omega_I + \omega_J & 0 \end{pmatrix} \begin{pmatrix} 1 & \\ \omega_K & 1 \end{pmatrix}.$$

In other words, the generalized Kähler structure is made up of two B-symplectic structures defined by the differential forms

$$\varphi_1 = e^{B+i\omega_1} \quad \varphi_2 = e^{-B+i\omega_2},$$

where $B = \omega_K$, $\omega_1 = \omega_I - \omega_J$, and $\omega_2 = \omega_I + \omega_J$.

The bi-Hermitian structure (g, I, J) obtained from a hyperkähler structure is an example of a *strongly* bi-Hermitian structure, i.e. a bi-Hermitian structure (g, J_+, J_-) such that J_+ is nowhere equal to $\pm J_-$. From expression (6.3), it is clear that in 4 dimensions, strongly bi-Hermitian structures with equal orientation correspond exactly to generalized Kähler structures where both generalized complex structures are B-symplectic. Without loss of generality, the B-field can be chosen to be $\pm B$ for the pair $\mathcal{J}_{1/2}$:

Example 6.31 (strongly bi-Hermitian). A strongly bi-Hermitian structure in 4 dimensions with equal orientation is equivalent to a generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ such that both $\mathcal{J}_1$ and $\mathcal{J}_2$ are B-symplectic, i.e. given by differential forms

$$\varphi_1 = e^{B+i\omega_1}, \quad \varphi_2 = e^{-B+i\omega_2}.$$

The algebraic conditions on these forms are as follows: first the requirements $L_1 \cap L_2 \neq 0$ and $L_1 \cap \overline{L_2} \neq 0$ imply that

$$(\varphi_1, \varphi_2) = (\varphi_1, \overline{\varphi_2}) = 0. \tag{6.15}$$

In 4 dimensions there is only one additional constraint: $-\mathcal{J}_1\mathcal{J}_2$ must be positive definite. This is equivalent to the requirement that ω_1^2, ω_2^2 have the same sign. Equation (6.15) is satisfied if and only if $(e^{B+i\omega_1}, e^{-B\pm i\omega_2}) = 0$, i.e. if and only if $(2B + i(\omega_1 \mp \omega_2))^2 = 0$. Therefore we conclude that a strongly bi-Hermitian structure in 4 dimensions with equal orientation is equivalent to the specification of three closed 2-forms B, ω_1, ω_2 such that

$$\begin{aligned} B\omega_1 = B\omega_2 = \omega_1\omega_2 = \omega_1^2 + \omega_2^2 - 4B^2 = 0, \\ \omega_1^2 = \lambda\omega_2^2, \quad \lambda > 0. \end{aligned} \tag{6.16}$$

This result appeared, of course without the generalized complex interpretation, in [1], and it was used to show, as we now do, that one can deform the hyperkähler example mentioned above in such a way to produce a genuine bi-Hermitian structure, i.e. a metric which admits exactly two distinct Hermitian complex structures:

Example 6.32 (Non-hyperhermitian bi-Hermitian structure). Let $B = \omega_K$, $\omega_1 = \omega_I - \omega_J$, and $\omega_2 = \omega_I + \omega_J$ be the forms defining the 4-dimensional bi-Hermitian structure from Example 6.30. Then let F_t be a 1-parameter family of diffeomorphisms generated by a ω_K-Hamiltonian vector field. For t sufficiently small the forms

$$B = \omega_K, \quad \omega_1 = \omega_I - F_t^*\omega_J, \quad \omega_2 = \omega_I + F_t^*\omega_J$$

satisfy the equations (6.16), and hence define a strongly bi-Hermitian structure. In [1] it is shown that the Hamiltonian vector field can be chosen so that the resulting deformed metric is not anti-self-dual, and hence does not admit more than two orthogonal complex structures. The idea of deforming a Hyperkähler structure in this way is due to D. Joyce.

The question remains as to whether one can find bi-Hermitian structures on manifolds which admit no hyperhermitian structure; this problem has up to now remained unsolved. The authors of [1] were unable to rule this out, but they were able to obtain much information about examples, should they exist. They reasoned that such a structure would not be strongly bi-Hermitian; indeed J_+ and $\pm J_-$ would coincide along a subvariety which would be an anticanonical divisor for both $J_\pm$. They went on to show that if the first Betti number is even, then the 4-manifold must be either $\mathbb{C}P^2$ or a minimal ruled surface of genus ≤ 1, or obtained from these by blowing up points along an anti-canonical divisor. In what follows we show that there is in fact a bi-Hermitian structure on $\mathbb{C}P^2$, answering the question posed in [1].

6.5.1 Deformation of generalized Kähler structures

Let $(\mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler structure. In this section we study deformations of this structure where $\mathcal{J}_1$ varies and $\mathcal{J}_2$ is kept fixed. As we saw in section 5, deformations of $\mathcal{J}_1$ as a generalized complex structure are given by small sections ε of $\wedge^2 L_1$ which satisfy

$$d_L\epsilon + \tfrac{1}{2}[\epsilon, \epsilon] = 0.$$

In the presence of $\mathcal{J}_2$ in a generalized Kähler structure, L_1 decomposes as $L_1 = L_1^+ \oplus L_1^-$. L_2 can then be written as $L_2 = L_1^+ \oplus \overline{L_1^-}$. It is easy to see, therefore, that small deformations of $\mathcal{J}_1$ keeping $\mathcal{J}_2$ fixed correspond to graphs of maps taking L_1^+ to $\overline{L_1^-}$, i.e. small sections of the bundle

$$\overline{L_1^+} \otimes \overline{L_1^-} < \wedge^2(L_1^+ \oplus L_1^-)^*,$$

satisfying the same condition as above. As we proved in section 5, assuming the obstruction vanishes, the locally complete family of deformations of $\mathcal{J}_1$ has tangent space given by $H_L^2(M)$. Therefore if the infinite-dimensional subspace $C^\infty(\overline{L_1^+} \otimes \overline{L_1^-})$ intersects the space of d_L-closed sections of $\wedge^2 L^*$

in a subspace with nontrivial image in $H^2_L(M)$, we can immediately conclude that there exist deformations of the generalized Kähler structure keeping $\mathcal{J}_2$ fixed.

As an example of this, we begin with a usual Kähler structure and attempt to deform it in the generalized sense. Let $\mathcal{J}_J, \mathcal{J}_\omega$ be a usual Kähler structure. Then

$$L_1^+ = \{X - i\omega X \ : \ X \in T_{1,0}\},$$

whereas

$$L_1^- = \{Y + i\omega Y \ : \ Y \in T_{1,0}\}.$$

As a consequence, the deformation ϵ which we seek is a linear combination of elements of the form

$$\epsilon = (X + i\omega X) \wedge (Y - i\omega Y), \quad X, Y \in T_{1,0}.$$

In particular, $\overline{L_1^+} \otimes \overline{L_1^-}$ has sections of the form

$$\tfrac{1}{2}((X + i\omega X) \wedge (Y - i\omega Y) - (Y + i\omega Y) \wedge (X - i\omega X)) = X \wedge Y + \omega X \wedge \omega Y,$$

i.e. sections of the form $\varepsilon = \beta + \omega^{-1}\beta\omega$ for β a $(2,0)$-bivector. Recall that in this situation the differential operator d_L is just $\overline{\partial}$, and so the equation $d_L\varepsilon = 0$ is satisfied if and only if β is holomorphic. So we see that if the Kähler manifold admits a holomorphic bivector, there must be a solution to the deformation equation within the subspace of sections $\beta + \omega^{-1}\beta\omega$. Such deformations are clearly not trivial, since the bivector component β will reduce the type of the complex structure wherever it is nonvanishing. Hence we conclude:

Proposition 6.33. *Let (M, J, ω) be a Kähler manifold, and assume that the obstruction to generalized deformations of J vanishes. If the Kähler manifold admits a holomorphic bivector β, then it has a deformation as a generalized Kähler structure such that the complex structure is deformed while the symplectic structure remains unchanged.*

Example 6.34 (Existence of bi-Hermitian structure on $\mathbb{C}P^2$). As we saw in Example 5.6, the complex structure of $\mathbb{C}P^2$ has vanishing obstruction map. Furthermore $\mathbb{C}P^2$ admits holomorphic bivectors as $\wedge^2 T_{1,0} = \mathcal{O}(3)$. Hence we conclude from the deformation theorem that there exists a nonzero generalized Kähler deformation keeping the symplectic structure constant, within the class of sections $\beta + \omega^{-1}\beta\omega$, where $\beta \in C^\infty(\wedge^2 T_{1,0})$. Hence we obtain a non-Kähler generalized Kähler structure on the real 4-manifold underlying $\mathbb{C}P^2$, and by Theorem 6.28, we see that this produces a genuine bi-Hermitian structure, i.e. a metric with exactly two distinct Hermitian complex structures with equal orientation.

6.6 Twisted generalized Kähler structures

In the list of bi-Hermitian structures studied by Apostolov, Gauduchon, and Grantcharov, there are some which do not satisfy the additional constraints $\mp d^c_\pm \omega_\pm = db$ of generalized Kähler geometry. For example, the hyperhermitian Hopf surface:

Example 6.35 (The Hopf surface: not generalized Kähler). Express $M = S^3 \times S^1$ as the quotient $(\mathbb{C}^2 - \{0\})/\{x \mapsto 2x\}$. As a complex surface this is known as the Hopf surface. The hyperkähler structure $(\tilde{g}, I, J, K)$ on $\mathbb{C}^2$ descends to a hyperhermitian structure on the Hopf surface once we rescale the metric $\tilde{g} \mapsto g = r^{-2}\tilde{g}$, where $r(x) = |x|$, the Euclidean length in $\mathbb{C}^2$. Then the associated 2-form ω_I is not closed (The Hopf surface has odd first Betti number and hence cannot be Kähler). The 3-form $d_I^c\omega_I$ is closed, however, and is proportional to $i_{\partial_r}(\omega_I \wedge \omega_I)$, the pull-back of the volume form on S^3. Hence we obtain that

$$\int_{S^3} d_I^c\omega_I \neq 0. \tag{6.17}$$

Therefore we see that $d_I^c\omega_I$ is not exact. Hence the standard hyperhermitian structure on the Hopf surface is not generalized Kähler.

We wish to show that no Hermitian metric could make the Hopf surface generalized Kähler. Suppose that ω is any other positive $(1,1)$-form with respect to the standard complex structure I. If I is to be one complex structure in a generalized Kähler pair, then $d^c\omega = db$, implying that $dd^c\omega = 0$. Hence $\partial\omega$ defines a cohomology class in the Dolbeault group $H^1(\Omega^2)$. This class must be nonzero for the following reason: if $\partial\omega = \overline{\partial}\sigma$ for some $(2,0)$-form σ, then let T be an elliptic curve in the Hopf surface (which must be homologically trivial, bounding the 3-manifold D). We have

$$\begin{aligned} 0 \neq \int_T \omega = \int_D d\omega &= \int_D \partial\omega + \overline{\partial}\omega \\ &= \int_D \overline{\partial}\sigma + \partial\overline{\sigma} = \int_D d(\sigma + \overline{\sigma}) \\ &= \int_T \sigma + \overline{\sigma} = 0, \end{aligned}$$

a contradiction. Furthermore since $H^1(\Omega^2) \cong \mathbb{C}$ for the Hopf surface (see [4]), $[\partial\omega]$ must be a nonzero multiple of $[\partial\omega_I]$, i.e.

$$\partial\omega = k\partial\omega_I + \overline{\partial}\tau,$$

for some $(2,0)$-form τ and nonzero k. Summing with the conjugate, we obtain

$$d\omega = k\partial\omega_I + \overline{k\partial\omega_I} + d(\tau + \overline{\tau}).$$

We have shown (6.17) that $\int_{S^3} \partial\omega_I = c \neq 0$, and so by integrating the last equation over S^3 we see that kc is imaginary. But recall that $d^c\omega = db$, i.e.

$$db = i(\overline{k\partial\omega_I} - k\partial\omega_I + d(\overline{\tau} - \tau)).$$

Therefore, integrating the last equation over S^3, we obtain that kc is real, a contradiction. We conclude that the Hopf surface may never enter into a generalized Kähler bi-Hermitian pair.

We now define a natural extension of generalized Kähler geometry, which allows non-exact torsion $h = db$, and which will accomodate the Hopf surface:

Definition 6.36 (Twisted generalized Kähler). *An almost generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ is said to be twisted generalized Kähler with respect to the closed 3-form H if both generalized complex structures $\mathcal{J}_1, \mathcal{J}_2$ are twisted generalized complex structures, as defined in section 4.9.*

In sections 6.4.3 and 6.4.4 we proved that the generalized Kähler integrability condition is equivalent to certain conditions on the bi-Hermitian data $(g, b, J_\pm)$ it defines. Using the same methods, together with our results on the twisted Courant bracket, it is easy to prove the following generalization of that result.

Theorem 6.37. *Let $(\mathcal{J}_1, \mathcal{J}_2)$ be an almost generalized Kähler structure, and let $(g, b, J_\pm)$ be the associated almost bi-Hermitian data. Also, let H be a closed 3-form. Then the following are equivalent:*

- *$(\mathcal{J}_1, \mathcal{J}_2)$ is an H-twisted generalized Kähler structure.*
- *$J_\pm$ are integrable complex structures and*
$$d^c_-\omega_- = -d^c_+\omega_+ = H + db.$$
- *$\nabla^\pm J_\pm = 0$, where*
$$\nabla^\pm = \nabla \pm \tfrac{1}{2}g^{-1}(H+db)$$
and $H + db$ is of type $(2,1)+(1,2)$ with respect to both $J_\pm$ (this last condition is implied by $J_\pm$ integrable).

Given this result, we revisit the Hopf surface to discover our first example of a twisted generalized Kähler structure:

Example 6.38 (The Hopf surface: twisted generalized Kähler). Let I be the standard complex structure on the Hopf surface defined by $(\mathbb{C}^2 - \{0\})/\{x \mapsto 2x\}$. We use the notation of example 6.35. We noted there that $H = d^c_I\omega_I$ was a closed, non-exact 3-form. Now let $\check{I}$ be the complex structure on the Hopf surface induced by the complex structure on $\mathbb{C}^2$ defined by holomorphic coordinates $(z_1, \overline{z_2})$; note that $\check{I}, I$ have opposite orientations, but are both Hermitian with respect to g. Also it is clear that $-d^c_{\check{I}}\omega_{\check{I}} = d^c_I\omega_I = H$, showing that
$$(g, I, \check{I}, H)$$
defines an H-twisted generalized Kähler structure of type (odd,odd) and with $b = 0$. Note that the complex structures $I, \check{I}$ commute. Also, it is remarkable that this geometry was discovered by Roček, Schoutens, and Sevrin [40] in the context of a supersymmetric $SU(2) \times U(1)$ Wess-Zumino-Witten model.

Finally, we provide a striking family of examples of twisted generalized Kähler manifolds: the even-dimensional semi-simple real Lie groups.

Example 6.39 (The even-dimensional compact semi-simple Lie groups). It has been known since the work of Samelson and Wang [44], [49] that any compact even-dimensional Lie group admits left- and right-invariant complex structures J_L, J_R, and that if the group is semi-simple, these can be chosen to be Hermitian with respect to the bi-invariant metric induced from the Killing form $\langle,\rangle$. The idea, then, is to use $(\langle,\rangle, J_L, J_R)$ as a bi-Hermitian structure with $b = 0$ and to show that it is integrable as an H-twisted generalized Kähler structure with respect to $H(X,Y,Z) = \langle [X,Y], Z\rangle$, the bi-invariant Cartan 3-form. To see that this works, let us compute $d^c_{J_L}\omega_{J_L}$:
$$\begin{aligned}
A = d^c_{J_L}\omega_{J_L}(X,Y,Z) &= d\omega_{J_L}(J_LX, J_LY, J_LZ)\\
&= -\omega_{J_L}([J_LX, J_LY], J_LZ) + c.p.\\
&= -\langle [J_LX, J_LY], Z\rangle + c.p.\\
&= -\langle J_L[J_LX, Y] + J_L[X, J_LY] + [X,Y], Z\rangle + c.p.\\
&= (2\langle [J_LX, J_LY], Z\rangle + c.p.) - 3H(X,Y,Z)\\
&= -2A - 3H(X,Y,Z),
\end{aligned}$$
Proving that $d^c_{J_L}\omega_{J_L} = -H$. Since the right Lie algebra is anti-isomorphic to the left, the same calculation with J_R yields $d^c_{J_R}\omega_{J_R} = H$, and finally we have
$$-d^c_{J_L}\omega_{J_L} = d^c_{J_R}\omega_{J_R} = H,$$
i.e. $(\langle,\rangle, J_L, J_R)$ forms an H-twisted generalized Kähler structure.

6.7 Generalized Calabi-Yau metrics

While we will not explore this geometry in detail in this thesis, we wish to define *generalized Calabi-Yau metric* geometry, which is a further reduction beyond generalized Kähler, and which is the analog of a Calabi-Yau manifold. This geometry has been studied in the 4-dimensional case on $K3$ by Huybrechts [20], and turns out to be precisely the geometrical structure parametrized by the moduli space of $N = (2, 2)$ superconformal field theories with $K3$ target, as studied by Aspinwall and Morrison [2].

Definition 6.40 (Generalized Calabi-Yau metric). *A generalized Calabi-Yau metric geometry is defined by a generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ where each generalized complex structure has holomorphically trivial canonical bundle, i.e. their canonical bundles have nowhere-vanishing closed sections $\rho_1, \rho_2 \in C^\infty(\wedge^\bullet T^* \otimes \mathbb{C})$. Finally, we require that the lengths of these sections are related by a constant, i.e.*

$$(\rho_1, \overline{\rho_1}) = c(\rho_2, \overline{\rho_2}), \tag{6.18}$$

where $c \in \mathbb{R}$ is a constant. By rescaling the differential forms by a constant, c may be chosen to be ± 1.

Of course, generalized Calabi-Yau metrics may also be twisted in the presence of a closed 3-form H, by requiring that $(\mathcal{J}_1, \mathcal{J}_2)$ is an H-twisted generalized Kähler structure defined by d_H-closed forms ρ_1, ρ_2 satisfying (6.18).

Example 6.41 (Calabi-Yau manifolds). A usual Calabi-Yau manifold is a Kähler manifold of complex dimension m with symplectic form ω and holomorphic volume form Ω, satisfying

$$\omega^m = \frac{i^m m!}{2^m} \Omega \wedge \overline{\Omega}.$$

In terms of the differential forms $e^{i\omega}, \Omega$ defining the generalized complex structures, this condition is simply that

$$(e^{i\omega}, e^{-i\omega}) = (-1)^{\frac{m(m-1)}{2}} (\Omega, \overline{\Omega}),$$

i.e. $c = (-1)^{\frac{m(m-1)}{2}}$. So we see that Calabi-Yau manifolds provide the basic examples.

Chapter 7

Generalized complex submanifolds

In this section we introduce the notion of generalized complex submanifold, which generalizes both the idea of complex submanifold and that of Lagrangian submanifold. In fact, even in the case of a usual symplectic manifold, there are generalized complex submanifolds besides the Lagrangian ones: these are the so-called co-isotropic A-branes discovered recently by Kapustin and Orlov [22] in the context of D-branes of topological sigma-models. We show that covariance with respect to B-field transformations dictates that generalized complex submanifolds are not simply submanifolds but carry extra data, consisting of a complex line bundle with connection, or more generally, a section of a gerbe with connection, which gives rise to a 2-form F on the submanifold. Such objects have been described by string theorists as D-branes; we show here that they arise completely naturally in the context of the generalized geometry of $T \oplus T^*$. We are thankful to Anton Kapustin for emphasizing the importance of gauge invariance in determining the correct definition of structured submanifold.

We should mention at the outset that generalized complex submanifolds do not necessarily inherit a generalized complex structure of their own; indeed they may be odd-dimensional. In this respect they resemble Lagrangian submanifolds. It is possible to study the generalization of symplectic submanifolds, as is done in [5], however we will not explore this topic here.

7.1 The generalized tangent bundle

The tangent bundle of a submanifold $M \subset N$ is a natural sub-bundle $T_M \leq T_N|_M$ of the restriction of the ambient tangent bundle. This sub-bundle has an associated annihilator $\mathrm{Ann}\, T_M \leq T_N^*$, also known as the conormal bundle of the submanifold. Taking the sum of these natural sub-bundles, we obtain a natural real maximal isotropic sub-bundle of $(T_N \oplus T_N^*)|_M$, which we call the *generalized tangent bundle*:

Definition 7.1. *The sub-bundle*

$$\tau_M = T_M \oplus \mathrm{Ann}\, T_M \leq (T_N \oplus T_N^*)\big|_M$$

is called the generalized tangent bundle of the submanifold $M \subset N$.

If the ambient manifold has a (possibly H-twisted) generalized complex structure $\mathcal{J}$, a natural condition on a submanifold M would be that its generalized tangent bundle is stable under $\mathcal{J}$. Indeed this definition reduces to familiar conditions in the complex and symplectic cases:

Example 7.2 (Complex submanifold). Let $(N, \mathcal{J}_J)$ be a complex manifold. Then $\tau_M = T_M \oplus \operatorname{Ann} T_M$ is stable under

$$\mathcal{J}_J = \begin{pmatrix} J & \\ & -J^* \end{pmatrix}$$

if and only if T_M is stable under J, which is equivalent to the condition that M be a complex submanifold.

Example 7.3 (Lagrangian submanifold). Let $(N, \mathcal{J}_\omega)$ be a symplectic manifold. Then $\tau_M = T_M \oplus \operatorname{Ann} T_M$ is stable under

$$\mathcal{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}$$

if and only if

- ω takes T_M into $\operatorname{Ann} T_M$, i.e. M is an isotropic submanifold, and
- ω^{-1} takes $\operatorname{Ann} T_M$ into T_M, i.e. M is coisotropic,

i.e. M must be isotropic and co-isotropic, hence Lagrangian.

The problem with this possible definition of a generalized complex submanifold is that it does not behave well with respect to B-field transformations. That is, if τ_M is stable under $\mathcal{J}$, then applying a B-field will modify $\mathcal{J}$ but not τ_M, and so under this definition M would cease to be a generalized complex submanifold when applying a B-field, which is supposed to be an underlying symmetry of the whole geometry.

The answer to this problem is to modify the definition of a generalized tangent space so that it is acted upon naturally by B-field transformations.

Let (M, F) be a pair consisting of a submanifold $M \subset N$ and a real 2-form $F \in \Omega^2(M)$ on M, and suppose that $dF = H|_M$, where H is the closed 3-form on N defining the twist. Then we will call this a generalized submanifold of (N, H). Using the language of gerbes (see section 3.8), (M, F) is really a submanifold on which the gerbe is trivializable, together with a trivialization with connection relative to the gerbe connection (whose curvature is H). While this language is more precise, we simplify matters in the following definition:

Definition 7.4 (Generalized submanifold). *Let (N, H) be a pair consisting of a manifold N and a closed 3-form H. Then the pair (M, F) of a submanifold $M \subset N$ together with a 2-form $F \in \Omega^2(M)$ is said to be a generalized submanifold of (N, H) iff $dF = H|_M$.*

The advantage of the gerbe interpretation is that a special case of a generalized submanifold (when $H|_M = 0$ and F is integral) is a triple (M, L, ∇) consisting of a submanifold together with a line bundle with unitary connection. The simplified definition above only sees the curvature $F^\nabla \in \Omega^2(M)$.

Now we define the generalized tangent space of the generalized submanifold:

Definition 7.5 (Generalized tangent bundle). *The generalized tangent bundle τ_M^F of the generalized submanifold (M, F) is*

$$\tau_M^F = \left\{ X + \xi \in T_M \oplus T_N^*\big|_M \ : \ \xi\big|_M = i_X F \right\},$$

a real, maximal isotropic sub-bundle $\tau_M^F < (T_N \oplus T_N^*)|_M$.

It is clear that since the generalized tangent bundle sits in $(T_N \oplus T_N^*)|_M$, it is acted upon naturally by B-field transforms of the ambient space; as a result the action of B-fields on generalized submanifolds is as follows:

$$e^B(M,F) = (M, F+B).$$

Clearly this transformation does not interfere with the condition $dF = H|_M$.

7.2 Generalized complex submanifolds

Finally we are able to define generalized complex submanifolds in such a way that the property of being one is covariant under B-field transformations.

Definition 7.6 (Generalized complex submanifold). *Let $(N, \mathcal{J}, H)$ be a H-twisted generalized complex manifold, where H is a real closed 3-form. Then the generalized submanifold $(M,F) \subset (N, \mathcal{J}, H)$ is said to be a generalized complex submanifold when τ_M^F is stable under $\mathcal{J}$.*

We now determine what the generalized complex submanifolds are in the standard cases of complex and symplectic geometry.

Example 7.7 (Complex case). Let $(M,F) \subset (N, \mathcal{J}_J)$ be a generalized submanifold of a complex manifold. Then it is generalized complex if and only if

$$\tau_M^F = \left\{ X + \xi \in T_M \oplus T_N^*\big|_M \ : \ \xi\big|_M = i_X F \right\}$$

is stable under the action of

$$\mathcal{J}_J = \begin{pmatrix} J & \\ & -J^* \end{pmatrix},$$

which happens if and only if

- T_M is closed under J, i.e. M is a complex submanifold, and
- $J^* i_X F + i_{JX} F \in \operatorname{Ann} T_M \ \forall X \in T_M$, i.e. F is of type $(1,1)$ on M.

In the special case where $H|_M = 0$ and F is integral, then we see that the generalized complex submanifold is a complex submanifold together with a unitary holomorphic line bundle on it (a line bundle with curvature of type $(1,1)$ has a unique compatible holomorphic structure).

Example 7.8 (Symplectic case). Let $(M,F) \subset (N, \mathcal{J}_\omega)$ be a generalized submanifold of a symplectic manifold. Then it is generalized complex if and only if

$$\tau_M^F = \left\{ X + \xi \in T_M \oplus T_N^*\big|_M \ : \ \xi\big|_M = i_X F \right\}$$

is stable under the action of

$$\mathcal{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}.$$

To aid with the calculation we choose a 2-form B on N such that $B|_M = F$ (only locally). Then τ_M^F is $\mathcal{J}_\omega$-stable if and only if $\tau_M^0 = T_M \oplus \operatorname{Ann} M$ is stable under

$$e^{-B} \mathcal{J}_\omega e^B = \begin{pmatrix} -\omega^{-1}B & -\omega^{-1} \\ \omega + B\omega^{-1}B & B\omega^{-1} \end{pmatrix},$$

which happens if and only if

- $\omega^{-1}(\xi) \in T_M \ \forall \xi \in \operatorname{Ann} T_M$, i.e. M is a coisotropic submanifold (note that $\operatorname{Ann} T_M$ always sits inside τ_M^F), and

- $\omega^{-1}(i_X B) \in T_M \ \forall X \in T_M$, i.e. $i_X F = 0 \ \forall X \in T_M^\perp$ ($T_M^\perp$ is the symplectic orthogonal bundle), i.e. F descends to $T_M/T_M^\perp$, and

- $\omega + B\omega^{-1}B$ sends T_M into $\operatorname{Ann} T_M$, i.e. $(\omega|_M)^{-1}F$ is an almost complex structure on $T_M/T_M^\perp$.

Note that $F + i\omega|_M$ defines a nondegenerate form of type $(2,0)$ on $T_M/T_M^\perp$, and so the complex dimension of $T_M/T_M^\perp$ must be even. Hence if $\dim N = 2n$, then $\dim M = n + 2k$ for some positive integer k. In the case where $H|_M = 0$, F is closed and we obtain a holomorphic symplectic structure transverse to the Lagrangian foliation of the coisotropic submanifold.

In the case that M is a Lagrangian submanifold, then the second condition implies that $F = 0$. This means that Lagrangian submanifolds can only be generalized complex submanifolds when $H|_M = dF = 0$, and in this case they always carry a flat line bundle on them.

It is remarkable that in the symplectic case, the notion of generalized complex submanifold coincides exactly with the recently discovered "coisotropic A-branes" of Kapustin and Orlov [22]. This connection is explored in further detail by Kapustin in [21].

Before we move on to discuss when generalized complex submanifolds can be *calibrated*, we provide one example of generalized complex submanifold in a case which is neither complex nor symplectic.

Example 7.9 (Deformed $\mathbb{C}P^2$). In example 5.6, we studied a generalized complex structure on $\mathbb{C}P^2$ obtained by deformation. This structure is B-symplectic outside a cubic, and along the cubic the generalized complex structure is none other than the original complex structure on $\mathbb{C}P^2$. Hence it is easy to see that since the cubic began as a complex submanifold and is fixed by the deformation, this means that it remains a generalized complex submanifold of the deformed structure.

7.3 Generalized calibrations

In this section we wish to describe briefly an idea which can be used to generalize the notion of special Lagrangian submanifold. In the usual Calabi-Yau case, a Lagrangian submanifold is said to be special Lagrangian when it is calibrated with respect to the real part of the Calabi-Yau volume form. That is, $\mathrm{Re}(\Omega)$ restricts to the Lagrangian to yield the volume form induced by the Riemannian metric.

Having established the machinery of generalized complex submanifolds and generalized Calabi-Yau metrics, the generalization is clear:

Definition 7.10 (Calibration in generalized Calabi-Yau manifolds). *Let $(\mathcal{J}_1, \mathcal{J}_2)$ define a generalized Calabi-Yau metric structure, as in definition 6.18. It is defined by two global closed complex differential forms ρ_1, ρ_2. By squaring the spinors we obtain top-degree skew forms*

$$\Omega_1 \in C^\infty(\det L_1), \quad \Omega_2 \in C^\infty(\det L_2).$$

Using the natural isomorphism $L_i = (T \oplus T^*, \mathcal{J}_i)$, $i = 1, 2$, we see that Ω_i are complex top-degree forms on $T \oplus T^*$.

Let (M, F) be a generalized complex submanifold with respect to $\mathcal{J}_1$. Then it is said to be calibrated with respect to $\mathcal{J}_2$ when the real part of Ω_2, restricted to the generalized tangent bundle of M, yields the natural volume induced by the Kähler metric $G = -\mathcal{J}_1\mathcal{J}_2$ on $T \oplus T^*$, i.e.

$$Re(\Omega_2)\big|_{\tau_M^F} = \det G\big|_{\tau_M^F}.$$

This condition can also be expressed directly in terms of ρ_2, by requiring that

$$\tau_M^F \cdot Im(\rho_2) = 0,$$

i.e. the Clifford product of any generalized tangent vector with $Im(\rho_2)$ vanishes.

This definition carries over without change to the twisted case, and specializes to the usual notion of SLAG in the presence of a usual Calabi-Yau structure. Of course this definition engenders many natural questions about moduli of such calibrated submanifolds as well as about their intersection theory, among others. It will be the subject of future work.

Chapter 8

Speculations on mirror symmetry

In this highly speculative chapter, we outline some thoughts concerning an approach to mirror symmetry through generalized complex geometry. It is clear that many aspects of mirror symmetry manifest themselves naturally in the context of generalized complex geometry, and so it is reasonable to expect that the "mirror relation", if it indeed exists, should be stated in the language of generalized complex manifolds.

Let (M_A, G_A, ∇_A) and (M_B, G_B, ∇_B) be manifolds with gerbes and gerbe connections, for which the curvatures are H_A, H_B: closed 3-forms, defining integral cohomology classes. Then on $M_A \times M_B$ we can put a gerbe $\pi_A^* G_A^{-1} \otimes \pi_B^* G_B$ (which we will denote $G_A^{-1} G_B$ for short) with the tensor product connection ∇. It has curvature $-\pi_A^* H_A + \pi_B^* H_B$. In general we needn't consider $M_A \times M_B$ but perhaps another gerbed manifold fibering over M_A and M_B in the appropriate way.

Suppose there is a generalized submanifold $(\mathcal{M}, F) \subset M_A \times M_B$ which submerges onto both M_A and M_B via π_A, π_B:

$$\begin{tikzcd}
(\mathcal{M}, F) \arrow[r, hook] & M_A \times M_B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\
M_A & & M_B
\end{tikzcd}$$

Recall that this means that the gerbe $G_A^{-1} G_B$ is trivializable over $\mathcal{M}$, i.e.

$$[-\pi_A^* H_A + \pi_B^* H_B]\big|_{\mathcal{M}} = 0,$$

and that we have a choice of trivialization with connection over $\mathcal{M}$, giving rise to a 2-form $F \in \Omega^2(\mathcal{M})$ satisfying

$$(-\pi_A^* H_A + \pi_B^* H_B)\big|_{\mathcal{M}} = dF.$$

Now $(\mathcal{M}, F)$ has generalized tangent bundle sitting in $T_{M\times N} \oplus T^*_{M\times N}$:

$$\tau_{\mathcal{M}} = \{X + \xi \in T_{\mathcal{M}} \oplus T^*_{M\times N} \;:\; \xi|_{T_{\mathcal{M}}} = i_X F\},$$

Of course M and N have generalized tangent bundles in $T_{M\times N} \oplus T^*_{M\times N}$ as well, namely $T_M \oplus T^*_M$ and $T_N \oplus T^*_N$. We require that $\tau_{\mathcal{M}}$ be transverse to both these. This is an analog of a transversality condition on the submanifold $\mathcal{M}$.

Then we propose that $(\mathcal{M}, F)$ defines a "mirror relation" between the manifolds $\mathcal{M}_A$ and $\mathcal{M}_B$, and there is a transform-like mapping (of Dirac structures, differential forms, and other objects, like generalized holomorphic bundles) defined by

$$\mathcal{F} = (\pi_A)_* \circ e^F \circ (\pi_B\big|_{\mathcal{M}})^*.$$

Now suppose we have generalized complex structures L_A, L_B, and $\mathcal{L}$ on the spaces (M_A, G_A), (M_B, G_B), and $(M_A \times M_B, G_A^{-1}G_B)$, such that $(\pi_A)_*\mathcal{L} = L_A$ and $(\pi_B)_*\mathcal{L} = L_B$. And suppose further that $(\mathcal{M}, F)$ is a generalized complex submanifold of $(M_A \times M_B, G_A^{-1}G_B, \nabla, \mathcal{L})$. Then we propose that $\mathcal{M}$ defines a mirror relation between the generalized complex structures on $\mathcal{M}_A$, $\mathcal{M}_B$ by the same transform $\mathcal{F}$. This transform can be thought of as a generalization of the following ideas:

- T-duality, where M_A and M_B are S^1-bundles via φ_A, φ_B over some base B and $\mathcal{M}$ is the correspondence space

$$\{(x, y) \in M_A \times M_B \ : \ \varphi_A(x) = \varphi_B(y)\} \subset M_A \times M_B$$

  on which is defined a natural 2-form F coming from the T-duality construction. This construction is described in the physics literature and especially in the remarkable paper of Bouwknegt, Evslin, and Mathai [7]. Understanding T-duality in the generalized complex framework is joint work between the author and Gil Cavalcanti; this will appear in [11].

- The Fourier-Mukai transform, where both generalized complex structures are complex. Also bi-meromorphic mappings in algebraic geometry.

- Bäcklund transformations, showing that a PDE system on M_A is equivalent to a PDE system on M_B through the use of a correspondence space.

- Poisson dual pairs and canonical transformations, where L_A, L_B are taken to be real Dirac structures.

In the case where both spaces M_A, M_B are generalized Calabi-Yau metric geometries, it may be possible to find a submanifold $\mathcal{M} \subset M_A \times M_B$ which satisfies the properties of a mirror relation with respect to both pairs of generalized complex structures, and furthermore is calibrated in the sense of section 7.3. The hope then would be that the known instances of mirror ensembles of Calabi-Yau 3-folds could be phrased using this description. It is with this wild speculation that we end the thesis.

Bibliography

[1] V. Apostolov, P. Gauduchon, and G. Grantcharov. Bihermitian structures on complex surfaces. *Proc. London Math. Soc.*, 79(3):414–428, 1999.

[2] P. Aspinwall and D. Morrison. *Mirror Symmetry II*, chapter String Theory on K3 Surfaces, pages 703–716. International Press, 1996.

[3] S. Barannikov and M. Kontsevich. Frobenius manifolds and formality of Lie algebras of polyvector fields. *Internat. Math. Res. Notices*, (4):201–215, 1998.

[4] W. Barth, C. Peters, and A. Van de Ven. *Compact complex surfaces.* Erg. Math. Grenzgebiete. Springer-Verlag, 1984.

[5] O. Ben-Bassat and M. Boyarchenko. Submanifolds of generalized complex manifolds. *math.DG/0309013*, 2003.

[6] J.-M. Bismut. A local index theorem for non Kähler manifolds. *Math. Ann.*, 284:681–699, 1989.

[7] P. Bouwknegt, J. Evslin, and V. Mathai. T-Duality: Topology change from H-flux. *hep-th/0306062*, June 2003.

[8] J.-L. Brylinski. A differential complex for Poisson manifolds. *J. Diff. Geom.*, 28:93–114, 1988.

[9] H. Bursztyn and O. Radko. Gauge equivalence of Dirac structures and symplectic groupoids. *Ann. Inst. Fourier (Grenoble)*, 53(1):309–337, 2003.

[10] G. Cavalcanti and M. Gualtieri. Generalized complex structures on nilmanifolds. *in preparation.*

[11] G. Cavalcanti and M. Gualtieri. T-Duality and generalized complex geometry. *in preparation.*

[12] D. Chatterjee. *On the construction of abelian gerbs.* PhD thesis, Trinity college, Cambridge, June 1998.

[13] C. Chevalley. *The Algebraic Theory of Spinors and Clifford Algebras. Collected Works.*, volume 2. Springer Verlag, 1996.

[14] T. Courant. Dirac manifolds. *Trans. Amer. Math. Soc.*, 319:631–661, 1990.

[15] T. Courant and A. Weinstein. Beyond Poisson structures. In *Action hamiltoniennes de groupes. Troisième théorème de Lie (Lyon, 1986)*, volume 27 of *Travaux en Cours*, pages 39–49. Hermann, Paris, 1988.

[16] I. Dorfman. Dirac structures of integrable evolution equations. *Phys. Lett. A*, 125(5):240–246, 1987.

[17] R. Fernandes. Lie algebroids, holonomy and characteristic classes. *Adv. Math.*, 170(1):119–179, 2002.

[18] N. Hitchin. Lectures on special Lagrangian submanifolds. In *Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999)*, volume 23 of *AMS/IP Stud. Adv. Math.*, pages 151–182. Amer. Math. Soc., Providence, RI, 2001.

[19] N. Hitchin. Generalized Calabi-Yau manifolds. *math.DG/0209099*, 2002.

[20] D. Huybrechts. Generalized Calabi-Yau structures, K3 surfaces, and B-fields. *math.AG/0306162*, 2003.

[21] A. Kapustin. Topological strings on noncommutative manifolds. *hep-th/0310057*, 2003.

[22] A. Kapustin and D. Orlov. Remarks on A-branes, Mirror Symmetry, and the Fukaya category. *Preprint*, hep-th/0109098, 2001.

[23] M. Karoubi. Algèbres de Clifford et K-théorie. *Ann. Scient. Ec. Norm. Sup.*, 1(1):161, 1968.

[24] P. Kobak. Explicit doubly-hermitian metrics. *Diff. Geom. and its Applications*, 10:179–185, 1999.

[25] M. Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 120–139, Basel, 1995. Birkhäuser.

[26] W. Kopczyński and A. Trautman. Simple spinors and real structures. *J. Math. Phys.*, 33(2):550–559, 1992.

[27] Y. Kosmann-Schwarzbach. Exact Gerstenhaber algebras and Lie bialgebroids. *Acta Appl. Math.*, 41(1-3):153–165, 1995. Geometric and algebraic structures in differential equations.

[28] J.L. Koszul. Crochet de Schouten-Nijenhuis et cohomologie, in "Elie Cartan et les Math. d'aujourd'hui". *Astérisque hors-série*, pages 251–271, 1985.

[29] M. Kuranishi. New proof for the existence of locally complete families of complex structures. In *Proc. Conf. Complex Analysis (Minneapolis, 1964)*, pages 142–154. Springer, Berlin, 1965.

[30] Z.-J. Liu, A. Weinstein, and P. Xu. Manin triples for Lie bialgebroids. *J. Diff. Geom.*, 45:547–574, 1997.

[31] K. Mackenzie. *Lie groupoids and Lie algebroids in differential geometry*, volume 124 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1987.

[32] K. Mackenzie and P. Xu. Lie bialgebroids and Poisson groupoids. *Duke Math. J.*, 73(2):415–452, 1994.

[33] D. McDuff and D. Salamon. *Introduction to symplectic topology.* Oxford University Press, 1998.

[34] C. Morrey. *Contributions to the theory of partial differential equations, Bers, Bochner, John eds.*, volume 33 of *Annals of mathematical studies*, chapter Second order elliptic systems of partial differential equations. Princeton UP, 1954.

[35] S. Mukai. Symplectic Structure of the Moduli Space of Sheaves on an Abelian or K3 Surface. *Invent. Math.*, 77:101–116, 1984.

[36] A. Newlander and L. Nirenberg. Complex analytic coordinates in almost complex manifolds. *Ann. of Math*, 65:391–404, 1957.

[37] R. Palais. *Foundations of Global Non-linear Analysis.* Benjamin and Co., New York, 1968.

[38] M. Pontecorvo. Complex structures on Riemannian four-manifolds. *Math. Ann.*, 309:159–177, 1997.

[39] J. Pradines. Théorie de Lie pour les groupoïdes différentiables. Relations entre propriétés locales et globales. *C. R. Acad. Sci. Paris Sér. A-B*, 263:A907–A910, 1966.

[40] M. Roček, K. Schoutens, and A. Sevrin. Off-shell WZW models in extended superspace. *Physics Letters B*, 265:303–306, 1991.

[41] D. Roytenberg. *Courant algebroids, derived brackets and even symplectic supermanifolds.* PhD thesis, University of California at Berkeley, 1999.

[42] Jr. S. Gates, C. Hull, and M. Roček. Twisted multiplets and new supersymmetric nonlinear σ-models. *Nuclear Phys. B*, 248(1):157–186, 1984.

[43] S. Salamon. Complex structures on nilpotent Lie algebras. *Preprint*, math.DG/9808025, 2000.

[44] H. Samelson. A class of complex analytic manifolds. *Port. Math.*, 5:129–132, 1953.

[45] D. Sundararaman. *Moduli, deformations and classifications of compact complex manifolds.* Pitman Publishing, 1990.

[46] H. Sussmann. Orbits of families of vector fields and integrability of distributions. *Trans. Amer. Math. Soc.*, 180:171–188, 1973.

[47] K. Uchino. Remarks on the definition of a Courant algebroid. *Lett. Math. Phys.*, 60(2):171–175, 2002.

[48] P. Ševera and A. Weinstein. Poisson geometry with a 3-form background. *Prog. Theor. Phys. Suppl.*, 144:145–154, 2001.

[49] H.C. Wang. Complex parallelizable manifolds. *Proc. Amer. Math. Soc.*, 5:771–776, 1954.

[50] R. Wells. *Differential Analysis on Complex Manifolds.* Springer-Verlag, second edition, 1980.