I. Introduction to theories without the independence property

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1 Preliminaries

Notation

- T will denote a complete theory in a language L, possibly many-sorted. There is no harm for now in assuming $T = T^{eq}$. Although subsequently we may want to work in a one-sorted theory for example and distinguish between 1-types and n-types.
- $M, N, ..$ denote models of T, $A, B, ..$ subsets of models of T, $a, b, c, ..$ (usually finite) tuples from models of T, and $x, y, ..$ (finite tuples of) variables.
- It is convenient to fix a "large" saturated model $\bar{M}$ of T, and now $M, N, ..$ will denote "small" elementary submodels, $A, B, ..$ small subsets of $\bar{M}$ etc.
- So for example for a sentence σ of $L_{\bar{M}}$ we may just write $\models \sigma$ for $\bar{M} \models \sigma$.
- Here "large" may mean of cardinality $\bar{\kappa}$ with $\bar{\kappa}$ inaccessible.
- By a definable set X we usually mean a subset of some sort of $\bar{M}$, definable with parameters in $\bar{M}$. (Remember the set of n-tuples from a given sort is also a sort.) We may say A-definable to exhibit the parameters over which the set is definable.
- By a type-definable set we mean the solution set in $\bar{M}$of a partial type over a small set of parameters, equivalently the intersection of a small collection of definable sets (in a given sort). We also have the notion "type-definable over A".
- By a "global complete type" we mean a complete type (usually in a finite tuple of variables) over $\bar{M}$.
- We sometimes may want to realize a global complete type in which case we do so in an elementary extension of $\bar{M}$.

Indiscernibles

Definition 1.1.
- (i) If α is an ordinal and b_i for $i < \alpha$ are tuples from $\bar{M}$ (of same sort), we say that $(b_i : i < \alpha)$ is an indiscernible sequence over A if if whenever $i_1 < ... < i_n < \alpha$ and $j_1 < .. < j_n < \alpha$, then $tp((b_{i_1}, b_{i_2}, .., b_{i_n})/A) = tp((b_{j_1}, ..., b_{j_n})/A)$. If $A = \emptyset$ we just say "indiscernible sequence".
- (ii) We have the same notion for an an arbitrary ordered set in place of $(\alpha, <)$.
- (iii) Given an infinite indiscernible sequence $(b_i)_i$ over A, by the EM-type of this sequence we mean the collection of $tp((b_{i_1}, ..b_{i_n})/A)$ for $n = 1, 2, ...$

Lemma 1.2. *If $(b_i)_i$ is an infinite indisceribIe sequence over A, then for any infinite totally ordered set $(J, <)$ there is an indiscernible sequence $(c_j : j \in J)$ over A with EM type the same as that of $(b_i)_i$.*

Proof. Compactness (exercise). □

The most powerful tool for producing indiscernible sequences uses the Erdos-Rado theorem, and is:

Lemma 1.3. *Given A, there is some λ such that whenever $(b_i : i < \lambda)$ is a set of tuples of the same sort then there is an indiscernible sequence $(c_i : i < \omega)$ over A such that for each n there are $j_1 < .. < j_n < \lambda$ such that $tp(c_1, .., c_n/A) = tp(b_{j_1}, .., b_{j_n}/A)$.*

The following special case can be proved using Ramsey's theorem

Lemma 1.4. *Suppose that for each n, $\Sigma_n(x_1, .., x_n)$ is a partial type over A, and that $(b_i : i < \omega)$ is a sequence (of suitable tuples) such that for each n, and $i_1 < ... < in < \omega$, $\models \Sigma_n(b_i, .., b_{i_n})$. Then we can find an indiscernible sequence $(c_i : i < \omega)$ over A with the same feature.*

Lascar strong types

The material here is relevant to later talks.

Definition 1.5.
- (i) a and b are said to have the same strong type over A, if $E(a, b)$ for each finite (finitely many classes) equivalence relation E definable over A.
- (ii) a and b have the same compact (or KP) strong type over A if $E(a, b)$ whenever E is an equivalence relation, type-definable over A, and with a bounded ($< \bar{\kappa}$) number of classes, equivalently with $\leq 2^{|L|+\omega+|A|}$ classes.
- (iii) a and b have the same Lascar strong type over A, if $E(a, b)$ whenever E is a bounded equivalence relation which is $Aut(\bar{M}/A)$-invariant.

Some remarks.

- We write $stp(a/A)$, $KPstp(a/A)$, $Lstp(a/A)$, for strong type of a over A etc.
- To be honest we have only really defined when e.g. $Lstp(a/A) = Lstp(b/A)$, but we can identify $Lstp(a/A)$ with the the class of a modulo the smallest bounded $Aut(\bar{M}/A)$-invariant equivalence relation, etc.
- $stp(b/A)$ "=" $tp(b/acl^{eq}(A))$ and $KPstp(b/A)$ "=" $tp(b/bdd^{heq}(A))$.
- Fleshing out the details of the above is left as an exercise.
- Of course $Lstp(b/A)$ implies (or refines) $KPst(b/A)$ implies $stp(b/A)$ implies $tp(b/A)$, and these all coincide when A is a model M.

Lemma 1.6. *$Lstp(a/A) = Lstp(b/A)$ if and only if there are $a = a_0, a_1, ..., a_n = b$ such that for each $i = 1, .., n-1$, a_i, a_{i+1} begin (are the first two members of) some infinite A-indiscernible sequence.*

We leave the full proof as an exercise (??). But note the easy part: suppose that $(a_i : i < \omega)$ is an indisceribile sequence over A. Then all members have the same Lascar strong type over A. For if not, then by "stretching" the sequence using Lemma 1.2 we could find arbitrarily many Lascar strong types over A, contradiction.

- A Lascar strong automorphism over A is an automorphism of $\bar{M}$ which fixes all Lascar strong types over A. We let $Autf(\bar{M}/A)$ denote the group of such automorphisms.
- Fact. $Lstp(a/A) = Lstp(b/A)$ iff there is a Lascar strong automorphism over A taking a to b.
- For a given theory T (or class of theories) it is important to known when various notions of strong type coincide.
- For example in a stable theory they are all the same.
- In a simple theory, $Lstp = KPstp$ but it is open whether this coincides with stp.

2 NIP

Definitions and equivalences

Definition 2.1.
- (i) The L-formula $\phi(x, y)$ is said to be unstable if there are a_i, b_i for $i < \omega$ such that for $i, j < \omega$, $\models \phi(a_i, b_j)$ iff $i < j$. T is said to be unstable if some formula $\phi(x, y)$ is unstable.

- (ii) $\phi(x,y) \in L$ has the independence property if there are a_i for $i < \omega$ and b_s for $s \subseteq \omega$ such that for all i, s, $\models \phi(a_i, b_s)$ iff $i \in s$. T is said to have the independence property if some formula $\phi(x,y)$ has it.
- (iii) We say that $\phi(x,y)$ is stable if it is not unstable, and has NIP (or is dependent) if does not have the independence property. Likewise for theories.

Lemma 2.2.
- *(i) If $\phi(x,y)$ has the independence property then it is unstable.*
- *(ii) $\phi(x,y)$ is stable iff there is $n_\phi < \omega$ such that for any indiscernible sequence $(a_i : i < \omega)$ and any b (of appropriate sorts), $|\{i < \omega :\models \phi(a_i, b)\}| \leq n_\phi$ or $|\{i < \omega :\models \neg\phi(a_i, b)\}| \leq n_\phi$.*
- *(iii) $\phi(x,y)$ has NIP iff there is n_ϕ such that for any indiscernible sequence $(a_i : i < \omega)$ and b there do not exist $i_1 < i_2 < .... < i_{n_\phi}$ such that for each $j = 1, .., n-1$, $\models \phi(a_{i_j}, b) \leftrightarrow \neg\phi(a_{i_j}, b)$. (i.e. the truth value of $\phi(a_i, b)$ cannot change its mind at least n_ϕ times.)*

Proof. Exercise. □

Corollary 2.3. *Suppose T has NIP and $(b_i : i < \omega)$ is an indiscernible sequence, and $\phi(x,y) \in L$. Then $\{\phi(x, b_i)\Delta\phi(x, b_{i+1}) : i = 0, 2, 4, ...\}$ is inconsistent.*
(Where $\phi(x,y)\Delta\phi(x,z)$ denotes $(\phi(x,y) \wedge \neg\phi(x,z)) \vee (\neg\phi(x,y) \wedge \phi(x,z))$.)

Proof. If not let a realize this set of formulas. Either there are infinitely many even i such that $\models \phi(a, b_i)$ or infinitely many even i such that $\models \neg\phi(a, b_i)$. In either case we contradict Lemma 2.2 (iii). □

Average types and eventual types

- In this mini-section we will assume that T has NIP.
- Suppose $I = (a_i : i < \omega)$ is an indiscernible sequence (or more generally an indiscernible sequence where the index set has no greatest element).
- By Lemma 2.2 for any formula $\phi(x, b)$, either for eventually all $i < \omega$, $\models \phi(a_i, b)$, or for eventually all $i < \omega$, $\models \neg\phi(a_i, b)$.
- So for any set B of parameters (or even for $B = \bar{M}$) we can define $Av(I/B)$, the average type of I over B to be those formulas $\phi(x)$ over B which are true of eventually all a_i.
- So $Av(I/B) \in S_x(B)$.

Definition 2.4. • (i) Let I be an infinite indiscernible sequence over A (where index set can be taken to be ω). We will call I A-special, if whenever I_1, I_2 realize $tp(I/A)$ there is J realizing $tp(I/A)$ such that each of I_1J, I_2J is indiscernible over A.

- (ii) Suppose I is A-special, and $B \supseteq A$. We define $Ev(I/B)$ (eventual type of I over B) to be the set of formulas $\phi(x)$ over B such that for any I' realizing $tp(I/A)$ there is J realizing $tp(I/A)$ such that $I'J$ is indiscernible over A and $\phi(x) \in Av(I'J/B)$ (equivalently $Av(J/B)$)

Lemma 2.5. *Let I be A-special. Then*

- *(i) For any $B \supseteq A$, $Ev(I/A) \in S(B)$.*
- *(ii) For $\phi(x)$ over B, $\phi(x) \in Ev(I/B)$ iff there is I' which realizes $tp(I/A)$ and witnesses a greatest possible alternation of truth values of $\phi(x)$, such that $\phi(x) \in Av(I'/B)$, i.e. $\phi(x)$ is true for eventually all elements of I'.*
- *(iii) If $A \subseteq B \subseteq C$ then $Ev(I/B) \subseteq Ev(I/C)$*
- *(iv) $Ev(I/B)$ depends only on $tp(I/A)$ (in fact on its EM-type over A).*

Proof.

- We start by proving (ii).
- Suppose $\phi(x) \in Ev(I/B)$. Let I' realize $tp(I/A)$ and witness a greatest possible alternation of truth values of $\phi(x)$. By definition of $Ev(I/A)$ there is J such that $I'J$ is indiscernible over A and $\phi(x) \in Av(I'J/B)$. But by choice of I' this implies that $\phi(x) \in Av(I'/B)$.
- On the other hand suppose the RHS holds, and let I' witness greatest alternation of truth values of $\phi(x)$. Let I_1 realize $tp(I/A)$ and let J be such that both $I'J$ and I_1J are A-indiscernible. So $\phi(x) \in Av(I'J/B) = Av(I_1J/B)$. Good.
- We now prove (i) using the above characterization.
- We want to show that for any $\phi(x)$ over B exactly one of $\phi(x)$, $\neg\phi(x)$ is in $Ev(I/B)$.
- By (ii) we have at least one.
- Suppose that I', I'' each witness the maximum alternation of truth values of $\phi(x)$. Let J be such that both $I'J$, $I''J$ are indiscernible over A. Then $Av(I'/B) = Av(I'J/B) = Av(J/B) = Av(I''J/B) = Av(I''/B)$. So $\phi(x) \in Av(I'/B)$ iff $\phi(x) \in Av(I''/B)$. Good.

The rest is left as an exercise.

Exercise: Give an example of an indiscernible sequence in an o-minimal theory which is independent (in the o-minimal sense) but not special.