II. Forking and Lascar strong types in NIP theories

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1 Forking

Introduction

- Until recently I believed that Shelah's theory of forking and dividing was only meaningful in simple theories.
- In fact in theories with NIP it turns out to be very meaningful (as Shelah saw some time ago).
- In many examples of NIP theories where there is already a well-behaved notion of independence and dimension (such as o-minimal theories) forking can come from a lowering of "order of magnitude" within a given dimension (and is related to finding the right notion of "infinitesimal").
- Forking also gives rise to very nice analogues of stable group theory within suitable NIP theories (although this won't figure in my current talks).
- The level of this lecture and these notes will be notched up a bit compared to the introductory lecture.

Forking in general

Definition 1.1.

- (i) Let $\Sigma(x)$ be a partial type (even over the big model), closed under under finite conjunctions. $\Sigma(x)$ is said to divide over A if there is $\phi(x,b) \in \Sigma$ and an A-indiscernible sequence $(b_i : i < \omega)$ of realizations of $tp(b/A)$ such that $\{\phi(x,b_i) : i < \omega\}$ is inconsistent.
- (ii) $\Sigma(x)$ is said to fork over A if Σ implies some finite disjunction of formulas (over $\bar{M}$)) each of which divides over A.

We begin with some trivialities (left to the reader) which also explain why the notion of forking, rather than just dividing, is introduced.

Lemma 1.2. *(i) $\Sigma(x)$ does not fork over A iff for any B containing $dom(\Sigma)$, (in particular for $B = \bar{M}$), Σ extends to a complete type $p(x)$ over B which does not divide over A.*
(ii) If $p(x)$ is a global complete type (or even a complete type over a sufficiently saturated model), and A is small, then p does not divide over A iff p does not fork over A.

- We first describe the situation when T is stable.
- Any partial type $\Sigma(x)$ does not fork over its domain.
- Forking equals dividing in the sense that a partial type $\Sigma(x)$ forks over A iff it divides over A.
- Let $p(x) \in S(\bar{M})$ be a global type, and A a (small) subset of $\bar{M}$.
- Then p does not fork over A iff p is definable over $acl^{eq}(A)$ iff p is almost finitely satisfiable in A (i.e. is finitely satisfiable in any model containing A).
- Moreover p is the unique global type which extends $p|(acl^{eq}(A))$ and does not fork over A.
- We also have the "algebraic" properties of (non)forking: symmetry, transitivity, existence of nonforking extensions,.., which are valid in any simple theory.

Invariant types

Definition 1.3. Let $p(x) \in S(\bar{M})$ (a global type) and A small.

- (i) We say that p does not split over A if p is $Aut(\bar{M}/A)$-invariant, equivalently whether or not $\phi(x, b) \in p$ depends on $tp(b/A)$.
- (ii) We say that p does not Lascar-split over A if p is $Autf(\bar{M}/A)$-invariant, equivalently whether or not $\phi(x, b) \in p$ depends on $Lstp(b/A)$.
- (iii) We say (with some abuse of language) that p is invariant if p is $Aut(\bar{M}/B)$-invariant for some small B.
- (So if p does not Lascar split over a small set then p is invariant (why?).)

- If the global complete type $p(x)$ is invariant then it has a kind of "infinitary" defining schema (over a small set A), namely for any $\phi(x,y) \in L$, there is a collection $d_p\phi$ of complete y-types over A such that $\phi(x,b) \in p$ iff $tp(b/A) \in d_p\phi$.

- So if B is any set containing A (and even containing $\bar{M}$ in some saturated elementary extension of $\bar{M}$, we can apply this schema to B to obtain $p|B$.

- Suppose $p(x), q(y)$ are global invariant types. Then by $pq \in S_{xy}(\bar{M})$ we mean $tp(a,b/\bar{M})$ where b realizes q and a realizes $p|Mb$. We can also form $qp \in S_{xy}(\bar{M})$ as $tp(a,b/\bar{M})$ where a realizes p and b realizes $q|Ma$.

- We let $pq = qp$.

- Let $p(x)$ be an invariant global complete type, and $x_1, x_2, ...$ disjoint copies of the variable x.

- Then $p^{(n)}(x_1, ...x_n)$ is defined inductively as $p^{(n-1)}(x_1, .., x_{n-1})p(x_n)$. And $p^{(\omega)}(x_i)_i$ is the union of the $p^{(n)}(x_1, .., x_n)$.

- So $p^{(\omega)}$ is $tp(a_1, a_2, a_3.../\bar{M})$ where a_1 realizes p, a_2 realizes $p|\bar{M}a_1$ etc.

- Any realization $(a_i)_{i<\omega}$ of $p^{(\omega)}(x_i)_i$ (with a_i corresponding to x_i, is easily seen to be an indiscernible sequence over $\bar{M}$.

- Assuming p to be $Aut(\bar{M}/A)$-invariant (or even $Autf(\bar{M}/A)$-invariant we may call any realization of $p^{(\omega)}|A$ in $\bar{M}$ a Morley sequence in p over A.

Lemma 1.4. • *i) Suppose that $p(x) \in S(\bar{M})$ is $Aut(\bar{M}/A)$-invariant. Then so is $p^{(\omega)}(x_i)_i$.*

- *(ii) If $p(x) \in S(\bar{M})$ is $Autf(\bar{M}/A)$-invariant, then so is $p^{(\omega)}$.*

- The proof is left as an exercise.

- Part (i) is easy.

- However part (ii) is a bit more tricky and uses Lemma 1.6 of Lecture I.

Back to NIP

The following is always true (not requiring NIP).

Lemma 1.5. *If $p(x) \in S(\bar{M})$ is $Autf(\bar{M}/A)$-invariant then p does not divide (fork) over A.*

Proof. If $(b_i : i < \omega)$ is an A-indiscernible sequence then all the b_i have the same Lascar strong type over A, so if $\phi(x, b_0) \in p$ then each $\phi(x, b_i) \in p$ hence $\{\phi(x, b_i) : i < \omega\}$ is consistent. □

The main point is that assuming NIP the converse to Lemma 1.5 holds.

Theorem 1.6. *Assume that T has NIP.*

- *(i) If $p(x) \in S(\bar{M})$ does not fork over (small) A, then p is $Autf(\bar{M}/A)$-invariant (so with earlier terminology, p is invariant).*
- *(ii) In particular if M is a model, and p does not divide (fork) over M then p is $Aut(\bar{M}/A)$-invariant.*

Proof.

- Suppose p does not fork over A. By Lemma 1.6 of Lecture I it is enough to show that if $\phi(x,y) \in L$ and $(b_i : i < \omega)$ is A-indiscernible, then $\phi(x,b_0) \in p$ iff $\phi(x,b_1) \in p$.
- If not then $\phi(x,b_0)\Delta\phi(x,b_1) \in p(x)$.
- But as $(b_0b_1, b_2b_3, ...)$ is also indiscernible over A, our asumptions imply that $\{\phi(x,b_i)\Delta\phi(x,b_{i+1}) : i = 0, 2, ...\}$ is consistent, contradicting Corollary 2.6 of Lecture I.

Lemma 1.7. *(NIP) Suppose $p(x) \in S(\bar{M})$ is $Aut(\bar{M}/A)$-invariant. Let I be a Morley sequence in p over A (namely a realization of $p^{(\omega)}|A$. Then I is A-special, and $p = Ev(I/\bar{M})$. In particular p is determined by the type of a Morley sequence in p over A.*

Proof.

- If I, J are realizations of of $p^{(\omega)}|A$ then clearly if I' is a realization of $p^{(\omega)}|IJ$ then II' and IJ are A-indiscernible.
- Moreover for any $B \supseteq A$ and realization I of $p^{(\omega)}$, if I' realizes $p^{(\omega)}|BI$ then obviously $Ev(I'/B) = p|B$. This suffices.

Remark: We could refine the notion of an A-indiscernible sequence I being A special, by requiring that whenever I' and I have the same Lascar strong type over A then there is J etc. With a suitable definition of $Ev(I/\bar{M})$ one could then write a version of Lemma 1.7 with "not forking over A" in place of $Aut(\bar{M}/A)$-invariant.

Corollary 1.8. *(NIP) For any A and $p(x) \in S(A)$, p has at most $2^{(|A|+\omega)}$ global nonforking extensions.*

Proof. Exercise.

On the other hand one can prove that if T has the independence property then for any $\lambda \geq |T|$ there is a model M of cardinality λ and a type $p(x)$ over M with at least $2^{2^{\lambda}}$ global nonforking extensions (in fact coheirs). So together with Corollary 1.8 this gives a nice characterization of theories with NIP.

- We can formulate the notion of a subset of (a sort in) $\bar{M}$ being "Borel".
- A set is "closed" if it is the solution set of a partial type over a small set.
- A set is Borel if it is in the σ-algebra generated by the closed sets.
- We could also call a set Borel over A, if it is in the σ-algebra generated by the solution sets of partial types over A.
- We call a set *strongly Borel* (over A) if it is a finite Boolean combination of closed (over A) sets.

Theorem 1.9. *(NIP) Let $p(x) \in S(\bar{M})$ be $Aut(\bar{M}/A)$-invariant. Then p is strongly Borel definable over A in the sense that it has a strongly Borel over A defining schema, namely for any formula $\phi(x,y)$ there is a strongly Borel set Y such that for any b, $\phi(x,b) \in p$ iff $b \in Y$.*

Proof. We use Lemma 2.5 of Lecture I.

- Let $\phi(x,y) \in L$, and $N = n_\phi$ given by Lemma I.2.2(iii). Let $Q(x_i)_i$ be $p^{(\omega)}|A$ and $Q_n(x_1, ..x_n)$ its restriction to $x_1, .., x_n$.
- Then $\phi(x,b) \in p$ iff for some $m \le N$ there is $(a_1, .., a_m)$ realizing Q_m with maximal possible successive alternations of truth values of $\phi(x,b)$, and moreover $\models \phi(a_m, b)$.
- I leave it as an exercise to express this as a strongly Borel over A condition.

2 Lascar strong types

Invariant types and KP-strong types

We begin with a result valid at the level of "invariant types", which then applies to the NIP context.

Lemma 2.1. *Let $p(x) \in S(\bar{M}/A)$ be $Autf(\bar{M}/A)$-invariant. Let c, d be realizations of p. Then $Lstp(c/A) = Lstp(d/A)$ iff there is some infinite sequence $\bar{a}$ such that both $(c\bar{a})$ and $(d\bar{a})$ realize $p^{(\omega)}|A$.*

Proof. By Lemma I.1.6, RHS implies LHS.
Now assume RHS. After possibly moving p by an A-automorphism, we may assume that $p(x)$ implies $Lstp(c/A)$. Let $(a_0, a_1, ...)$ realize $p^{(\omega)}$. Then $(a_1, a_2, ...)$ also realizes $p^{(\omega)}$. By Lemma 1.4(ii), $p^{(\omega)}$ is $Aut(\bar{M}/A)$-invariant. As $Lstp(c/A) = Lstp(a_0/A) = Lstp(d/A)$ we conclude that $tp(c, a_1, a_2, .../A) = tp(a_0, a_1, a_2, .../A) = tp(d, a_1, a_2, .../A)$, completing the proof.

Corollary 2.2. *Suppose $p(x) \in S(\bar{M})$ is $Autf(\bar{M}/A)$-invariant. Then for any realization a of $p|A$, $KPstp(a/A)$ equals $Lstp(a/A)$*

Proof. Lemma 2.1 implies that on realizations of $p|A$, the relation $Lstp(x_1/A) = Lstp(x_2/A)$ is type-definable over A, which suffices.

Restatement for NIP theories.

Corollary 2.3. *(NIP) Suppose $p(x) \in S(A)$ does not fork over A. Then on realizations of p, $KPstp$ over A equals $Lstp$ over A.*

Proof. Exercise using earlier results and propositional calculus.

Corollary 2.4. *(NIP) Suppose that T is 1-sorted, and that for every A, every complete 1-type over A does not fork over A (equivalently has a global nondividing extension). Then over any set A, $KPstp = Lstp$.*

Proof. Use a result of Shelah: assuming NIP if $tp(a/B)$ does not fork over A and $tp(b/Ba)$ does not fork over A, then $tp(a,b/B)$ does not fork over A, to conclude that any complete type over any A does not fork over A. Then use the previous Corollary.

Forking and $bdd(A)$-invariance

In this final minisection we aim to prove the following strengthening of Theorem 1.6.

Theorem 2.5. *(NIP) Let $p(x) \in S(\bar{M})$ and A a small set. Then p does not fork over A iff p is $bdd^{heq}(A)$-invariant (namely p is fixed by all automorphisms which fix all KP-strong types over A).*

Sketch of proof.

- This really falls out of of our previous results.
- (i) Note that (by Lemma 1.4 for example) $p^{(\omega)} \in S_\omega(\bar{M})$ does not fork over A.
- (ii) So by (i) and Corollary 2.3 if I realizes $p^{(\omega)}|A$ then $KPstp(I/A)$ implies $Lstp(I/A)$.
- Let $Q(x_i)_i = p^{(\omega)}|bdd^{heq}(A)$ (a KP-strong type over A).
- (iii) So (ii) says that Q implies a Lascar strong type over A.

- *Claim (iv).* Q is bdd^{heq}-special, more precisely $tp(I/bdd^{heq}(A))$ is $bdd^{heq}(A)$-special for some (any) I realizing Q.

- *Proof of Claim (iv).* Let I_1, I_2 realize Q, and let J realize $p^{(\omega)}$ (in a saturated model containing $\bar{M}$.

- Then $Lstp(I_1/A) = Lstp(I_2/A) = Lstp(J/A)$, hence as $p^{(\omega)}$ is $Autf(\bar{M}/A)$-invariant, both I_1J and I_2J are A-indiscernible. End of proof of Claim (iv).

- Deduce as in Lemma 1.7 that $p = Ev(I/\bar{M})$ for any realization I of Q.

- Hence p is $Aut(\bar{M}/bdd^{heq}(A))$-invariant as required.

Corollary 2.6. *(NIP) Suppose $p(x) \in S(\bar{M})$ does not fork over A. Then p is strongly Borel definable over $bdd^{heq}(A)$.*