Extending Definable Functions on Undefinable Curves

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Background

- We work here with o-minimal structures.
- A structure is o-minimal iff it is linearly ordered and any definable subset is a finite union of points and intervals.
- In an o-minimal structure, M, for any definable n-ary function, there exists a decomposition of M^n into finitely many definable "cells" such that the function is continuous on each cell.
- A consequence: every definable function in an o-minimal structure is "eventually" continuous, monotone, and unchanging in sign.
- To verify that a function is continuous on a definable set, it suffices to show that, for any two definable curves with the same endpoint, the limit of the function along the curves is the same.

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Let φ be a bounded definable function on M^n. Cell decomposition allows us to partition M^n into definable sets on which φ is continuous. In one dimension, φ can be extended continuously to the closure of each such set.

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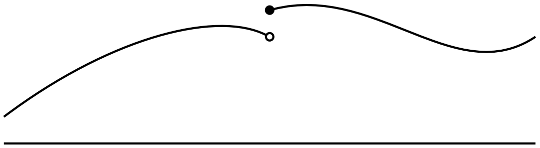

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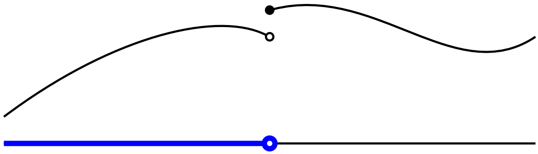

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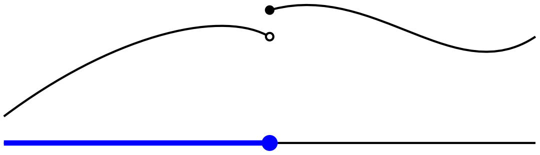

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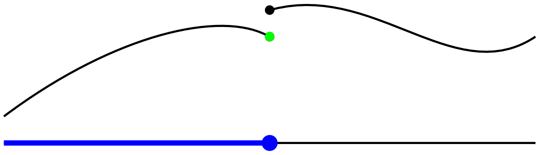

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An Example When Naive Extension Fails

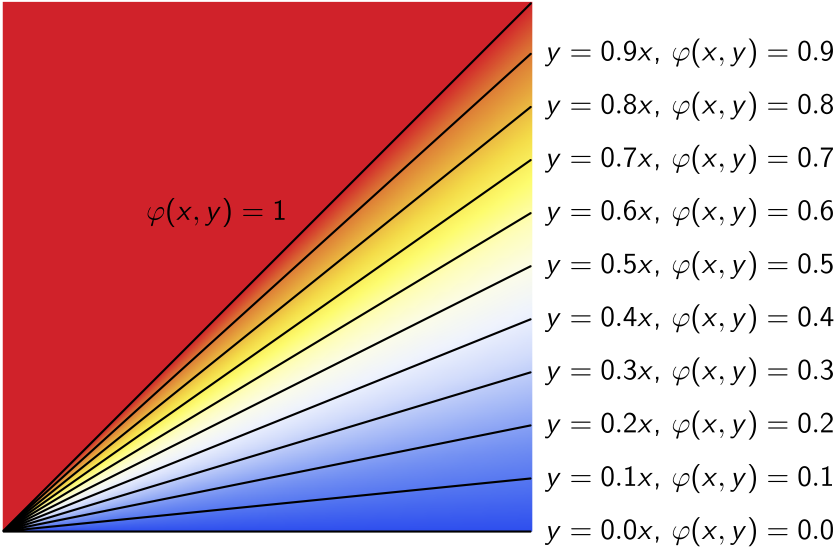
Example

Let $\varphi(x, y)$ be the function defined on the first quadrant by

$$\varphi(x, y) = \begin{cases} \frac{y}{x} & y < x \\ 1 & O.W. \end{cases}$$

Then φ is continuous on the cell $\{\langle x, y\rangle \mid x > 0, y > 0\}$, but not on its closure.

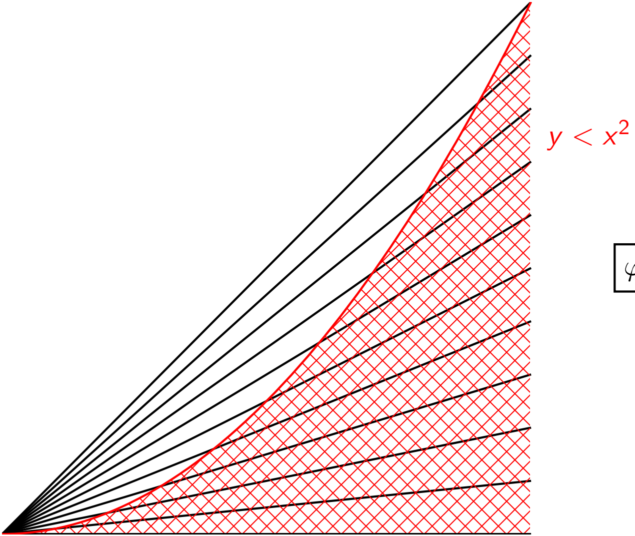
A Pretty Picture of Failure



$$\boxed{\varphi(x, y) = \min(1, y/x)}$$

However, in this example, for any point, a, in the closure, it is possible to give an open cell on which φ is continuous, φ can be continuously extended to the closure, and whose closure includes a.
For instance, if $a = \langle 0, 0 \rangle$, consider the cell $\{\langle x, y \rangle \mid x > 0, y < x^2\}$:

A Cell That Works



$$\varphi(x, y) = \min(1, y/x)$$

For Any Definable Curve, There Is a Cell That Works

This cell's closure contains the x-axis. Every definable curve in this cell has slope equal to 0 at 0, so φ goes to 0 along every definable curve in the cell.

In general, given any definable curve going to the origin, we can take a pair of parabolas whose derivatives at 0 are the same as the curve's at 0, giving us a cell on which φ extends continuously to the closure.

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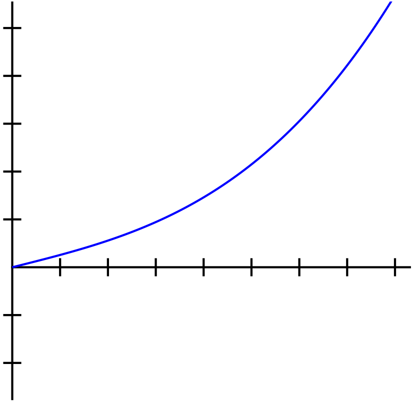

$\gamma(t) = \langle t, t/4 + t^3/2 \rangle$

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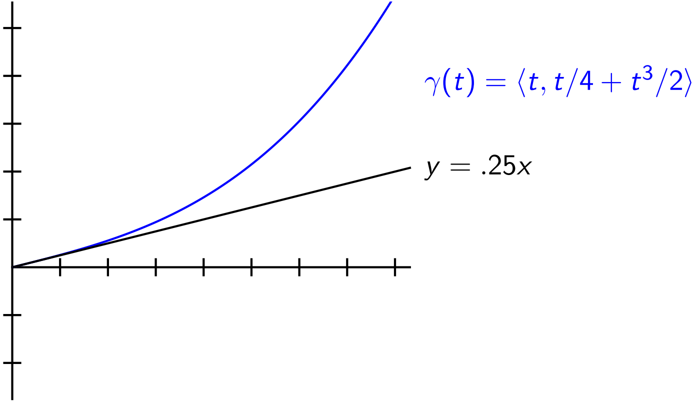

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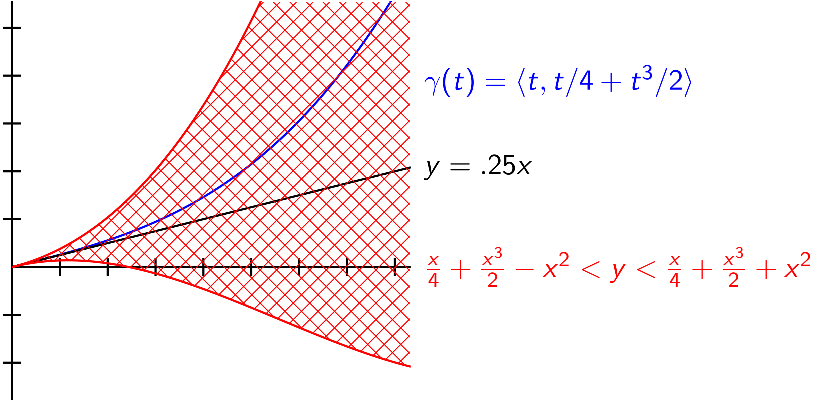

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What about the question for non-definable curves? Given a (non-definable) curve, can we find a set on which the function is continuous, which contains the curve, and on whose closure the function extends continuously.

There are easy examples of failure.

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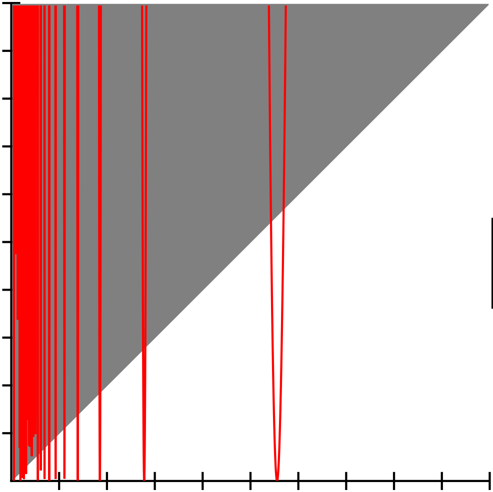
Oscillating Disaster



$\gamma(t) = \langle t, \sin^2(1/t) \rangle$

$$\varphi(x,y) = \begin{cases} 1 & y \geq x \\ 0 & y < x \end{cases}$$

Restricting to Good Curves

- Clearly, we must impose some conditions on our curves.
- Requiring that the curve lie in a Hardy field would prevent oscillation.
- Not clear why derivatives should matter.
- Important point is that components of curve (and every definable function of them) are eventually *comparable* to any definable function. Motivates:

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Comparing Curves

Definition

Let f and g be unary functions (not necessarily definable), each of whose domains includes some positive neighborhood of 0. f and g are comparable if, for some $s > 0$, one of a) for all $t \in (0, s)$, $f(t) < g(t)$; b) for all $t \in (0, s)$, $f(t) = g(t)$; or c) for all $t \in (0, s)$, $f(t) > g(t)$.

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Definition

Let M be any o-minimal structure expanding a real closed field, and let $\mathcal{F}$ be the set of all germs of functions in a positive neighborhood of 0. Let Γ be a subset of $\mathcal{F}$. Let $F_\Gamma \subset \mathcal{F}$ be the structure in the language $\{f \mid f \text{ an } M\text{-definable function}\}$, generated from the elements of Γ, where, for instance, the definable function $x + y$, when applied to $\gamma_1(t), \gamma_2(t) \in \Gamma$, yields the function $\gamma_1(t) + \gamma_2(t)$ – note that the operations are well-defined on the germs, by choosing representatives. (We will not distinguish between germs and representatives henceforth.) We say that F_Γ is ordered if all functions in F_Γ are pairwise comparable.
If $\gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$ is a curve, we say that γ is orderable if F_γ is ordered.

Ordered Failure

- Unfortunately, requiring that γ be orderable is not enough to make any bounded definable function continuous on γ's closure.
- Let $M = (\mathbb{R}, +, \cdot, <)$. Let $\varphi(x, y)$ be as before – $\min(1, y/x)$, and let $\gamma(t) = \langle t, -t/\ln t\rangle$, so γ is undefinable. Note, though, that since γ is definable in the expansion of M, $(\mathbb{R}, +, \cdot, <, \exp)$, γ is certainly orderable.
- $\varphi(\gamma(t)) = -1/\ln t$, so $\lim_{t\to 0^+} \varphi(\gamma(t)) = 0$.

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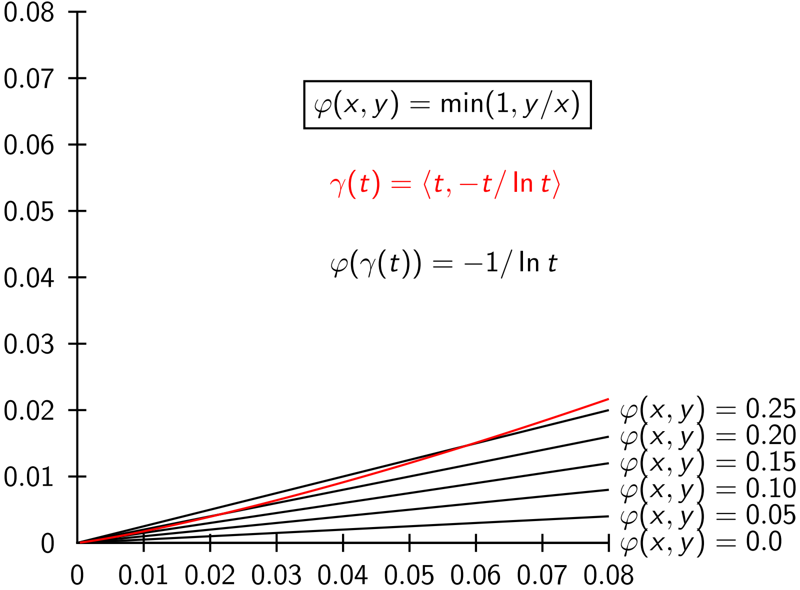

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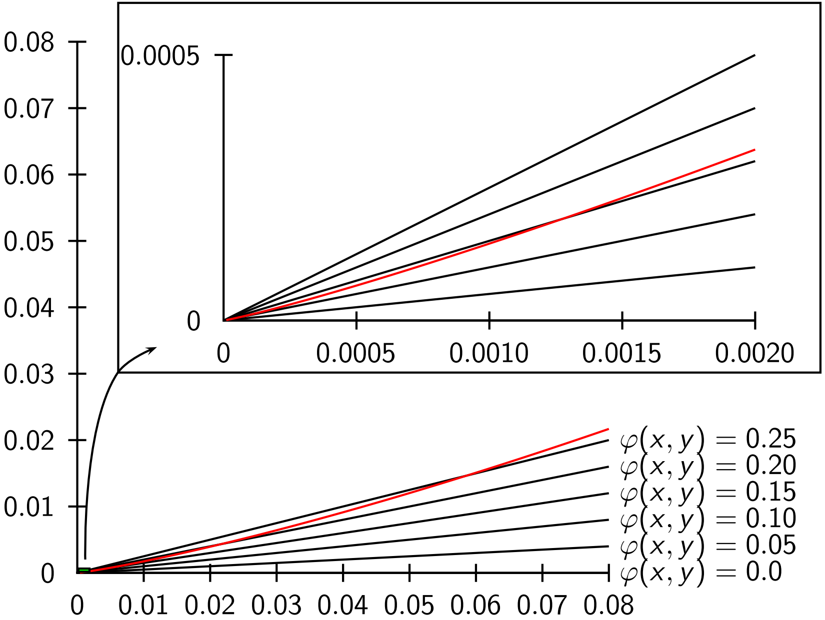

γ Cannot Be Squeezed

- $-1/\ln t$ goes to 0, but it is also greater than t^d, for any $d > 1$, for sufficiently small t:
- Quick check:

$$\frac{-1}{\ln t} > t^d \iff \frac{1}{t^d} > -\ln t \iff \frac{1}{t} < e^{1/t^d} \iff x^{1/d} < e^x,$$

  where $x = \frac{1}{t^d}$. As t goes to 0, x goes to ∞, and thus the last inequality will hold.
- It is not hard to see that any definable set in $(\mathbb{R}, +, \cdot)$ that contains γ must contain the curve $\langle t, at \rangle$, for some real positive a, as well as the curve $\langle t, bt^{1+q} \rangle$, for some real positive b and positive rational q.
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Why Did γ Fail?

The failure of γ can be seen as coming from the fact that we could not squeeze γ sufficiently. The gap between a linear function and a higher-power function is too great. To more closely analyze this, we can abstract out the "type" of γ.

The Limit Type of a Curve

Lemma

Let $\gamma = \langle \gamma_1, \ldots, \gamma_k \rangle \subset \mathcal{F}$*. Suppose* γ *is orderable. Let* $\gamma(t)$ *denote the sequence* $\langle \gamma_1(t), \ldots, \gamma_k(t) \rangle \in M^k$*, for* $t \in M$*. Then* $\lim_{t \to 0^+} \mathrm{tp}(\gamma(t)/M)$ *exists, in the following sense: for each formula* $\psi(x_1, \ldots, x_k)$ *in* M*, there is some* $s > 0$ *such that either* $\psi(\gamma(t))$ *holds for all* $t \in (0, s)$*, or* $\neg\psi(\gamma(t))$ *holds for all* $t \in (0, s)$*.*

Curve Limit Type Determines Definable Set Membership

Definition

With γ as above, let $\text{tp}(\gamma/M)$ denote $\lim_{t\to 0^+} \text{tp}(\gamma(t)/M)$. We can then talk about the type of γ_i over $\gamma_{<i}M$.

Proposition

Let γ be an orderable curve. Then, for any definable C, there exists an $s > 0$ such that $\gamma((0,s)) \subseteq C$ if and only if $C \in \text{tp}(\gamma/M)$.

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Back to Bad γ

We now return to $\gamma(t) = \langle t, -t/\ln t\rangle$, and examine $\mathrm{tp}(\gamma)$.

- For every $r > 0 \in \mathbb{R}_+$, $x_1 < r$ is in $\mathrm{tp}(\gamma)$.
- For every $r \in \mathbb{R}_+$, $x_2 < rx_1$ is in $\mathrm{tp}(\gamma)$.
- For every $r \in \mathbb{R}$, $q \in \mathbb{Q}_+$, $x_2 > rx_1^{1+q}$ is in $\mathrm{tp}(\gamma)$.

Notice that, for any $\langle c_1, c_2\rangle \models \mathrm{tp}(\gamma)$, $\langle c_1, rc_2\rangle \models \mathrm{tp}(\gamma)$, for any real positive r.

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Failure of Continuity Extension for a Type

Since we have equivalence of definable set membership for curves and their types, we can rephrase our failure with γ as follows:

Example

Take our model to be $(\mathbb{R}, +, \cdot, <)$. Let $p(x, y)$ be the type which says that x is greater than 0 but less than every real, and that y is less than rx, for any $r \in \mathbb{R}_+$, but greater than rx^{1+q}, for any $r \in \mathbb{R}$, $q \in \mathbb{Q}_+$. It is easy to see that these conditions generate a complete consistent type. Let φ be as before: $\varphi(x, y) = y/x$ if $y < x$, and 1 otherwise.
There is no definable set, C, such that $C \in p$, φ is continuous on C, and φ extends continuously to $\overline{C}$.

To say precisely what is wrong with p, we will need to give a classification of o-minimal types.

Failure of Continuity Extension for a Type

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Take our model to be $(\mathbb{R}, +, \cdot, <)$. Let $p(x, y)$ be the type which says that x is greater than 0 but less than every real, and that y is less than rx, for any $r \in \mathbb{R}_+$, but greater than rx^{1+q}, for any $r \in \mathbb{R}$, $q \in \mathbb{Q}_+$. It is easy to see that these conditions generate a complete consistent type. Let φ be as before: $\varphi(x, y) = y/x$ if $y < x$, and 1 otherwise.
There is no definable set, C, such that $C \in p$, φ is continuous on C, and φ extends continuously to $\overline{C}$.

To say precisely what is wrong with p, we will need to give a classification of o-minimal types.

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Order Type Implies Type

Dave Marker, in a 1986 paper, "Omitting Types in O-Minimal Theories," pointed out a basic classification of one-dimensional types in o-minimality – that of cuts and noncuts. While this classification is valid for any structure with a transitive binary relation, it is especially effective in o-minimal structures because of the following:

Lemma (Marker)

Let M be o-minimal, let $A = \mathrm{acl}(A)$ be a subset of M, and let $p \in S_1(A)$. Then the formulas in p of the form $x > a$, $x < a$, and $x = a$ generate p.

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Definitions of Cuts and Noncuts

Given this characterization of o-minimal types, we can classify non-algebraic types in an apparently trivial way:

Definition

For $A \subset M$, $A = \text{acl}(A)$, $p \in S_1(A)$ is a cut iff it is non-algebraic and (1) there are formulas of the form $a < x$ and $x < a$ in p, and (2) for every formula of the form $a < x$ in p, there is $b > a$ such that $b < x$ is in p, and similarly for $x < a$. p is a noncut if it is non-algebraic and not a cut.

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Characterization of Noncuts

- Despite their negative definition, noncuts are actually quite simple to describe. A noncut has one of the following four forms, for some $a \in A$:
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  - $\{x > a\} \cup \{x < b \mid b > a, b \in A\}$
  - $\{x < a\} \cup \{x > b \mid b < a, b \in A\}$
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Dichotomy of Cuts – Uniquely Realizable or Not

Cuts come in two varieties – uniquely realizable and non-uniquely realizable. We note here that any set A in an o-minimal structure has a unique prime model containing it, denoted $\mathrm{Pr}(A)$.

Definition

A cut p is uniquely realizable over A iff for any c realizing p, c is the only realization of p in $\mathrm{Pr}(Ac)$. We use "u.r." as shorthand for "uniquely realizable" and "n.r." for "non-uniquely realizable."

For example, $\mathrm{tp}(\pi)$ is u.r. over $\overline{\mathbb{Q}} \cap \mathbb{R}$. On the other hand, in $\mathrm{Pr}(\mathbb{R}\epsilon)$ as an ordered group, where ϵ is an infinitesimal, $\{x < r \mid r \in \mathbb{R}_+\} \cup \{x > n\epsilon \mid n \in \mathbb{N}\}$ is n.r.

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$p = \text{tp}(\gamma)$ is n.r.

As we pointed out before, if $\langle c_1, c_2 \rangle \models p$ (where $p = \text{tp}(\gamma)$ from before), then so does $\langle c_1, 2c_2 \rangle$, for example. Thus, $\text{tp}(c_2/c_1)$ is an n.r. cut.

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Main Result

Proposition

Let M be an o-minimal structure, φ a bounded definable function on M^n, and $p(x_1, \ldots, x_n)$ an n-type over M. Let $c = \langle c_1, \ldots, c_n \rangle$ realize p. Suppose that $\mathrm{tp}(c_i/Mc_1 \ldots c_{i-1})$ is not an n.r. cut, $i = 1, \ldots, n$. Then there is an M-definable subset of M^n, C, such that φ is continuous on C, φ can be continuously extended to $\overline{C}$, and $c \in \overline{C}$.

Outline of Proof

- C is constructed "from the top down." We assume that C has been constructed if we restrict to $\langle x_{i+1}, \ldots, x_n \rangle$, taking $\langle x_1, \ldots, x_i \rangle$ as parameters, and then show that it can be extended to x_i.
- We then hold $\langle x_1, \ldots, x_{i-1} \rangle$ fixed, and consider φ_{x_i} as a function on $\langle x_{i+1}, \ldots, x_n \rangle$, with parameter x_i.
- By a result of van den Dries, since φ_{x_i} is continuous on $\overline{C_{x_i}}$, for each x_i, we can restrict to an interval in x_i on which φ is continuous as a function of $\langle x_i, \ldots, x_n \rangle$.

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Extending to x_i Boundary - Noncut

There are two possibilities – either c_i is a noncut over $c_1 \dots c_{i-1}$, or it is a cut.

In the noncut case, we ensure continuity as follows:

- For any n-tuple, y, the distance between y_i and any definable function of the previous y_j's is definable.
- Each j such that c_j is a noncut over $Mc_1 \dots c_{j-1}$ gives a definable function, α_j.
- Consider the function that is the minimum, over j, of the difference between a point's jth coordinate and α_j of the preceding coordinates.
- In constructing C, we can require that, if y and y' agree on their first i coordinates, then the difference of $\varphi(y)$ and $\varphi(y')$ is bounded by the above function, and thus, if we take any curve whose x_i coordinate goes to the noncut boundary, the value of φ along that curve will converge to the same value, so we can extend φ to the boundary.

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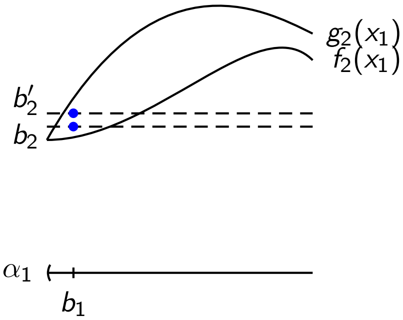

$$|\varphi(b_1, b_2) - \varphi(b_1, b_2')| < b_1 - \alpha_1$$

Squeezing - Cut

- If $\mathrm{tp}(c_i/c_{<i})$ is a cut, both boundary elements can be chosen inside the interval of continuity.
- The values of φ shouldn't differ too much on the two boundaries, as they did in the counterexample. If they did, then the assumption that points that agree on their first i coordinates have φ values close together would fail.
- We accomplish this by choosing a small δ, such that, if two i-tuples differ only in their ith coordinate, and by less than δ, then they can be extended by the same $(n-i)$-tuple to form two n-tuples that are both in C. This requires a u.r. cut, since otherwise we may not be able to bound the ith coordinate by δ.
- By continuity in the ith coordinate, we can ask that, if two points differ only on their ith coordinate, and by less than δ, then the difference when φ is evaluated at them is very small.
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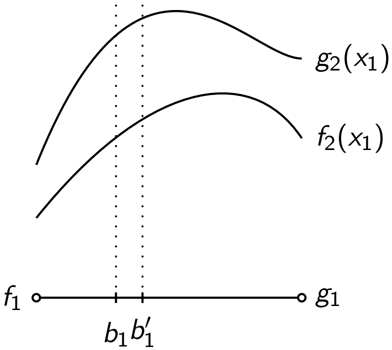

$b_1' - b_1 < \delta$

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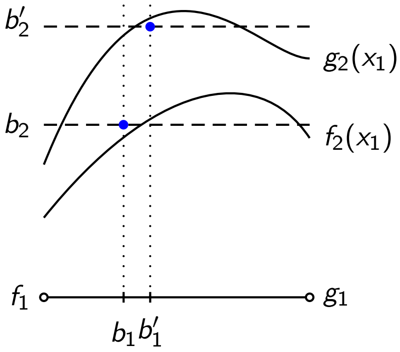

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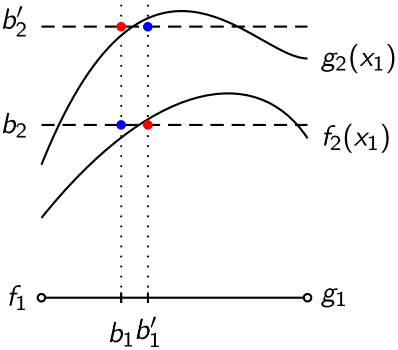

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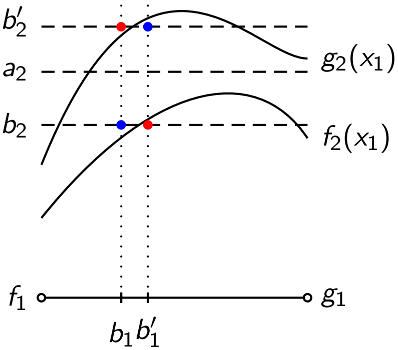

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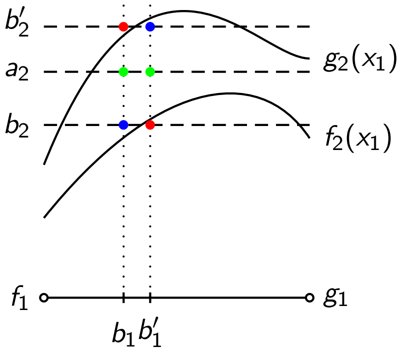

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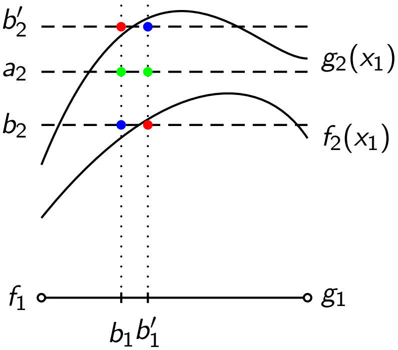

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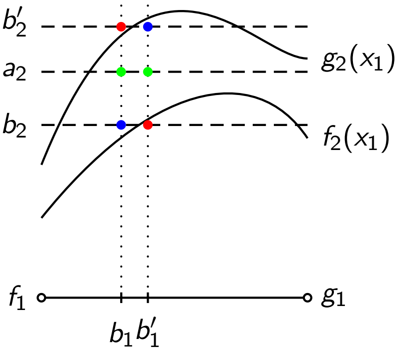

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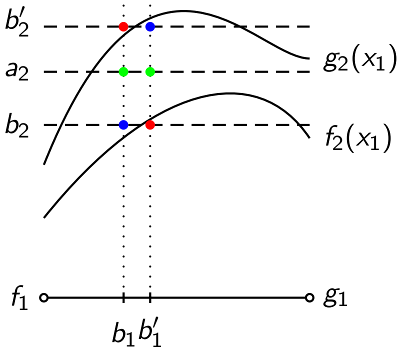

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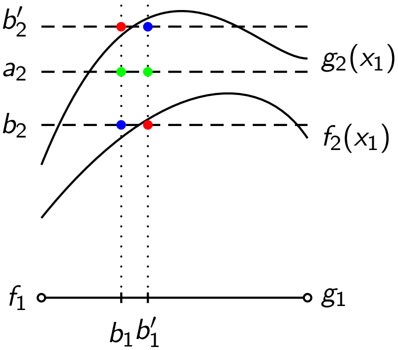

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$$b_2 \notin C_{b_1'}, b_2' \notin C_{b_1}$$
$$a_2 \in C_{b_1}, a_2 \in C_{b_2}$$

$|\varphi(b_1, b_2) - \varphi(b_1, a_2)| < \epsilon/3$ (By induction – same through i)

$|\varphi(b_1, a_2) - \varphi(b_1', a_2)| < \epsilon/3$ (By selection of δ – same except at i)

$|\varphi(b_1', a_2) - \varphi(b_1', b_2')| < \epsilon/3$ (By induction – same through i)

$|\varphi(b_1, b_2) - \varphi(b_1', b_2')| < \epsilon$

Conclusion

With the proposition, our original case of a curve is resolved. If the curve can be presented so that its type contains no n.r. cuts, then, for any bounded definable function, we can find a definable set containing the curve, onto whose closure the function extends continuously.