TUTORIAL ON DEPENDENT THEORIES

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1. Strict order property

PART I - Independence property and strict order property

- We say that a formula $\varphi(\bar{x}, \bar{y})$ has the *strict order property* if there exists an indiscernible sequence $\langle \bar{b}_i : i < \omega \rangle$ such that
$$\left[\exists \bar{x} \neg \varphi(\bar{x}, \bar{b}_i) \wedge \varphi(\bar{x}, \bar{b}_j)\right] \iff i < j$$
- A theory T has the strict order property if some formula (maybe with parameters) does.
- ***Exercise:*** show that T has the strict order property if and only if there exists a formula $\theta(\bar{x}, \bar{y})$ which defines on the monster model of T a partial order with infinite chains.

- **Theorem**(Shelah) T is unstable if and only if it has the independence property or the strict order property.
- **Exercise** Show that if T has the independence property or the strict order property, then T is unstable. Moreover, if a formula $\varphi(\bar{x}, \bar{y})$ has the strict order property or the independence property, then it is unstable (that is, it has the order property).
- **More precisely, we will prove**: Let $\varphi(\bar{x}, \bar{y})$ be an *unstable dependent* formula, the instability witnessed by indiscernible sequences $I = \langle \bar{a}_i : i \in \mathbb{Q} \rangle$, $J = \langle \bar{b}_i : i \in \mathbb{Q} \rangle$. Then there exists a formula $\psi(\bar{x}, \bar{y}, \bar{c})$ such that
  - $\psi(\bar{x}, \bar{y}, \bar{c})$ implies $\varphi(\bar{x}, \bar{y})$
  - ψ has the strict order property exemplified by a finite subsequence of J
  - $\bar{c} \subseteq \cup J$

(*) By dependence there exists k such that
$$\{\varphi^{i(\text{mod } 2)}(\bar{x}, \bar{b}_i) \colon i \in \mathbb{N}, i < k\}$$
is inconsistent.

(**) On the other hand, by instability, for every $\ell < k$ we have

$$\{\neg\varphi(\bar{x}, \bar{b}_i) \colon i < \ell\} \cup \{\varphi(\bar{x}, \bar{b}_i) \colon i \geq \ell\}$$

is consistent witnessed by $\bar{a}_{\ell-\frac{1}{2}}$.

✓ Clearly we can get from (*) to (**) by replacing $\varphi(\bar{x}, \bar{b}_i) \& \neg\varphi(\bar{x}, \bar{b}_{i+1})$ with $\neg\varphi(\bar{x}, \bar{b}_i) \& \varphi(\bar{x}, \bar{b}_{i+1})$ one at a time.

- This means that there exists $\eta \colon k \to 2$ and $\ell < k$ such that
- $$\{\varphi^{\eta(i)}(\bar{x}, \bar{b}_i) \colon i \neq \ell, \ell+1\} \cup \{\varphi(\bar{x}, \bar{b}_\ell), \neg\varphi(\bar{x}, \bar{b}_{\ell+1})\}$$
  is inconsistent, but
- $$\{\varphi^{\eta(i)}(\bar{x}, \bar{b}_i) \colon i \neq \ell, \ell+1\} \cup \{\neg\varphi(\bar{x}, \bar{b}_\ell), \varphi(\bar{x}, \bar{b}_{\ell+1})\}$$
  is consistent.
- Let us define
  $$\psi_1(\bar{x}) = \bigwedge_{i \neq \ell, \ell+1} \varphi^{\eta(i)}(\bar{x}, \bar{b}_i)$$

- By indiscernibility we have the following for any $i < j \in \mathbb{Q} \cap (\ell, \ell+1)$:
- $$\psi_1(\bar{x}) \bigwedge \{\varphi(\bar{x}, \bar{b}_i), \neg\varphi(\bar{x}, \bar{b}_j)\}$$
  is inconsistent, but
- $$\psi_1(\bar{x}) \bigwedge \{\neg\varphi(\bar{x}, \bar{b}_i), \varphi(\bar{x}, \bar{b}_j)\}$$
  is consistent

- Let us define
  $$\psi(\bar{x}, \bar{y}) = \psi_1(\bar{x}) \bigwedge \varphi(\bar{x}, \bar{y})$$
- and denote $J' = \langle \bar{b}_i : i \in \mathbb{Q} \cap (\ell, \ell+1) \rangle$
- So on J' we have:
  $$\exists \bar{x} \neg\psi(\bar{x}, \bar{b}_i) \wedge \psi(\bar{x}, \bar{b}_j) \iff i < j$$
- This completes the proof.

Recall: a (partial) type p is called *stable* if every extension of it is definable.
The following are equivalent for a dependent theory T:

- p is stable.
- For every $B \supseteq A$, p has at most $|B|^{\aleph_0}$ extensions in $\mathrm{S}(B)$.

- There is no formula $\varphi(\bar{x}, \bar{y})$ (with parameters from $\mathfrak{C}$) exemplifying the order property with respect to indiscernible sequences $I = \langle \bar{a}_i : i < \omega \rangle$ and $J = \langle \bar{b}_i : i < \omega \rangle$ with $\cup J \subseteq p^{\mathfrak{C}}$. We call this "$p$ does not admit the order property".
- On the set of realizations of p there is no definable (maybe with external parameters) partial order with infinite chains.

2. Morley sequences in dependent theories

- Part II - Morley sequences in dependent theories.
- *From now on we assume that the theory T is dependent and $T = T^{eq}$.*
- The source of the current presentation: *"On generically stable types in dependent theories", "A note on Morley sequences in dependent theories"*, can be found on my web-page.

- We write $\bar{a} \equiv_A \bar{b}$ for $\text{tp}(\bar{a}/A) = \text{tp}(\bar{b}/A)$.
- We say that $\bar{a}$ and $\bar{b}$ are of Lascar distance 1 over a set A if there exists an A-indiscernible sequence containing both. This is not an equivalence relation, but its transitive closure $E^L_A(\bar{x}, \bar{y})$ is. We say that $\bar{a}$ and $\bar{b}$ have the same Lascar strong type if they are E^L_A-equivalent (this is equivalent to Anand's definition).
- We write $\text{Lstp}(\bar{a}/A) = \text{Lstp}(\bar{b}/A)$ or $\bar{a} \equiv_{\text{Lstp},A} \bar{b}$.
- **Exercise:** Let I be an indiscernible sequence over a set A. Then $\bar{a} \models \text{Av}(I, A \cup I)$ if and only if $I^\frown\{\bar{a}\}$ is indiscernible over A.

- Recall: we call an A-indiscernible type sequence I *special* if for every two realizations I_1 and I_2 of $\text{tp}(I/A)$, there exists $\bar{c}$ such that $I_1^\frown\bar{c}$ and $I_2^\frown\bar{c}$ are A-indiscernible.
- We call an A-indiscernible sequence *weakly special* if two realizations I_1 and I_2 of $\text{Lstp}(I/A)$, there exists $\bar{c}$ such that $I_1^\frown\bar{c}$ and $I_2^\frown\bar{c}$ are A-indiscernible.
- Let $\varphi(\bar{x}, \bar{b})$ be a formula. We say that an indiscernible sequence J *eventually determines* $\varphi(\bar{x}, \bar{b})$ if $\lim_{J'} \varphi(\bar{x}, \bar{b})$ is constant for all J' continuing J.

Let I be a weakly special sequence over A, $\varphi(\bar{x}, \bar{b})$ a formula. The following is very similar to Anand's treatment of special sequences:

- There exists $J \equiv_{\text{Lstp},A} I$ which eventually determines $\varphi(\bar{x}, \bar{b})$. Moreover, every $J_0 \equiv_{\text{Lstp},A} I$ can be extended to J that eventually determines $\varphi(\bar{x}, \bar{b})$.
- For every $J, J' \equiv_{\text{Lstp},A} I$ which eventually determine $\varphi(\bar{x}, \bar{b})$ we have $\lim_J \varphi(\bar{x}, \bar{b}) = \lim_{J'} \varphi(\bar{x}, \bar{b})$, that is, the "eventual value" of $\varphi(\bar{x}, \bar{b})$ depends only on Lascar strong type of J over A, and not on the choice of J.

- Let I be a weakly special sequence over A. We define (exactly like in Anand's lecture) the *Eventual type* of I over a set C, $\mathrm{Ev}(I, C)$: the truth value of a formula $\varphi(\bar{x}, \bar{b})$ equals the "eventual value" of $\varphi(\bar{x}, \bar{b})$ as in the previous slide (depends only on $\mathrm{Lstp}(I/A)$). We denote $\mathrm{Ev}(I) = \mathrm{Ev}(I, \mathfrak{C})$.
- **Important (easy) Exercise(!)**: Prove that if I is a weakly special sequence over A, then $\mathrm{Ev}(I)$ extends $\mathrm{Av}(I, A \cup I)$.
- **Important Exercise(!)**: Prove that if I is a weakly special sequence over A which is also an indiscernible *set* over A, then $\mathrm{Ev}(I) = \mathrm{Av}(I, \mathfrak{C})$.
- **Exercise/Example**: Show that an increasing sequence of elements in the structure $(\mathbb{Q}, <)$ is weakly special and $\mathrm{Ev}(I) \neq \mathrm{Av}(I, \mathfrak{C})$.

- A type $p \in \mathrm{S}(B)$ *does not split* over a set A if whenever $\bar{b}, \bar{c} \in B$ have the same type over A, we have $\varphi(\bar{x}, \bar{b}) \in p \iff \varphi(\bar{x}, \bar{c}) \in p$ for every formula $\varphi(\bar{x}, \bar{y})$.
- A type $p \in \mathrm{S}(B)$ *does not split strongly* over a set A if whenever $\bar{b}, \bar{c} \in B$ are of Lascar distance 1 over A, we have $\varphi(\bar{x}, \bar{b}) \in p \iff \varphi(\bar{x}, \bar{c}) \in p$ for every formula $\varphi(\bar{x}, \bar{y})$.
- A type $p \in \mathrm{S}(B)$ *does not Lascar-split* over a set A if whenever $\bar{b}, \bar{c} \in B$ have the same Lascar strong type over A, we have $\varphi(\bar{x}, \bar{b}) \in p \iff \varphi(\bar{x}, \bar{c}) \in p$ for every formula $\varphi(\bar{x}, \bar{y})$.
- Note that a global type doesn't split over a set A if it is invariant under the action of the automorphism group of $\mathfrak{C}$ over A.

Exercises (no use of dependence):

- A type p over B does not split over A if and only if whenever $\bar{b}, \bar{c} \in B$ have the same type over A and $\bar{a} \models p$, we have $\bar{a}\bar{b} \equiv_A \bar{a}\bar{c}$.
- A type p over B does not Lascar-split over A if and only if whenever $\bar{b}, \bar{c} \in B$ have the same *Lascar strong* type over A and $\bar{a} \models p$, we have $\bar{a}\bar{b} \equiv_A \bar{a}\bar{c}$.
- Let M be a $(|A| + \aleph_0)^+$-saturated model containing A, $p \in \mathrm{S}(M)$. Then p does not Lascar-split over A if and only if p does not split strongly over A.
- Let A be a set. Then there are at most $2^{2^{|A|+|T|}}$ types over $\mathfrak{C}$ which do not split over A. Same is true for splitting replaced with Lascar splitting or strong splitting.

- If I is a weakly special sequence over A, then $\mathrm{Ev}(I)$ does not Lascar-split over A.
- Assume $\bar{b} \equiv_{\mathrm{Lstp},A} \bar{b}'$, and let $\varphi(\bar{x}, \bar{y})$ be a formula such that $\varphi(\bar{x}, \bar{b}) \in \mathrm{Ev}(I)$. Let $J \equiv_{\mathrm{Lstp},A} I$ eventually determine $\varphi(\bar{x}, \bar{b})$. So we know that $\varphi(\bar{x}, \bar{b}) \in \mathrm{Av}(J, A\bar{b})$.
- Choose J' such that $J\bar{b} \equiv_{\mathrm{Lstp},A} J'\bar{b}'$. Then J' eventually determines $\varphi(\bar{x}, \bar{b}')$ and clearly $\varphi(\bar{x}, \bar{b}') \in \mathrm{Av}(J, A\bar{b}')$, so (by uniqueness of the eventual value) $\varphi(\bar{x}, \bar{b}') \in \mathrm{Ev}(J') = \mathrm{Ev}(I)$, as required.

- Let $I = \langle \bar{a}_i : i < \lambda \rangle$ be such that

  – $\text{tp}(\bar{a}_i/A\bar{a}_{<i})$ does not Lascar-split over A
  – $\text{Lstp}(\bar{a}_i/A\bar{a}_{<i}) = \text{Lstp}(\bar{a}_j/A\bar{a}_{<i})$ for every $j \geq i$.

  Then I is a indiscernible over A.
- We prove by induction on k that $\text{Lstp}(\bar{a}_{i_1} \dots \bar{a}_{i_k}/A) = \text{Lstp}(\bar{a}_{j_1} \dots \bar{a}_{j_k}/A)$ for every $i_1 < \dots < i_k, j_1 < \dots < j_k$. For $k = 1$ this is given.

  For $k > 1$, assume wlog $j_k \geq i_k$. By the assumption $\text{Lstp}(\bar{a}_{j_k}/A\bar{a}_{i_1} \dots \bar{a}_{i_{k-1}}) = \text{Lstp}(\bar{a}_{i_k}/A\bar{a}_{i_1} \dots \bar{a}_{i_{k-1}})$. By the induction hypothesis $\text{Lstp}(\bar{a}_{i_1} \dots \bar{a}_{i_{k-1}}/A) = \text{Lstp}(\bar{a}_{j_1} \dots \bar{a}_{j_{k-1}}/A)$ and by the lack of Lascar splitting $\text{Lstp}(\bar{a}_{j_k}/A\bar{a}_{i_1} \dots \bar{a}_{i_{k-1}}) = \text{Lstp}(\bar{a}_{j_k}/A\bar{a}_{j_1} \dots \bar{a}_{j_{k-1}})$, which completes the proof.

- Let O a linear order, A a set. We call a sequence $I = \langle \bar{a}_i : i \in O \rangle$ a *Morley sequence over* A if it is an indiscernible sequence over A of realizations of p and $\text{tp}(\bar{a}_i/A\bar{a}_{<i})$ does not fork over A for all $i \in O$.
- If a sequence I is indiscernible over B and Morley over $A \subseteq B$, we sometimes say that I is *based on* A.
- Let $p \in S(B)$ be a type. We call a sequence I a *Morley sequence in* p if it is a Morley sequence over B of realizations of p.
- (*Existence of Morley sequences*). Let $\bar{a}, A \subseteq B$ be such that $\text{tp}(\bar{a}/B)$ does not fork over A. Then there exists a Morley sequence in $\text{tp}(\bar{a}/B)$ based on A.

- Strong splitting implies dividing, hence forking (Anand proved something very similar for a global type).
- Assume $p \in S(B)$ splits strongly over A, that is, there exists a sequence $I = \langle \bar{b}_i : i < \omega \rangle$ indiscernible over A with $\varphi(\bar{x}, \bar{b}_0), \neg\varphi(\bar{x}, \bar{b}_1) \in p$; then $\psi(\bar{x}, \bar{b}_0\bar{b}_1) = \varphi(\bar{x}, \bar{b}_0) \wedge \neg\varphi(\bar{x}, \bar{b}_1) \in p$ divides over A, since the set
$$\{\varphi(\bar{x}, \bar{b}_{2i}), \neg\varphi(\bar{x}, \bar{b}_{2i+1}) : i < \omega\}$$
  is inconsistent by the dependence of T.
- **Exercise**: Deduce that Lascar-splitting implies forking (Hint: recall that for global types strong splitting coincides with Lascar-splitting).

- There are boundedly many global types which do not fork over a given set A.
- Let $I = \langle \bar{a}_i : i < \lambda \rangle$ be such that
  – $\text{tp}(\bar{a}_i/A\bar{a}_{<i})$ does not fork over A
  – $\text{Lstp}(\bar{a}_i/A\bar{a}_{<i}) = \text{Lstp}(\bar{a}_j/A\bar{a}_{<i})$ for every $j \geq i$.

  Then I is a Morley sequence over A (that is, it is indiscernible over A).
- A Morley sequence over A is weakly special over A.

- **Exercise:** Let $I = \langle \bar{b}_i : i < \omega \rangle$ be an indiscernible sequence in $p \in S(A)$. Prove that the following are equivalent:
  ♢ I is a Morley sequence in p.

♢ $\mathrm{Av}(I, I \cup A)$ is a nonforking extension of p.
♢ There exists a global extension of $\mathrm{Av}(I, I \cup A)$ which does not fork over A.

- A natural question is: what can be said about global extensions of $\mathrm{Av}(I, A \cup I)$ as above? How many such extensions are there? Can we describe them?
- The answer has been in fact given by Anand already: there is only one (!), and we understand what it looks like pretty well.

- Let I be a weakly special sequence over A. Recall that $\mathrm{Ev}(I)$ is a global type which does not Lascar-split over A. Hence it does not *fork* over A.
- Recall that $\mathrm{Ev}(I)$ extends $\mathrm{Av}(I, I \cup A)$. It follows (**why?**) that I is a *Morley sequence* over A.
- On the other hand, if I is a Morley sequence, then it is weakly special.
- We have established: I is a Morley sequence over A if and only if it is weakly special over A! Moreover, if I is a Morley sequence, then $\mathrm{Ev}(I)$ is *the unique global type extending* $\mathrm{Av}(I, A \cup I)$ *which does not fork over* A.

(will be omitted in the lecture)

- Let $I = \langle \bar{a}_i : i \in O \rangle$ be an indiscernible sequence over a set A and let p be a global type which extends $\mathrm{Av}(I, A \cup I)$ and does not fork over A. Suppose that $I' = \langle \bar{a}'_i : i \in O' \rangle$ satisfies $\bar{a}'_i \models p \restriction AI\bar{a}'_{<i}$. Then $J = I^\frown I'$ is indiscernible over A.
- Let $I = \langle \bar{a}_i : i \in O \rangle$ be an indiscernible sequence over a set A and let p be a global type which extends $\mathrm{Av}(I, A \cup I)$ and does not fork over A. Suppose that $I' \equiv_{\mathrm{Lstp},A} I$. Then $p \restriction AI' = \mathrm{Av}(I', A \cup I')$.

(will be omitted in the lecture)

- Let I be an indiscernible sequence over a set A, p a global type extending $\mathrm{Av}(I, A \cup I)$ which does not fork over A. Then for every A-indiscernible sequence I' continuing I, we have $p \restriction AII' = \mathrm{Av}(I', AII')$.
- Let I be an indiscernible sequence over a set A and let p, q be global types extending $\mathrm{Av}(I, A \cup I)$, both do not fork over A. Then $p = q$.
- Let I be a Morley (nonforking) sequence over a set A. Then there exists a unique global types extending $\mathrm{Av}(I, A \cup I)$ which does not fork over A. In other words, $\mathrm{Av}(I, A \cup I)$ is stationary over A.

(will be omitted in the lecture)

- Assume towards contradiction that $q \neq p$, so there is $\varphi(\bar{x}, \bar{b})$ such that $\varphi(\bar{x}, \bar{b}) \in p$ but $\neg\varphi(\bar{x}, \bar{b}) \in q$.
- Construct by induction on $\alpha < \omega$ sequence $J_\alpha = \langle \bar{a}^\alpha_i : i < \omega \rangle$ such that
  - $\bar{a}^{2\alpha}_i \models p \restriction A\bar{b}IJ_{<\alpha}\bar{a}^{2\alpha}_{<i}$
  - $\bar{a}^{2\alpha+1}_i \models q \restriction A\bar{b}IJ_{<\alpha}\bar{a}^{2\alpha+1}_{<i}$

- We claim that $J = J_0^\frown J_1^\frown \cdots$ is an indiscernible sequence. Once we have shown this, it yields an immediate contradiction to dependence.

(will be omitted in the lecture)

- So we show by induction on α that $J^\alpha = I^\frown J_0^\frown \cdots ^\frown J_\alpha$ is indiscernible (even over A). For $\alpha = 0$ this is true.
- Let us take care of $\alpha = 1$ (the continuation is the same). Recall that q extends $\mathrm{Av}(J^0, A \cup J^0)$. Now continue as in the case $\alpha = 0$.

3. Generic stability

- Part III - Generic stability.

- **Exercise:** Let $\varphi(\bar{x}, \bar{y})$ be a formula, $k = k_\varphi$ (as defined by Anand). Show that if $I = \langle \bar{b}_i : i \in O \rangle$ is an infinite indiscernible set, then for every $\bar{c} \in \mathfrak{C}$, either
$$|\{i \in O \colon \varphi(\bar{b}_i, \bar{c})\}| < k$$
or
$$|\{i \in O \colon \neg\varphi(\bar{b}_i, \bar{c})\}| < k$$
- **Exercise:** Show that if $\varphi(\bar{x}, \bar{y})$ is an unstable formula witnessed by indiscernible sequences I and J, then neither I nor J is an indiscernible set. In other words, if I is an indiscernible set, then every formula is *stable with respect to* I.
- Recall: if I is a weakly special indiscernible set over A, then $\mathrm{Ev}(I) = \mathrm{Av}(I, \mathfrak{C})$. Hence $\mathrm{Av}(I, \mathfrak{C})$ *does not fork over* A.

- We call a type $p \in \mathrm{S}(A)$ *generically stable* if there exists a Morley sequence $\langle \bar{b}_i : i < \omega \rangle$ in p (over A) which is an indiscernible set.
- Recall: a type $p \in \mathrm{S}_m(B)$ is said to be *definable* over A if for every formula $\varphi(\bar{x}, \bar{y})$ with $\mathrm{len}(\bar{x}) = m, \mathrm{len}(y) = k$ there exists a formula $d_p\bar{x}\varphi(\bar{x}, \bar{y})$ with free variables $\bar{y}$ such that for every $\bar{b} \in B^k$
$$\varphi(\bar{x}, \bar{b}) \in p \Longleftrightarrow \models d_p\bar{x}\varphi(\bar{x}, \bar{b})$$
- A definition schema d_p is said to be *good* is for every set C the set
$$\{\varphi(\bar{x}, \bar{c}) \colon \varphi(\bar{x}, \bar{y}) \text{ is a formula}, \mathrm{len}(\bar{x}) = m, \bar{c} \in C, \models d_p\bar{x}\varphi(\bar{x}, \bar{c})\}$$
is a complete type over C (denotes by $p|C$).

($\diamond$) Let $I = \langle \bar{b}_i : i < \omega \rangle$ be an indiscernible set over a set A, $C \supseteq A$. Then $p = \mathrm{Av}(I, C)$ is definable over $\cup I$.

($\diamond\diamond$) Let $p \in \mathrm{S}(A)$ be generically stable. Then p is (well-) definable almost over A.

- Let $\varphi(\bar{x}, \bar{y})$ be a formula and let $k = k_\varphi$. Now clearly for every $\bar{c} \in C$
$$\varphi(\bar{x}, \bar{c}) \in \mathrm{Av}(I, C)$$
if and only if
$$|\{i < 2k\colon \models \varphi(\bar{b}_i, \bar{c})\}| \geq k$$
if and only if
$$\bigvee_{u \subset 2k, |u|=k} \bigwedge_{i \in u} \varphi(\bar{b}_i, \bar{c})$$
So p is definable over I by the schema
$$d_p \bar{x} \varphi(\bar{x}, \bar{y}) = \bigvee_{u \subset 2k_\varphi, |u|=k_\varphi} \bigwedge_{i \in u} \varphi(\bar{b}_i, \bar{y})$$

- Let $I = \langle \bar{b}_i : i < \omega \rangle$ be a nonforking indiscernible (over A) set in p.
Let $\varphi(\bar{x}, \bar{y})$ be a formula, then p is definable over I as before by
$$\vartheta(\bar{y}, \bar{b}_{<2k}) = d_p \bar{x} \varphi(\bar{x}, \bar{y}) = \bigvee_{u \subset 2k_\varphi, |u|=k_\varphi} \bigwedge_{i \in u} \varphi(\bar{b}_i, \bar{y})$$
.
- *Claim*: $\vartheta(\bar{x}, \bar{b}_{<2k})$ as above is almost over A.
- Note that once we have proven the Claim we are done: p is definable almost over A by a definition which is clearly good (it defines $\mathrm{Av}(I, \mathfrak{C})$).

- For the proof of the Claim note that otherwise we would have unboundedly many pairwise nonequivalent automorphic copies of ϑ over A. In other words, we would have an unbounded sequence of automorphisms $\langle \sigma_\alpha \rangle$ over A such that $\{\vartheta_\alpha = \sigma_\alpha(\vartheta)\}$ are pairwise nonequivalent. Let $I_\alpha = \sigma_\alpha(I)$, $p_\alpha = \mathrm{Av}(I_\alpha, A \cup I_\alpha)$.
- Recall that $q_\alpha = \mathrm{Av}(I_\alpha, \mathfrak{C})$ all do not fork over A (because they equal $\mathrm{Ev}(I_\alpha)$, since I_α are *indiscernible sets*!).
- Note that q_α is definable by ϑ_α and therefore are all distinct. So $\langle q_\alpha \rangle$ is an unbounded sequence of global types all of which do not fork (**why?**) over A, a contradiction.

- Let $p \in \mathrm{S}(A)$ be a generically stable type witnessed by a nonforking indiscernible set I such that the definition schema d_p as before is *over* A (e.g. $A = \mathrm{acl}(A)$). Then p is stationary.
- We aim to show that p has a unique nonforking extension to any superset of A. By existence of nonforking extensions and stationarity over A of the average type, it is enough to show that the only nonforking extension of p to $A \cup I$ is $\mathrm{Av}(I, A \cup I)$. In fact, it is enough to show that $\mathrm{Av}(I, A \cup I)$ is the only extension of p to $A \cup I$

which does not *split strongly* over A. Denote $B = A \cup I$, $B_k = A \cup \langle \bar{b}_i : i < k \rangle$ for $k \leq \omega$.

- Let $\bar{b}' \models p$, $\mathrm{tp}(b'/B)$ does not split strongly over A. We show by induction on k that $\mathrm{tp}(\bar{b}'/B_k) = \mathrm{Av}(I, B_k)$.

- There is nothing to show for $k = 0$.
  Assume the claim for k, and suppose $\varphi(\bar{b}', \bar{b}_0, \ldots, \bar{b}_k, \bar{a})$ holds. Let $\psi(\bar{x}, \bar{b}_{<k}, \bar{b}', \bar{a}) = \varphi(\bar{b}', \bar{b}_0, \ldots, \bar{b}_{k-1}, \bar{x}, \bar{a})$, so
  $\psi(\bar{b}_k, \bar{b}_{<k}, \bar{b}', \bar{a})$ holds.
- Note that since $\mathrm{tp}(\bar{b}'/B)$ doesn't split strongly over A, the set $\langle \bar{b}_i \colon i \geq k \rangle$ is indiscernible over $B_k \bar{b}'$ (**why?**).
- We see that $\psi(\bar{b}_\ell, \bar{b}_{<k}, \bar{b}', \bar{a})$ holds for all ℓ big enough, and therefore

$$\psi(\bar{x}, \bar{b}_{<k}, \bar{b}', \bar{a}) \in \mathrm{Av}(I, B\bar{b}')$$

- Therefore (denoting $q = \mathrm{Av}(I, \mathfrak{C})$), $d_q \bar{x} \psi(\bar{x}, \bar{b}_{<k}, \bar{b}', \bar{a})$ holds, where the definition is over A. So we get $\theta(\bar{y}) = d_q \bar{x} \psi(\bar{x}, \bar{b}_{<k}, \bar{y}, \bar{a})$ is in $\mathrm{tp}(\bar{b}'/B_k)$ and therefore (by the induction hypothesis) is in $\mathrm{Av}(I, B_k)$, which we think now as of a type in $\bar{y}$.
- This means that $d_q \bar{x} \psi(\bar{x}, \bar{b}_{<k}, \bar{b}_\ell, \bar{a})$ holds for almost all ℓ, and therefore (since d_q defines $\mathrm{Av}(I, \mathfrak{C})$) we have $\psi(\bar{x}, \bar{b}_{<k}, \bar{b}_\ell, \bar{a}) \in \mathrm{Av}(I, B)$ for almost all ℓ.
- Let ℓ be such, so by the definition of average type, there exists an m such that $\psi(\bar{b}_m, \bar{b}_{<k}, \bar{b}_\ell, \bar{a})$, that is, $\varphi(\bar{b}_\ell, \bar{b}_{<k}, \bar{b}_m, \bar{a})$ holds.

- Since I is an indiscernible set, we get $\varphi(\bar{b}_m, \bar{b}_{<k}, \bar{b}_k, \bar{a})$ for all m big enough, and therefore
-
$$\varphi(\bar{x}, \bar{b}_{\leq k}, \bar{a}) \in \mathrm{Av}(I, B)$$
- This finishes the proof.

- A type p is generically stable if and only if it is extensible (does not fork over its domain) and *every* Morley sequence in it is an indiscernible set.
- Let $p \in \mathrm{S}(A)$ be generically stable, $q \in \mathrm{S}(B)$ extending p. Then q does not fork over A if and only if it is definable almost over A.

From now on we write $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{b}$ for "$\mathrm{tp}(\bar{a}/A\bar{b})$ does not fork over A". *Caution*: unlike in simple theories, this relation does not need to be symmetric (**find an example!**). Still:

Let $p \in \mathrm{S}(A)$ be generically stable, $q \in \mathrm{S}(A)$ does not fork over A, $\bar{a} \models p$, $\bar{b} \models q$. Then

- $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{b} \implies \bar{b} \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{a}$. Moreover, if $A = \mathrm{acl}(A)$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{b}$, then there exists a unique nonforking extension of q to $\mathrm{S}(A\bar{a})$ which equals $\mathrm{tp}(\bar{b}/A\bar{a})$.
- $\bar{b} \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{a} \implies \bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{b}$.

- We prove the first item. Clearly, it is enough to prove the statement for $A = \operatorname{acl}(A)$. Let q^* be a global nonforking extension of q. We will show that $q^* \restriction A\bar{a} = \operatorname{tp}(\bar{b}/A\bar{a})$, proving the moreover part as well.
- Suppose not. Then there is a formula $\varphi(\bar{x}, \bar{y})$ such that $\varphi(\bar{a}, \bar{b})$ (so $d_p \bar{x} \varphi(\bar{x}, \bar{b})$ holds), but $\neg\varphi(\bar{a}, \bar{y}) \in q^*$.
- Let $\bar{a}_0 = \bar{a}$, $\bar{b}_0 = \bar{b}$. Construct sequences $\langle \bar{a}_i \rangle$, $\langle \bar{b}_i \rangle$ for $i < \omega$ as follows:

$$\bar{a}_i \models p | A \langle \bar{a}_j : j < i \rangle \langle \bar{b}_j : j < i \rangle$$
$$\bar{b}_i \models q^* \restriction A \langle \bar{a}_j : j < i+1 \rangle \langle \bar{b}_j : j < i \rangle$$

Now note:

- $j < i \Rightarrow \varphi(\bar{a}_i, \bar{b}_j)$: since $\models d_p \bar{x} \varphi(\bar{x}, \bar{b})$, $\bar{b} \equiv_A \bar{b}_j$ and $\bar{a}_i$ is chosen generically over $A\bar{b}_j$
- $j \geq i \Rightarrow \neg\varphi(\bar{a}_i, \bar{b}_j)$: since $\neg\varphi(\bar{a}, \bar{y}) \in q^*$, q^* does not fork hence does not Lascar split over A, $\bar{a} \equiv_{Lstp,A} \bar{a}_i$ (in fact, they are of Lascar distance 1) and $\bar{b}_j$ was chosen to realize q^* over $A\bar{a}_i$.
- This is a contradiction to generic stability of p, that is, $\langle \bar{a}_i : i < \omega \rangle$ being an indiscernible set.
- **Exercise:** Deduce the second item of the symmetry lemma.

Let $p, q \in S(A)$ be generically stable, $\bar{a}, \bar{b}$ realize p, q respectively, and let $\bar{c}, \bar{d}$ be any tuples (maybe infinite). Then:

- *Irreflexivity* $\bar{a} \downarrow_A \bar{a}$ if and only if p is algebraic
- *Monotonicity* If $a \downarrow_A \bar{b}\bar{c}\bar{d}$, then $a \downarrow_A \bar{c}\bar{b}$.
- *Symmetry* $\bar{a} \downarrow_A \bar{b}$ if and only if $\bar{b} \downarrow_A \bar{a}$
- *Transitivity* $\bar{a} \downarrow_A \bar{c}\bar{d}$ if and only if $\bar{a} \downarrow_{A\bar{c}} \bar{d}$ and $\bar{a} \downarrow_A \bar{c}$
- *Existence* Let $B \supseteq A$, then there exists $\bar{a}' \equiv_A \bar{a}$ such that $\operatorname{tp}(\bar{a}'/B)$ is generically stable and $\bar{a}' \downarrow_A B$.
- *Uniqueness* If $\bar{a} \downarrow_A \bar{c}$, $\bar{a}' \downarrow_A \bar{c}$ and $\bar{a}' \equiv_{\operatorname{acl}(A)} \bar{a}$, then $\bar{a} \equiv_{A\bar{c}} \bar{a}'$
- *Local Character* If $\bar{a} \downarrow_A \bar{c}$, then for some subset A_0 of A of cardinality $|T|$, $\bar{a} \downarrow_{A_0} \bar{c}$.

Let $p \in \mathrm{S}(A)$. The Following Are Equivalent:

- p is stable.
- Every extension of p is stable.
- Every extension of p is generically stable.
- Every indiscernible sequence in p is an indiscernible set.

- Let us consider the theory of $\mathbb{Q}$ with a predicate P_n for every interval $[n, n+1)$ ($n \in \mathbb{Z}$) and the natural order $<_n$ on P_n. It is easy to see that the "generic" type "at infinity" (that is, the type of an element not in any of the P_n's) is stable, hence generically stable.

- Let us consider the theory of a two-sorted structure (X, Y): on X there is an equivalence relation $E(x_1, x_2)$ with infinitely many infinite classes and each class densely linearly ordered, while Y is just an infinite set such that there is a definable function f from X onto Y with $f(a_1) = f(a_2) \iff E(a_1, a_2)$.

  In other words, Y is the sort of imaginary elements corresponding to the classes of E.

  Let M a model and p the "generic" type in X over M, that is, a type of an element in a new equivalence class. It is easy to see that p is generically stable, but clearly not stable. In fact, in this example p is "stably dominated".

Generically stable types which are not stable or stably dominated:

- Similar to Example I: $(Q, P_0, <_0, +)$, p the "infinity" type. Then it is generically stable, but there is a definable order on it, so it is unstable.
- Let RV be a two-sorted theory of a real closed (ordered) field R and an infinite dimensional vector space V over it. There is a definable partial order on V:

  $$v_1 \leq v_2 \iff \exists r \in R, r \geq 1_R \text{ such that } v_2 = r \cdot v_1$$

  Let M be a model and $p \in \mathrm{S}(M)$ be the type of a generic vector. Then p is generically stable and every Morley sequence is an indiscernible linearly independent set.