TUTORIAL ON DEPENDENT THEORIES - PART IV

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1. Strong dependence

- Part IV - Strongly dependent theories.

- A theory T is *not* strongly dependent if there exists an array $\langle \bar{a}_i^\alpha \colon i < \omega, \alpha < \omega \rangle$ and formulas $\langle \varphi_\alpha(\bar{x}, \bar{y}_\alpha) \colon \alpha < \omega \rangle$ (note that $\bar{x}$ does not depend on α) such that for every $\eta \in {}^\omega\omega$ the set
$$\left\{ [\varphi_\alpha(\bar{x}, \bar{a}_i^\alpha)]^{(\eta(\alpha)=i)} : \alpha < \omega, i < \omega \right\}$$
is consistent.
- One can add in addition that $I^\alpha = \langle \bar{a}_i^\alpha : i < \omega \rangle$ is indiscernible over $\cup\{I^\beta \colon \beta \neq \alpha\}$ for every $\alpha < \omega$. Then there is no need to demand "for all η", it is enough to say, for example:
$$\{[\varphi_\alpha(\bar{x}, \bar{a}_0^\alpha) \wedge \neg\varphi_\alpha(\bar{x}, \bar{a}_1^\alpha)] \colon \alpha < \omega\}$$
is consistent.

Exercises:

- A theory T is independent if and only of there exists an array $\langle \bar{a}_i^\alpha \colon i < \omega, \alpha < \omega \rangle$ and a formula $\varphi(\bar{x}, \bar{y})$ (so it does not depend on α) such that for every $\eta \in {}^\omega\omega$ the set
$$\left\{ [\varphi(\bar{x}, \bar{a}_i^\alpha)]^{(\eta(\alpha)=i)} : \alpha < \omega, i < \omega \right\}$$
is consistent.
- A theory T is independent if and only of there exists an array $\langle \bar{a}_i^\alpha \colon i < \omega, \alpha < |T|^+ \rangle$ and formulas $\langle \varphi_\alpha(\bar{x}, \bar{y}_\alpha) \colon \alpha < |T|^+ \rangle$ such that for every $\eta \in {}^\omega\omega$ the set
$$\left\{ [\varphi_\alpha(\bar{x}, \bar{a}_i^\alpha)]^{(\eta(\alpha)=i)} : \alpha < |T|^+, i < \omega \right\}$$
is consistent.

Using similar arguments to the ones that appeared in Lou's first talk one can show that the following are equivalent for a theory T:

- T is dependent.

- For every set A, an infinite A-indiscernible sequence I, a finite tuple $\bar{b}$ and a finite set of formulas Δ, there is an infinite convex subset of I which is an indiscernible sequence over $A\bar{b}$ with respect to formulas in Δ.
- For every set A, an A-indiscernible sequence I of order type $|T|^+$ and a set B of cardinality $|T|$, I is eventually indiscernible over $A \cup B$.

Strong dependence is in a sense a "global" version of dependence, namely, the following are equivalent for a theory T:

- T is strongly dependent.
- For every set A, an infinite A-indiscernible sequence $I = \langle \bar{a}_i : i < \omega \rangle$ (maybe the length of $\bar{a}$ is ω!) and a finite tuple $\bar{b}$, there is an infinite convex subset J of I such that all elements of J have the same type over $A\bar{b}$.
- For every set A, an infinite A-indiscernible sequence $I = \langle \bar{a}_i : i < \omega \rangle$ (maybe the length of $\bar{a}$ is ω!) and a finite tuple $\bar{b}$, there is an infinite convex subset J of I which is an indiscernible sequence over $A\bar{b}$.

A natural question is: what are stable strongly dependent theories? It is easy to see that a superstable theory is strongly dependent. Are there others?

We recall the notion of *weight* in stable theories. For simplicity, we give the definition for $n < \omega$ (in general, can be an ordinal).

- Let $p(x)$ be any type over some set A. We will say that $a, \langle b_i \rangle_{i=1}^n$ witnesses (*pre-weight of* p is at least n) if $a \models p(x)$, $\langle b_i \rangle_{i=1}^n$ is A-independent and $a \not\perp_A b_i$ for all i. If n is maximal such that such a witness exists, we will say that that p *has pre-weight* n and that $a, \langle b_i \rangle_{i=1}^n$ witness this.
- The *weight* of a type p is defined to be the supremum over the pre-weights of all nonforking extensions of p.

- T is strongly dependent and stable (called *strongly stable*) if and only if every type in finitely many variables has finite weight [this is easy].
- Hence in a strongly stable theory every type is domination equivalent to a free product of types of weight 1 (not necessarily regular). [This is a nontrivial result, probably originally due to Anand(?)].
- Lachlan's Theorem is true for strongly stable theories, namely: a countable strongly stable theory has either 1 or infinitely many countable models.