Shelah's reduction of dependence to one-variable dependence

Below, $x_1, \dots, x_m, y_1, \dots, y_n, z_1, \dots, z_p$ are distinct variables, and

$$x = (x_1, \dots, x_m), \qquad y = (y_1, \dots, y_n), \qquad z = (z_1, \dots, z_p).$$

We fix a complete L-theory T, and "formula" means "L-formula" unless specified otherwise. To prove in concrete cases that T is dependent the following result due to Shelah is crucial.

Theorem 0.1. *If every formula $\phi(x; y)$ with $n = 1$ is dependent in T, then T is dependent. Note: $n = 1$ means that y is a single variable.*

Let us take this on faith for the moment, and see how it is applied. Let M be your favorite model of T. Suppose you can show that for every formula $\phi(x; y)$ with $n = 1$ there is an integer $d = d(\phi) \geq 1$ with the following property: every d-element set $D \subseteq M$ has a subset that is not of the form $\phi(a; M) \cap D$ with $a \in M^m$. Then T is dependent. (Exercise.)

Example. Let $T :=$ RCF, the theory of real closed fields. So T has QE in the language L of ordered rings, and the ordered field $\mathbb{R}$ of real numbers is a model of T. Let $\phi(x; y)$ be a formula with $n = 1$. Each set $\phi(a; \mathbb{R})$ is a finite union of intervals, where "interval" means "nonempty connected subset of $\mathbb{R}$". In fact we have a number $N = N(\phi) \in \mathbb{N}$ such that each set $\phi(a; \mathbb{R})$ is a union of at most N intervals. (Exercise.) Then the above holds with $d := 2N + 1$: let $D \subseteq \mathbb{R}$ be a d-element set, so $D = \{r_0, \dots, r_{2N}\}$ with $r_0 < \cdots < r_{2N}$; then

$$\{r_0, r_2, r_4, \dots, r_{2N}\} \neq \phi(a, \mathbb{R}) \cap D$$

for all $a \in \mathbb{R}^m$. (Exercise.) Thus RCF is dependent.

Likewise, every complete o-minimal theory is dependent.

Shelah's proof of the theorem uses set theory and absoluteness. The proof below is due to Poizat. We let i and j (sometimes subscripted) range over the integers ≥ 1. It is convenient to reformulate Shelah's theorem as follows where we remind you of the symmetry of (in)dependence.

Theorem 0.2. *If there is an independent formula $\phi(x; y)$ in T, then there is one where x is a single variable, that is, $m = 1$.*

Proof. (Sketch) We fix a monster model $\mathbb{M}$ of T, and do all our work in $\mathbb{M}$. Let $\phi(x; y)$ be an independent formula in T. Towards a contradiction we can assume that $m = \text{length}(x)$ is minimal and $m > 1$. Take $a \in \mathbb{M}$, $b \in \mathbb{M}^{m-1}$ and an indiscernible sequence $c_1, c_2, c_3, \dots$ in $\mathbb{M}^n$ such that for all i,

$$\models \phi(a, b; c_{2i}), \qquad \models \neg\phi(a, b; c_{2i-1}).$$

By the minimality of m, if $\psi(x_2, \dots, x_m; z)$ is a formula and $d_1, d_2, d_3, \dots$ is an indiscernible sequence in $\mathbb{M}^p$, then either $\models \psi(b; d_i)\}$ for all sufficiently large i, or $\models \neg\psi(b; d_i)\}$ for all sufficiently large i.

Pick new variables y^i_j, put $y^i = (y^i_1, \dots, y^i_n)$ and consider the partial type $\Sigma(y^1, y^2, y^3, \dots)$ over (a, b) consisting of the formulas

(1) $$\theta(y^1, \dots, y^k) \longleftrightarrow \theta(y^{i_1}, \dots, y^{i_k})$$
where $\theta(y^1, \dots, y^k)$ is a formula and $i_1 < \cdots < i_k$;

(2) $$\phi(a, b; y^{2i}) \text{ and } \neg\phi(a, b; y^{2i-1});$$

(3) $$\psi(b; y^1, \dots, y^k) \longleftrightarrow \psi(b; y^{i_1}, \dots, y^{i_k})$$
where $\psi(x_2, \dots, x_m; y^1, \dots, y^k)$ is a formula and $i_1 < \cdots < i_k$.

Claim. $\Sigma(y^1, y^2, y^3, \dots)$ has a realization over (a, b).

If $(d_1, d_2, d_3, \dots)$ is such a realization, then by (3) the sequence

$$(b, d_1), (b, d_2), (b, d_3), \dots$$

is indiscernible, so by (2) the formula $\phi(x_1; (x_2, \dots, x_m, y))$ is independent in T, and we are done. So it only remains to prove the claim.

Let Σ_k be Σ with (3) restricted to to sequences $i_1 < \cdots < i_k$ of length k, and note that Σ_0 is realized by $(c_1, c_2, \dots)$. Assume inductively that for a certain k the partial type Σ_k can be realized; it suffices to derive from this assumption that Σ_{k+1} can be realized. We can of course rename things so that Σ_k is realized by $(c_1, c_2, \dots)$. To show the basic idea with minimal notational burden, we just do the induction step for $k = 0$ and for $k = 1$, and leave the general case from k to $k + 1$ as an exercise.

Let $k = 0$; to realize Σ_1 is to realize Σ_0 together with all $L(b)$-formulas

$$\psi(b; y^1) \longleftrightarrow \psi(b; y^i)$$

where $\psi(x_2, \dots, x_m; y)$ is a formula. Consider just a single such ψ, and recall that either $\models \psi(b; c_i)$ for all sufficiently large i, or $\models \neg\psi(b; c_i)$ for all sufficiently large i. Thus $\Sigma_0(y^1, y^2, y^3, \dots)$ together with the equivalence displayed is realized by $(c_j, c_{j+1}, c_{j+2}, \dots)$ for sufficiently large odd j. This is also true for any *finite set* of equivalences as displayed, and then saturation of $\mathbb{M}$ takes care of all such equivalences at once.

Next we let $k = 1$, and show how to realize Σ_2. By (1) we have for each i the indiscernible sequence

$$(c_i, c_{i+1}), (c_i, c_{i+2}), (c_i, c_{i+3}), \dots,$$

so for each i and formula $\psi(x_2, \dots, x_m; y^1, y^2)$ we have either $\models \psi(b; c_i, c_j)$ for all sufficiently large $j > i$, or $\models \neg\psi(b; c_i, c_j)$ for all sufficiently large $j > i$. By saturation and after renaming we can arrange that "for all sufficiently large $j > i$" can be replaced here by "for all $j > i$", that is, for each ψ as above we have

$$\models \psi(b; c_i, c_{i+1}) \longleftrightarrow \psi(b; c_i, c_j)$$

for all i and all $j > i$. So for each i we have a complete type $p_i(y^1, y^2)$ over b realized by all (c_i, c_j) with $i < j$. Using saturation we can extend the sequence $c_1, c_2, c_3, \dots$ by an extra element $c_\omega \in M^n$ such that $c_1, c_2, c_3, \dots, c_\omega$

is still indiscernible and (c_i, c_ω) realizes p_i for all i. Then the sequence

$$(c_1, c_\omega), (c_2, c_\omega), (c_3, c_\omega), \ldots$$

is indiscernible. Thus for any formula $\psi(x_2, \ldots, x_m; y^1, y^2)$,

$$\begin{aligned} \text{either } &\models \psi(b; c_i, c_\omega) \quad \text{for all sufficiently large } i, \\ \text{or } &\models \neg\psi(b; c_i, c_\omega) \quad \text{for all sufficiently large } i. \end{aligned}$$

Therefore:

Claim. For each formula $\psi(x_2, \ldots, x_m; y^1, y^2)$, either $\psi(b; y^1, y^2) \in p_i$ for all sufficiently large i, or $\neg\psi(b; y^1, y^2) \in p_i$ for all sufficiently large i.

By this claim, each finite subset of $\Sigma_2(y^1, y^2, y^3, \ldots)$ is realized by the tuple $(c_i, c_{i+1}, c_{i+2}, \ldots)$ for all sufficiently large odd i. Thus Σ_2 can be realized. $\square$

We gave the proof above for one-sorted structures, but it works just as well in the many-sorted setting. A more constructive version of Poizat's proof was given by Laskowski, and a fully constructive and combinatorial descendant is in Chapter 5 of *Tame Topology and O-minimal Structures.*