Math 401 - Exam 1 Solutions

1. Suppose $\beta(t) = (1 + 3\cos t, 2 + 5\sin t, 3 + 4\cos t)$ for $t \in \mathbf{R}$.

(a) Compute the following functions of t: Velocity, speed, acceleration, tangent, normal, binormal, curvature, torsion.

velocity: $\beta'(t) = (-3\sin t, 5\cos t, -4\sin t)$

speed: $\frac{ds}{dt} = |\beta'(t)| = \sqrt{9\sin^2 t + 25\cos^2 t + 16\sin^2 t} = \sqrt{25} = 5$

acceleration: $\beta''(t) = (-3\cos t, -5\sin t, -4\cos t)$

tangent: $T(t) = \frac{\beta'(t)}{|\beta'(t)|} = \frac{1}{5}(-3\sin t, 5\cos t, -4\sin t)$

With $s(t) = 5t$, $t(s) = \frac{s}{5}$,

$$\kappa(s)N(s) = \frac{d}{ds}T\big(t(s)\big) = \frac{1}{25}(-3\cos\frac{s}{5}, -5\sin\frac{s}{5}, -4\cos\frac{s}{5})$$

so that the normal: $N(s) = \frac{1}{5}(-3\cos 5s, -5\sin 5s, -4\cos 5s)$

curvature: $\kappa(s) = \frac{1}{5}$

binormal: $B = T \wedge N = (\frac{1}{25}(-20, 0, 15) = \frac{1}{5}(-4, 0, 3)$

torsion: $\tau(s) = 0$ because $\tau(s)N(s) = \frac{d}{ds}B(s) = 0$

(b) Find an arc-length reparameterization of β.

$\beta(s) = \big(1 + 3\cos(\frac{s}{5}), 2 + 5\sin(\frac{s}{5}), 3 + 4\cos(\frac{s}{5})\big)$.

(c) Describe the geometric shape of the image of β.

This is a planar circle because the torsion is identically zero and the curvature is constant.

2. Suppose $\alpha(s)$ is a regular curve parameterized by arc-length. Find a formula for $\alpha'''(s)$ in terms of κ (curvature), κ', τ (torsion), N (principal normal), T (tangent), and B (binormal).

The acceleration is $\alpha''(s) = T'(s) = \kappa(s)N(s)$ so

$$\alpha'''(s) = \kappa'(s)N(s) + \kappa(s)N'(s) = \kappa'(s)N(s) + \kappa(s)\big(-\kappa(s)T(s) - \tau(s)B(s)\big)$$

by the Frenet formulas.

3. (a) Find a curve in $\mathbf{R}^3$ with curvature $\equiv 1$ everywhere and torsion $\equiv 0$ everywhere.

Any planar circle of radius 1. e.g. $\alpha(t) = (\cos t, \sin t, 0)$ for $t \in \mathbf{R}$.

(b) Find a curve in $\mathbf{R}^3$ with constant curvature and constant torsion.

A planar circle has torsion identically 0. A helix has constant curvature and nonzero constant torsion, e.g. $\beta(t) = (\cos t, \sin t, t)$ for $t \in \mathbf{R}$.

(c) Does there exist a curve in $\mathbf{R}^3$ with curvature $\equiv 1$ everywhere and torsion not being constant? Why or why not?

Yes, the Fundamental Theorem of Curves allows one to prescribe smooth curvature and torsion functions arbitrarily.

4. Suppose $\alpha : [0, \ell] \to \mathbf{R}^3$ is a smooth curve parameterized by arc-length, and $\alpha(0) = \alpha(\frac{\ell}{3}) = \alpha(\ell)$. What is the maximum possible total area of all the bounded regions determined by α?

The curve α restricted to $[0, \frac{\ell}{3}]$ is a closed curve. It may enclose, by the isoperimetric inequality, an area at most $\frac{1}{4\pi}(\frac{\ell}{3})^2$ and only if it is a circle. The strict inequality $\sum_{i=1}^{j} a_i^2 < (\sum_{i=1}^{j} a_i)^2$ for $j \geq 2$ and positive a_i shows that, to enclose the most area, this first curve will not cross itself before $\frac{\ell}{3}$ and make more regions. Similar reasoning shows that the restriction of α to $[\frac{\ell}{3}, 1]$ will also be a simple closed curve and it encloses an area of at most $\frac{1}{4\pi}(\frac{2\ell}{3})^2$. Since there exists a single such curve α consisting of 2 touching circles with circumferences $\frac{\ell}{3}$ and $\frac{2\ell}{3}$, the maximum possible area of all the bounded regions is

$$\frac{1}{4\pi}\ell^2\left(\frac{1}{9} + \frac{4}{9}\right) = \frac{5\ell^2}{36\pi}.$$

5. Suppose $g : \mathbf{R}^3 \to \mathbf{R}$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3) : \mathbf{R} \to \mathbf{R}^3$ are smooth and the maximum value of g on the image of α occurs at $p = \alpha(0)$. Show that

(a) $(\operatorname{grad} g)(p) \cdot \alpha'(0) = 0$ (where $\operatorname{grad} g = (\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z})$).

Since $g \circ \alpha(t)$ has a maximum at $t = 0$,

$$0 = \frac{d}{dt}|_{t=0}\big(g \circ \alpha(t)\big) = (\operatorname{grad} g)\big(\alpha(0)\big) \cdot \alpha'(0)$$

by the chain rule.

(b) $\sum_{i,j=1}^{3} \frac{\partial^2 g}{\partial x_i \partial x_j}(p)\alpha_i'(0)\alpha_j'(0) \leq -(\operatorname{grad} g)(p) \cdot \alpha''(0)$.

In fact the 2nd derivative test shows that

$$\begin{aligned} 0 \geq \frac{d^2}{dt^2}|_{t=0}\big(g \circ \alpha(t)\big) &= \frac{d}{dt}|_{t=0}\Big[\big((\operatorname{grad} g) \circ \alpha(t)\big) \cdot \alpha'(t)\Big] \\ &= \sum_{i,j=1}^{3} \frac{\partial^2 g}{\partial x_i \partial x_j}(p)\alpha_i'(0)\alpha_j'(0) + (\operatorname{grad} g)(p) \cdot \alpha''(0) \end{aligned}$$

6. Suppose $g : \mathbf{R}^3 \to \mathbf{R}$ is smooth, $\vec{x} = (x_1, x_2, x_3) : \mathbf{R}^2 \to \mathbf{R}^3$ defines a regular surface S, and the maximum value of g on S occurs at $p = \vec{x}(0, 0)$. Show that

(a) $(\operatorname{grad} g)(p)$ is parallel to the normal to S at p.

Since $g \circ \vec{x}(t,0)$ and $g \circ \vec{x}(0,t)$ have maxima at $t = 0$, the chain rule shows that

$$0 \;=\; \frac{d}{dt}|_{t=0}\big(g \circ \vec{x}(t,0)\big) \;=\; (\operatorname{grad} g)\big(p\big) \cdot \vec{x}_u(0,0)$$

and similarly that $(\operatorname{grad} g)\big(p\big) \cdot \vec{x}_v(0,0) = 0$. Thus $(\operatorname{grad} g)(p)$ is normal to the tangent space of S at p and so parallel to the normal to S at p.

(b) $\frac{\partial^2 (g \circ \vec{x})(tu,tv)}{\partial t^2}(p) \leq 0$ for all $(u,v) \in \mathbf{R}^2$.

This also follows from the 2nd derivative test because each function $(g \circ \vec{x})(tu,tv)$ has a maximum at $t = 0$.

7. True-False (no proofs necessary but draw pictures)

(a) There exists a smooth simple closed curve with 5 vertices.

True. Add one vertex as an inflection point for curvature between a curvature maximum and minimum in an ellipse.

(b) Suppose p is a point in the interior region of a simple closed curve α. The most distant point from p on the image of α is a vertex of α.

False. Consider a "French curve".

As a very explicit example (following Florin's solution), the ellipse E defined by $\frac{x^2}{3} + (y-1)^2 = 1$ has exactly 4 vertices, $(0,0)$, $(0,2)$, $(\sqrt{3},1)$, $(-\sqrt{3},1)$. The distance from $(0,0)$ to each of the other 3 vertices is 2. But since the tangent line to E is vertical at the side vertices $(\pm\sqrt{3},1)$, there are points in E outside the circle $x^2 + y^2 = 4$ and so $(0,0)$ (and $(0,\epsilon)$ for ϵ small) has points in E farther than the vertices.

8. (a) Show that the length $(\alpha \cdot v) \leq$ length (α) for any smooth curve α in $\mathbf{R}^3$ and unit vector v in $\mathbf{R}^3$.

Since $|(\alpha \cdot v)'(t)| = |(\alpha'|\alpha'(t) \cdot v| \leq |\alpha'(t)| \cdot 1$, integration over the domain of α gives the desired length inequality.

(b) Find length (β) where $\beta(0) = (0,0,0)$, $\beta(t) = \big(t^2\cos(t^{-2}), t^2\sin(t^{-2}), t^2\big)$ for $0 < t < \sqrt{\pi}$.

Using part (a) we project the curve β onto the X-axis. For $t^2 \sin(t^{-2})$, the local maximum and minima occur when $t^{-2} = n\pi$ and the values there are $\pm\frac{1}{n\pi}$. Thus the length of the curve β is at least $\sum_{n=1}^{\infty} \frac{1}{n\pi}$ which equals infinity.

9. Suppose $\vec{x}(u,v) = \big((3+\sin v)\sin u, (3+\sin v)\cos u, \cos v\big)$ for $-\pi \le u < \pi$, $-\pi \le v < 2\pi$, and S is the image of $\vec{x}$.

(a) Find a formula for the outward pointing unit normal $N(u,v)$ of S at $\vec{x}(u,v)$.

$\vec{x}_u(u,v) = \big((3+\sin v)\cos u, -(3+\sin v)\sin u, 0\big)$

$\vec{x}_v(u,v) = \big((\cos v)\sin u, (\cos v)\cos u, -\sin v\big)$

$\vec{x}_u(u,v) \wedge \vec{x}_v(u,v) = \big(-(3+\sin v)\sin u \sin v, -(3+\sin v)\cos u \sin v, (3+\sin v)\cos v\big)$

$N(u,v) = \big(\sin u \sin v, \cos u \sin v, \cos v\big)$

Checking at $\vec{x}_u(u,v)(\frac{\pi}{2},\frac{\pi}{2}) = (4,0,0)$ that $N(\frac{\pi}{2},\frac{\pi}{2}) = (1,0,0)$, we see that N is outward pointing.

(b) Find the area of S. Arguing as on page 99 or page 101 (Ex.11) we see that the area equals $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(3+\sin v)\,du\,dv = 12\pi^2$.

(c) What are the all the values of the normal curvatures of S at $(4,0,0)$ and at $(2,0,0)$? (Hint: look at the picture.)

At both points one uses the intersections with the $X-Y$ and the $Y-Z$ planes.

At $(4,0,0)$ the resulting circles have radii 4 and 1 and the normal curvatures vary vary between $\frac{1}{4}$ and 1.

At $(2,0,0)$ the resulting circles have radii 2 and 1 and the normal curvatures vary vary between $-\frac{1}{2}$ and 1.

(d) What are all the possible values for $|\kappa(p)|$ where $p \in S$ and κ is the curvature of a *curve* lying on S?

The smallest is $\frac{1}{4}$, and there is no upper bound on the size of curvature of a curve in S.

10. Suppose f is a smooth real-valued function on the unit interval $[0,1]$ and $G = \{(x,y) \in \mathbf{R}^2 \ : 0 \le x \le 1,\ y = f(x)\}$.

(a) Find a formula for the length of G.

With $\alpha(t) = \big(t, f(t)\big)$, the length equals $\int_0^1 \sigma(t)dt$ where

$$\sigma(t) \ = \ |\alpha'(t)| \ = \ \sqrt{1 + f'(t)^2}\ .$$

(b) Find a formula for the curvature κ of G at each point $\big(x, f(x)\big)$ with $0 < x < 1$.

Using p.25, Ex.12(b)(d), we let $\kappa(t)$ denote the curvature of G at the point $\big(t, f(t)\big)$. Then $\kappa(t) = |k(t)|$ where $k(t)$ is the signed curvature

$$k(t) \ = \ \sigma^{-3}(t)(1 \cdot f'' - 0 \cdot f') \ = \ \sigma^{-3}(t)f''(t) \ .$$

(c) For a fixed small positive number ϵ, find a formula for the length of the set

$$G_\epsilon = \{p \in \mathbf{R}^2 \ : \ \text{dist}\,\big(p, G\big) = \epsilon\} \ .$$

(Recall that $\text{dist}\,(p, A) = \min\{|p - a| \ : \ a \in A\}$ for any closed set A.)

Here we proceed as in Ex.6,p.47 except that the curve is no longer necessarily convex. With $n(t) = \sigma^{-1}(-f', 1)$ being the upward unit normal to G at $\big(t, f(t)\big)$, we see that the two parallel curves in G_ϵ and the images of $\alpha_\pm = \alpha \pm \epsilon n$. Then, with s being the arc-length parameter for α,

$$n' \ = \ \frac{dn}{ds}\frac{ds}{dt} \ = \ (-k\frac{\alpha'}{\sigma})\sigma \ ,$$

$$\sigma_\pm \ = \ |\alpha'_\pm| \ = \ |\alpha' \pm \epsilon k \alpha'| \ = \ (1 \pm \epsilon k)\sigma \ = (1 \pm \epsilon \sigma^{-3} f'')\sigma \ .$$

Including the two end semi-circles, we find that the total length of G_ϵ is

$$2(\pi\epsilon) \ + \ \int_0^1 \big(\sigma(t) + \epsilon\sigma^{-2}(t)f''(t)\big)\,dt \ + \ \int_0^1 \big(\sigma(t) - \epsilon\sigma^{-2}(t)f''(t)\big)\,dt \ .$$

(d) Find, wherever possible, a formula for the curvature of G_ϵ.

For $0 < t < 1$, the normal to G_ϵ at $\alpha_\pm(t)$ is simply $n_\pm(t) = n(t)$ because $n'(t) \cdot (\alpha_\pm)'(t) = 0$. Thus, with $s_\pm$ being the arc-length parameter for $\alpha_\pm$, the curvature there is given by

$$\kappa_\pm(t) = |\frac{dn_\pm}{ds_\pm}(t)| = |n'(t)|\frac{dt}{ds_\pm}(t) = |n'(t)|\sigma_\pm^{-1} = \frac{\kappa\sigma}{(1 \pm \epsilon\sigma^{-3}f'')\sigma} \ = \frac{|\sigma^{-3}f''|}{1 \pm \epsilon\sigma^{-3}f''} \ .$$

On the two open semicircles of G_ϵ the curvature is $\frac{1}{\epsilon}$. The curvature is not defined at the 4 points joining the semi-circles to the top and bottom curves because the normal is not differentiable at these points.