Part III Differential Geometry exam 2004

Answer **THREE** *questions. There are* **five** *questions in total. The questions carry equal weight.*

All manifolds and related concepts should be assumed to be smooth.

1. Define smooth vector fields on a manifold. Show that the tangent bundle TM over an n-dimensional manifold M is isomorphic to the product vector bundle $M \times \mathbb{R}^n$ if and only if M admits a collection of n smooth vector fields $X_1, \dots, X_n$, such that $X_1(p), \dots, X_n(p)$ is a basis of the tangent space T_pM, for any $p \in M$.

Explain what is meant by a left-invariant vector field on a Lie group G and show that any left-invariant vector field is smooth. Show that the tangent bundle TG is isomorphic to the product vector bundle $G \times \mathbb{R}^{\dim G}$.

[*Basic properties of the differential of a smooth map between manifolds can be used without proof provided these are clearly stated.*]

2. Define a smooth free right action of a Lie group G on a manifold P. Give a definition of a principal G-bundle $\pi : P \to B$ over a manifold B. Show that if Φ_1, Φ_2 are two local trivializations of π over some (overlapping) neighbourhoods N_1, N_2 in B then $\Phi_2 \circ \Phi_1^{-1}(x, g) = (x, \psi(x)g)$, for some smooth map $\psi : N_1 \cap N_2 \to G$, where $x \in B$, $g \in G$.)

Let the map $p : \mathbb{R}P^3 \to \mathbb{C}P^1$ be given by $p(x_0 : x_1 : x_2 : x_3) = (x_0 + ix_1) : (x_2 + ix_3)$. Identify a smooth free right action of $U(1)$ on $\mathbb{R}P^3$ whose orbits are precisely the fibres of p over point in $\mathbb{C}P^1$. Show that p is a principal $U(1)$-bundle by constructing appropriate local trivializations over the neighbourhoods $\{z : 1 \mid z \in \mathbb{C}\}$ and $\{1 : z \mid z \in \mathbb{C}\}$ in $\mathbb{C}P^1$. Calculate the transition function between these local trivializations.

3. Let A be a connection on a vector bundle E. Using local coordinates on the base manifold and a local trivialization of E, give an explicit local formula for the covariant derivative d_A induced by A and acting on the sections of E. Explain how to extend d_A, using an appropriate version of the Leibniz rule, to the differential forms with values in E and to the differential forms with values in the endomorphism bundle $\operatorname{End} E$. For both cases, include explicit formuli for d_A in local trivializations.

Define the curvature $F(A)$ of a connection A, showing that $F(A)$ is a well-defined 2-form with values in $\operatorname{End} E$. Prove the Bianchi identity $d_A F(A) = 0$.

By using the Bianchi identity or otherwise, show that if E is a vector bundle of rank 1 and then $F(A)$ is a closed form and its de Rham cohomology class is independent of the choice of connection A.

[*Preliminary results on connections may be used without proof provided these are clearly stated.*]

4. Explain what is meant by a geodesic on a Riemannian manifold and deduce a system of ordinary differential equation satisfied by geodesics in local coordinates.

Given a geodesic $\gamma(t)$, defined for $|t| < \varepsilon$ ($\varepsilon > 0$), show that for some $\delta > 0$ the family of velocity vectors $\dot\gamma(t)$, $|t| < \delta$, can be extended to a smooth vector field defined on a neighbourhood of $\gamma(0)$. Prove that the velocity vectors $\dot\gamma(t)$ have constant length.

Find all geodesics on the sphere S^n with the metric induced by the standard embedding in Euclidean $\mathbb{R}^{n+1}$.

5. Define the Levi–Civita connection on a Riemannian manifold and prove the every Riemannian manifold has a unique Levi–Civita connection.

Define the curvature of a Riemannian manifold and state the symmetry relations satisfied by the curvature components. Define the Ricci curvature and show that it is a symmetric bilinear form on the tangent spaces. Show that on a Riemannian manifold of dimension *three* the value of the (full) curvature at any point is determined by the Ricci curvature at that point.