LECTURE NOTES ON NON-HOLONOMIC GEOMETRY

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1. Distributions on manifolds

Let M be an n-dimensional smooth manifold. We denote by $\mathfrak{X}(U)$ the Lie algebra of vector fields defined on an open $U \subset M$.

A *k-dimensional distribution* Δ on M is a smooth field of k-dimensional subspaces $\Delta(p) \subset T_pM$, $p \in M$. The smoothness of this field means that, for any $p \in M$, there exists a neighbourhood $U(p)$ and vector fields $E_1, E_2, \dots, E_k \in \mathfrak{X}(U(p))$ such that, for each $q \in U(p)$, $\Delta(q) = \operatorname{span}\{E_1(q), E_2(q), \dots, E_k(q)\}$.

A distribution Δ on a manifold M is called *totally integrable* if for each $p \in M$ there exists a submanifold Σ passing through p such that $T_q\Sigma = \Delta(q)$, for any $q \in \Sigma$. This submanifold is called an *integral submanifold* of Δ. A totally integrable distibution is called a foliation, and the integral submanifolds are called the leaves.

Locally, a distribution Δ can be given either by vector fields $E_1, \dots, E_k$ spanning the subspaces $\Delta(p) \subset T_pM$, or by differential forms $\omega^1, \omega^2, \dots, \omega^{n-k}$ such that, for any $V \in T_pM$, $V \in \Delta(p)$ if and only if $\omega^\alpha(V) = 0$, $\alpha = 1, \dots, n-k$.

The following theorem gives necessary and sufficient conditions for a distribution be completely integrable.

Theorem 1 (Frobenius)**.** *Let Δ be a distribution on a manifold M locally given by vector fields E_a, $a = 1, \dots, k$, or by fields of 1-forms ω^α, $\alpha = 1, \dots, n-k$. The following conditions are equivalent:*

a) *The distribution Δ is completely integrable*;

b) $[E_a, E_b] = Q^c_{ab}E_c$, *where Q^c_{ab} are functions*;

c) $d\omega^\gamma = Q^\gamma_\beta \wedge \omega^\beta$, *where Q^γ_β are 1-forms.*

Example 1.1. Consider a distribution Δ on $\mathbb{R}^3$ given by $\omega = dz - xdx - ydy$. Then $d\omega = 0$, and, by Theorem 1, Δ is completely integrable. In fact, this distribution consists of planes tangent to hyperbolic paraboloids $z = \frac{1}{2}(x^2 + y^2) + c$.

Example 1.2. Let a distribution Δ on $\mathbb{R}^3$ be given by $\omega = dy - zdx$. Then the vector fields $E_1 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}$, $E_2 = \frac{\partial}{\partial z}$ span Δ. We have $[E_1, E_2] = -\frac{\partial}{\partial y}$, and this vector field does not

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lie in Δ because $\omega(\frac{\partial}{\partial y}) = -1 \neq 0$. Hence this distribution is not completely integrable. This means that one cannot find a surface tangent to Δ.

2. Non-holonomic surface in E^3

Let E_3 be the three-dimensional Euclidean space, we denote by $(\cdot,\cdot)$ the scalar product in E_3. Any distribution Δ on E_3 can be given by a unit vector field $\vec{n}$ normal to Δ. The integrability condition for Δ can be formulated in terms of $\vec{n}$.

Theorem 2. *Δ is integrable if and only if $(\vec{n}, \operatorname{rot}\vec{n}) = 0$.*

2.1. **Geometric sense of non-holonomity.** The non-holonomity of a distribution in E^3 can be visualized in the following manner. Take a small disk L in the distribution plane and draw the straight line $l(A)$ through any point A of the disk boundary ∂L in the direction of $\vec{n}(A)$. Thus we obtain a ruled surface. Now take a curve γ passing through A_0 which is orthogonal to the straight lines $l(A)$ (see Fig. 1). This curve again meets

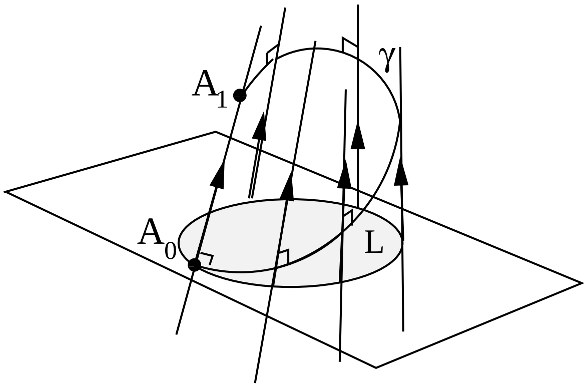

Figure 1. Geometric sense of non-holonomity. I

$l(A_0)$ at a point A_1, then $\frac{|A_0A_1|}{\sigma(L)}$, where $\sigma(L)$ is the area of L, converges to $(\vec{n}, \operatorname{rot}\vec{n})_{A_0}$ as L contracts to A_0. If Δ is holonomic (totally integrable), then γ lies on the integral surface passing through A_0, hence $A_1 = A_0$, and $|A_0A_1| = 0$.

Another way to visualize the non-holonomity is to take two unit orthogonal vector fields $\vec{a}$, $\vec{b}$ spanning Δ. Let $\gamma(t)$ be an integral curve of the field $\vec{b}$, and for each t we take the integral curve $\delta_t(s)$ of the field $\vec{a}$ passing through $\gamma(t)$. Thus we obtain a surface Σ (see Fig. 2). Now at a point $A(s) = \delta_{t(s)}(s)$ we take the vector $\vec{\nu}(A(s))$ normal to $T_{A(s)}\Sigma$ and the vector $\vec{n}(A(s))$ normal to $\Delta(A(s))$. Denote by $\varphi(s)$ the angle between $\vec{\nu}(A(s))$ and $\vec{n}(A(s))$. Then

$$\frac{d\varphi}{ds} = -(\vec{n}, \operatorname{rot}\vec{n}). \tag{1}$$

If Δ is holonomic (totally integrable), then Σ is an integral manifold of Δ, and $T_p\Sigma = \Delta(p)$. Hence $\vec{\nu}(p) = \vec{n}(p)$, and $\varphi = 0$.

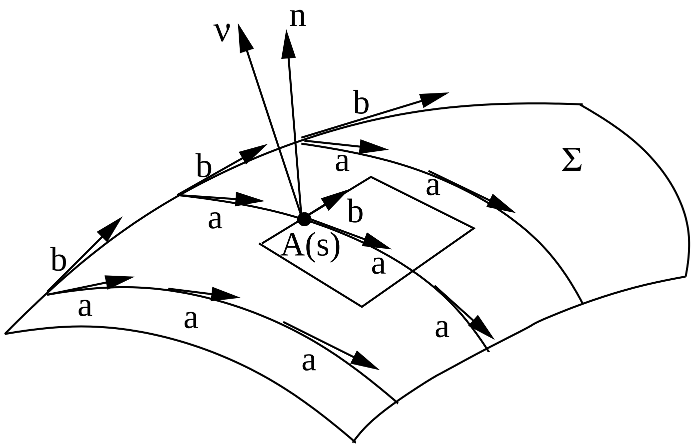

FIGURE 2. Geometric sense of non-holonomity. II

2.2. Curvatures of non-holonomic surface.

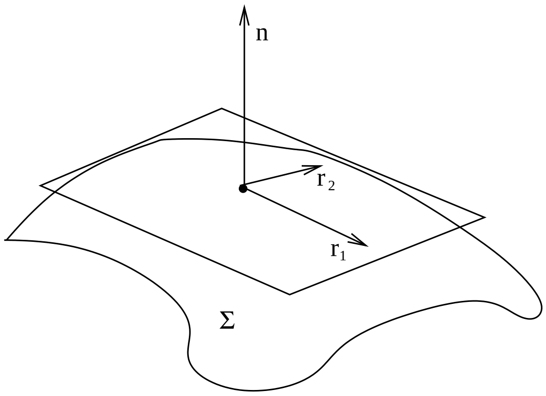

FIGURE 3. Moving frame

2.2.1. *Curvatures of holonomic surface.* Let a surface Σ in E_3 be given by the parametric equation $\vec{r} = \vec{r}(u^1, u^2)$. Let us set $\vec{r}_i = \partial\vec{r}_i = \frac{\partial}{\partial u^i}\vec{r}$. The vectors $\vec{r}_1$, $\vec{r}_2$ give a frame of the tangent plane $T\Sigma$, and the coordinates of the metric tensor (the first fundamental form) with respect to this frame are $g_{ij} = (\vec{r}_i, \vec{r}_j)$. We denote by g^{ij} the tensor inverse to g_{ij}, this means that $g^{is}g_{js} = \delta^i_j$. Now let $\vec{n}$ be the unit normal to $T\Sigma$. Then we obtain a moving frame $\{\vec{r}_1, \vec{r}_2, \vec{n}\}$ (see Fig. 3) and the derivation equations:

$$\partial_i\vec{r}_j = \Gamma^k_{ij}\vec{r}_k + h_{ij}\vec{n} \tag{2}$$

$$\partial_i\vec{n} = -h^j_i\vec{r}_j \tag{3}$$

Here

$$\Gamma^k_{ij} = \frac{1}{2}g^{ks}(\partial_i g_{js} + \partial_j g_{is} - \partial_s g_{ij}) \tag{4}$$

are the *connection coefficients* (we will discuss them later), which depend only on the metric tensor, h_{ij} is the *second fundamental form*, and $h^i_j = g^{is}h_{js}$ is the *shape operator*.

The shape operator is a symmetric linear operator on $T\Sigma$, therefore it has real eigenvalues k_1, k_2 which are called *principal curvatures* of Σ. $H = \frac{1}{2}(k_1 + k_2)$ is called the *mean curvature* of Σ and $K = k_1k_2$ the *total (Gaussian) curvature* of Σ.

Geometrically, the principal curvatures can be described as follows. Take a point $p \in \Sigma$.

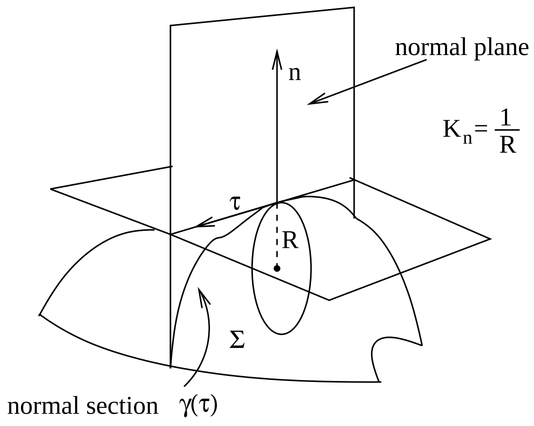

FIGURE 4. Normal curvature

For any direction tangent to Σ given by a unit vector $\vec{\tau}$ at p, we can take the plane passing through $\vec{n}$ and $\vec{\tau}$ (a normal plane), and this plane meets the surface Σ along a normal section $\gamma(\vec{\tau})$. Let $\vec{\rho}(s)$ be the parametric equation of $\gamma(\vec{\tau})$ referred to the natural parameter s. Then, $k_n(\vec{\tau}) = (\frac{d^2}{ds^2}\vec{\rho}(s), \vec{n})$ is called the *normal curvature of* Σ *with respect to the direction* $\vec{\tau}$. As k_n is a continuous function on the circle, it attains its extremal values. One can easily prove that the principal curvatures k_1 and k_2 are just the extremal values of the normal curvature $k_n(\vec{\tau})$. In addition, the normal curvature can be written in terms of the second fundamental form: $k_n(\vec{\tau}) = h(\vec{\tau}, \vec{\tau})$. Also, using (3), we get

$$k_n(\vec{\tau}) = -(\tau^i \partial_i \vec{n}, \vec{\tau}) = -(\nabla_{\vec{\tau}}\vec{n}, \vec{\tau}). \tag{5}$$

where ∇ stands for the directional derivative.

The total curvature can be defined in another way. The unit normal $\vec{n}$ determines the Gaussian map $G : \Sigma \to \mathbb{S}^2$, $p \in \Sigma \mapsto \vec{n}(p)$. Let U be a neighborhood of $p \in \Sigma$, and $V = G(U)$ be its image on $\mathbb{S}^2$. We denote by $\sigma(U)$ the area of U on Σ, and by $S(V)$ the area of V on $\mathbb{S}^2$. Then the total curvature K at p equals $\lim \frac{S(V)}{\sigma(U)}$ as U contracts to p. From this property one can obtain another formula for the total curvature:

$$K = \frac{(\partial_1\vec{n}, \partial_2\vec{n}, \vec{n})}{\sqrt{g_{11}g_{22} - g_{12}^2}} \tag{6}$$

If we have a foliation in E^3 whose leaves are given by an implicit equation $\Phi(x^1, x^2, x^3) = const$, then the vector field $\vec{n}$ is defined in the entire E^3. In this case, from (6) it follows that

$$K = (\vec{P}, \vec{n}), \tag{7}$$

where $\vec{P}$ is the *curvature vector* defined as follows:

$$\vec{P} = \big((\frac{\partial}{\partial x^2}\vec{n}, \frac{\partial}{\partial x^3}\vec{n}, \vec{n}), (\frac{\partial}{\partial x^3}\vec{n}, \frac{\partial}{\partial x^1}\vec{n}, \vec{n}), (\frac{\partial}{\partial x^1}\vec{n}, \frac{\partial}{\partial x^2}\vec{n}, \vec{n}) \big), \tag{8}$$

where $(\cdot, \cdot, \cdot)$ denotes the mixed product of three vectors. In particular, (7) implies the von Neumann formula:

$$K = \frac{\begin{vmatrix} \partial_{11}\Phi & \partial_{12}\Phi & \partial_{13}\Phi & \partial_1\Phi \\ \partial_{21}\Phi & \partial_{22}\Phi & \partial_{23}\Phi & \partial_2\Phi \\ \partial_{31}\Phi & \partial_{32}\Phi & \partial_{33}\Phi & \partial_3\Phi \\ \partial_1\Phi & \partial_2\Phi & \partial_3\Phi & 0 \end{vmatrix}}{(\partial_1\Phi)^2 + (\partial_2\Phi)^2 + (\partial_3\Phi)^2)^2},$$

where the partial derivatives are taken with respect to x^1, x^2, x^3.

2.2.2. *Curvatures of non-holonomic surface. First type.* Let a two-dimensional distribution (a non-holonomic surface) Δ be given in E^3, and $\vec{n}$ be the unit normal vector field of Δ. The formula (5) gives us the normal curvature of a surface in terms of the normal vector $\vec{n}$ and a tangent vector $\vec{\tau}$, so it can be generalized to the non-holonomic case. We say that the *normal curvature of non-holonomic surface* Δ *in the direction of a unit vector* $\tau \in \Delta$ is

$$k_n(\tau) = -(\nabla_{\vec{\tau}}\vec{n}, \vec{\tau}). \tag{9}$$

Let $A \in E^3$, and $\vec{e}_1$, $\vec{e}_2$ be an orthonormal frame of $\Delta(A)$. Then $\vec{\tau}(\varphi) = \cos\varphi \vec{e}_1 + \sin\varphi \vec{e}_2$, and one can easily see that $k_n(\vec{\tau})$ defined by (9) has the form

$$k_n(\vec{\tau}(\varphi)) = k_1 \cos\varphi + k_2 \sin\varphi, \tag{10}$$

and it is clear that k_1 and k_2 are extremal values of $k_n(\vec{\tau}(\varphi))$. k_1 and k_2 are called *the principal curvatures of* Δ *of first type.* Then we can define the *mean curvature of first type* $H_I = \frac{1}{2}(k_1 + k_2)$ and the *total curvature of first type* $K_I = k_1 k_2$.

2.2.3. *Curvatures of non-holonomic surface. Second type.* For a non-holonomic distribution Δ we can define the *total curvature of second type* using (7), where $\vec{P}$ is given by (8). We will denote it by K_{II}.

At the same time, this curvature can be obtained in the following way. Let us consider eigenvalues of the Jacobi matrix $J(\vec{n}) = ||\frac{\partial n^i}{\partial x^j}||$, where n^i are the Cartesian coordinates of the vector-function $\vec{n}$. Since $\vec{n}$ maps the three-dimensional space into the two-dimensional sphere, the determinant of $J(\vec{n})$ is zero, hence one of the eigenvalues of $J(\vec{n})$ is zero. We will call the other two eigenvalues λ_1, λ_2 the principal curvatures of second type (they can be complex conjugate) and one can prove that $K_{II} = \lambda_1\lambda_2$. Also, we can define the *mean curvature of second type* to be $H_{II} = \frac{1}{2}(\lambda_1 + \lambda_2)$.

Theorem 3. *For a non-holonomic surface* Δ *we have*

$$H_I = H_{II} \qquad K_I - K_{II} = \frac{(\vec{n}, \operatorname{rot}\vec{n})^2}{4}. \tag{11}$$

Example 2.1. An example of distribution which has $K_{II} = 0$, $H_{II} = 0$, and is non-holonomic can be constructed as follows. We take the family of concentric spheres in E^3 with center at the origin, and each sphere $\mathbb{S}^2(r)$ of the family is uniquely determined by its radius $r > 0$. For each $r > 0$, we take a straight line $l(r)$ lying in the plane XOY such that the angle between $l(r)$ and OX is $\alpha(r)$, where $\alpha(r)$ is a function (see Fig. 5). Now,

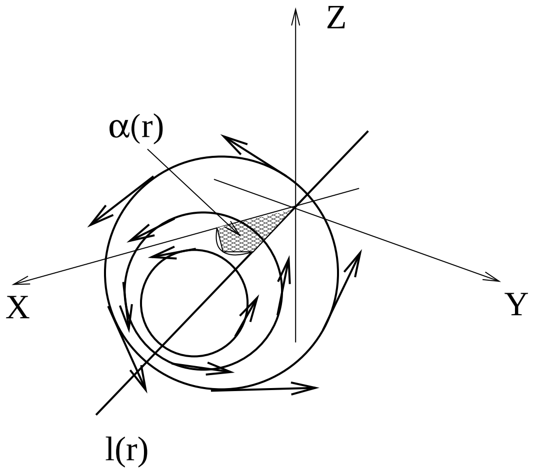

FIGURE 5. non-holonomic distribution with zero curvatures

for each $\mathbb{S}^2(r)$, we take the unit vector field $\vec{n}$ tangent to this sphere which is generated by rotations about the axis $l(r)$ (certainly this field is not defined in the points where $l(r)$ meets $\mathbb{S}^2(r)$). Thus we obtain a unit vector field $\vec{n}$ which is defined everywhere in E^3 except for points where $l(r)$ meets $\mathbb{S}^2(r)$. If $\frac{d\alpha}{dr} \neq 0$, at points with $z \neq 0$ we have $(\vec{n}, \operatorname{rot}\vec{n}) \neq 0$, so the distribution Δ orthogonal to $\vec{n}$ is non-holonomic. At the same time, the second type mean and total curvature of Δ vanish.

Example 2.2. Another way to obtain a distribution Δ with zero total curvature of second type is to consider a surface Σ with a vector field $\vec{a}$ along it (we assume that $\vec{a}$ is not tangent to Σ). Then, for each point $p \in \Sigma$ we take a plane curve which is tangent to $\vec{a}(p)$ at p. Thus, a neighborhood of Σ is foliated by these curves, and for the vector field $\vec{n}$ we take the unit vector field tangent to these curves. The distribution Δ orthogonal to $\vec{n}$ has $K_{II} = 0$.

2.2.4. *Gauss-Bonnet formula.* For a closed surface Σ in E^3, we have the Gauss-Bonnet formula

$$\int_{\Sigma} K dS = 2\pi\chi(\Sigma), \tag{12}$$

where K is the total curvature of Σ, and $\chi(\Sigma)$ is the Euler characteristic of Σ.

For a non-holonomic surface Δ, we have the following generalization of the Gauss-Bonnet formula.

Theorem 4. *Let Q be a closed oriented surface which lies in the domain of Δ. Then*

$$\int_{Q} (K_{II}\vec{n} + 2H_{II}\vec{k} + \nabla_{\vec{k}}\vec{n}, \vec{\nu}) dS = 4\pi\theta, \tag{13}$$

where $\vec{k}$ is the curvature vector of integral curves of $\vec{n}$, $\vec{\nu}$ is the unit normal to the surface Q, $\nabla_{\vec{k}}$ is the derivative in E^3 with respect to $\vec{k}$, and θ is the degree of the map $\vec{n}|_Q : Q \to \mathbb{S}^2$.

Let us recall that the degree of a map $f : Q \to P$ between oriented surfaces is an integer which equals zero if f is homotopic to a constant map. Under assumptions of Theorem 4, Q is a boundary of a region $V \subset E^3$. If Δ is defined everywhere in V, and V is contractible, then $\vec{n}$ is homotopic to a constant vector field, hence $\theta = 0$.

2.3. **The shortest and the straightest curves on a non-holonomic surface.** Let us recall that a geodesic on a surface Σ in the three-dimensional Euclidean space can be defined in two ways: a) a geodesic is a straightest curve; b) a geodesic is a locally shortest curve. Consider these definitions in more details.

Let us consider a curve on a surface Σ given by a parametric equation $\vec{r} = \vec{\rho}(s)$, where s is the natural parameter. The length of projection of the curvature vector $\frac{d^2}{ds^2}\vec{\rho}$ onto the plane tangent to Σ is called the *geodesic curvature*. If the geodesic curvature is zero, or equivalently the curvature vector is collinear to the normal vector of the surface, we say that this curve is a *straightest curve* on Σ because it has no "acceleration" along the surface.

At the same time, if a curve γ joining sufficiently close points p and q on the surface Σ has the minimal length among all the curves joining these points, then γ is called the *shortest curve*.

For a surface $\Sigma \subset E^3$, the definition a) is equivalent to definition b). However, for a non-holonomic surface Δ the curves defined by a) and b) are different. We will call a curve γ in E^3 an *admissible curve* if its tangent vector $\frac{d}{dt}\gamma$ lies in $\Delta(\gamma(t))$.

2.3.1. *Straightest lines on non-holonomic surfaces.* Let a non-holonomic surface Δ be given. An admissible curve γ is called a *straightest curve* if its curvature vector is collinear to the normal vector $\vec{n}$ of Δ.

Theorem 5. *Let $\vec{r} = \vec{\rho}(s)$ be parametric equation of an admissible curve γ. Then γ is a straightest line if and only if $\vec{\rho}(s)$ satisfies the following second order differential equation:*

$$\frac{d^2\vec{\rho}}{ds^2} = -\left(\frac{d\vec{\rho}}{ds}, \nabla_{\frac{d\vec{\rho}}{ds}}\vec{n}\right)\vec{n}. \tag{14}$$

Thus, given a point $p \in E^3$ and a vector $V_0 \in \Delta(p)$ one can find a unique straightest curve γ which passes through p and whose tangent vector at p is V_0.

2.4. **Shortest lines on non-holonomic surfaces.** Let us consider two points p and q in E^3, and let $P_a(p,q)$ be the set of all admissible curves joining p and q. We say that a curve $\gamma \in P_a(p,q)$ such that $\gamma(a) = p$, $\gamma(b) = q$, is a shortest admissible curve if the length $L(\gamma) = \int_a^b |\frac{d\gamma}{dt}|dt$ is less than or equal to the length of any other curve in $P_a(p,q)$.

Theorem 6. *Let a curve $\gamma \in P_a(p,q)$ be given by a parametric equation $\vec{r} = \vec{\rho}(s)$, $s \in [a,b]$. Then, if γ is a shortest curve, then $\vec{\rho}(s)$ satisfies the third order ordinary differential equation*

$$\frac{d^2\vec{\rho}}{ds^2} + \frac{d}{ds}\left\{\frac{(\frac{d^2\vec{\rho}}{ds^2}, \frac{d\vec{\rho}}{ds}, \vec{n})}{(\vec{n}, \operatorname{rot}\vec{n})}\right\}\vec{n} + \frac{(\frac{d^2\vec{\rho}}{ds^2}, \frac{d\vec{\rho}}{ds}, \vec{n})}{(\vec{n}, \operatorname{rot}\vec{n})}[\operatorname{rot}\vec{n}, \frac{d\vec{\rho}}{ds}] = 0. \tag{15}$$

Therefore, given a point $p \in E^3M$ and a vector $V_0 \in \Delta(p)$ one can find an infinite number of shortest curves joining p and q which pass through p and whose tangent vector at p is V_0.

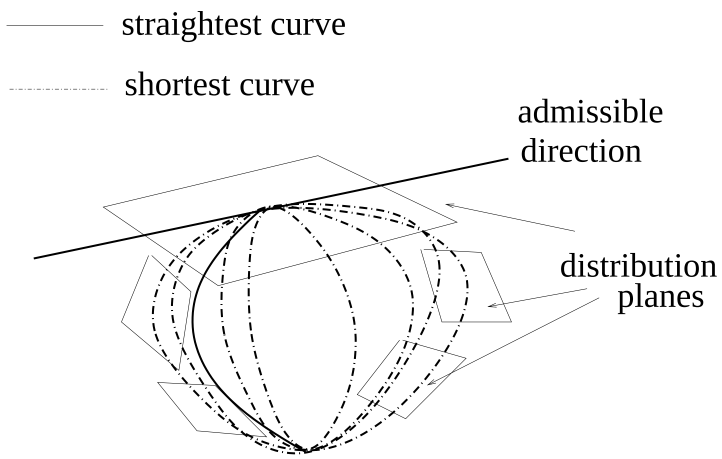

FIGURE 6. A unique straightest curve and a bundle of shortest curves

3. PROPERTIES OF METRIC OF SUB-RIEMANNIAN MANIFOLD

3.1. **Accessibility of points.** Let M be an n-dimensional differential manifold, and Δ be an m-dimensional distribution on M. We will call a curve γ in M *admissible* if its tangent vector lies in Δ. Let us denote by $P_a(p,q)$ the set of all admissible curves joining p and q (this set can be empty!). We say that a point q is *accessible* from a point p if $P_a(p,q)$ is not empty.

Theorem 7 (Sussman)**.** *The set of points accessible from a given point p is an immersed submanifold in M.*

3.2. **Completely non-integrable distributions.** First let us define a distribution with singularities on an n-dimensional manifold M. A *distribution with singularities* Δ on M is a smooth field of subspaces $\Delta(p) \subset T_pM$, $p \in M$. The smoothness of this field means that, for any $p \in M$, there exists a neighbourhood $U(p)$ and vector fields $E_1, E_2, \dots, E_k \in \mathfrak{X}(U(p))$ such that, for each $q \in U(p)$, $\Delta(q) = \operatorname{span}\{E_1(q), E_2(q), \dots, E_k(q)\}$. Note that we do not assume that the vectors $E_1(q), E_2(q), \dots, E_k(q)$ are linearly independent (this makes difference from the definition of k-dimensional distribution, see Section 1). Certainly, each k-dimensional distribution is a distribution with singularities.

Now let Δ be a distribution with singularities on M. Let $\mathfrak{X}_\Delta(U)$ be the set of vector fields X on an open set $U \subset M$ such that, for each $q \in U$, $X(q) \in \Delta(q)$. Now, for the distribution Δ we can construct another distribution Δ^1 with singularities in the following way. For any point $p \in M$, we set

$$\Delta^1(p) = \{X(p), [X,Y](p) \mid X, Y \in \mathfrak{X}_\Delta(U_p), \text{ where } U_p \text{ is a neighborhood of } p\} \tag{16}$$

From this definition it follows that $\Delta(p) \subset \Delta^1(p)$, for any $p \in M$.

It is clear that, if Δ is completely integrable if and only if $\Delta^1 = \Delta$.

For a distribution Δ we can sequentially construct the series Δ^k of distributions with singularities by setting $\Delta^{k+1} = (\Delta^k)^1$. Since $\Delta^k \subset \Delta^{k+1}$, and $\Delta^k = \Delta^{k+1}$ if and only if Δ^k is completely integrable, there exists N such that $\Delta^N = \Delta^{N+1} = \Delta^{N+2} = \dots$, and Δ^N is completely integrable. Thus, if the subspaces $\Delta^N(p)$ do not coincide with T_pM, then we get a foliation (with singularities) on M. If, for each $p \in M$, we have $\Delta^N(p) = T_pM$, then we say that the distribution Δ is *completely non-integrable*.

If a distribution Δ is completely non-integrable, then the following theorem implies that any point of M is accessible from any other point of M.

Theorem 8 (Caratheodory, Rashevskii, Chow)**.** *Let a distribution Δ on a connected manifold M be completely non-integrable. Then, for any p, q in M, $P_a(p,q) \neq \emptyset$.*

3.3. **Sub-Riemannian metric of completely non-integrable distributions.** A *sub-Riemannian manifold* is a completely non-integrable distribution Δ endowed by a metric g. This means that, for any p in M, on $\Delta(p)$ we have the scalar product $g_p : \Delta(p) \times \Delta(p) \to \mathbb{R}$ such that for any $X, Y \in \mathfrak{X}_\Delta(M)$ the function $p \mapsto g_p(X(p), Y(p))$ is smooth. In other words, we can say that we have a metric on Δ considered as a vector subbundle of the vector bundle TM.

If $\gamma(s)$, $s \in [a, b]$, is an admissible curve in M, then we can define the length of $\gamma(s)$ as follows:

$$L(\gamma) = \int_a^b g_{\gamma(s)}(\frac{d\gamma}{ds}, \frac{d\gamma}{ds})ds \tag{17}$$

From theorem 8 it follows that we can define a distance d on M:

$$d(p, q) = \inf_{\gamma \in P_a(p,q)} L(\gamma). \tag{18}$$

This distance has some nice properties like the distance defined by a Riemannian metric. For example, we have the following theorem.

Theorem 9. *The topology defined by the distance d coincides with the original topology on M.*

A curve $\gamma(s)$, $s \in [a, b]$, is called a minimizing geodesic if $L(\gamma) = d(\gamma(a), \gamma(b))$.

Theorem 10 (Hopf-Rinow theorem for sub-Riemannian manifolds)**.** a) *Sufficiently near points can be joined by a minimizing geodesic;*

b) *If (M, d) is a complete metric space, then any two points can be joined by a minimizing geodesic.*

4. References

The detailed exposition of results on geometry of non-holonomic surfaces in the three-dimensional Euclidean space (stuff of section 2) can be found in the very interesting and profound book by Yu. Aminov:

Yu. Aminov: *The geometry of vector fields.* Amsterdam: Gordon and Breach Publishers. (2000).

For the introduction to the sub-Riemannian geometry (stuff of section 3), we can recommend the book:

R. Montgomery *A Tour of Subriemannian geometries, Their Geodesics and Applications* , Mathematical Surveys and Monographs, Vol. 91, AMS, 2002.