LECTURES ON DEFORMATIONS OF G-STRUCTURES

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1. Pseudogroups. Sheaves

1.1. **Pseudogroups.** Recall that a *transformation group* G is a subgroup in a group $Diff(M)$, so each $g \in G$ is a diffeomorphism of M.

A *local diffeomorphism* is a diffeomorphism $f : U \to V$, where U and V are open subsets in M. We set $D(f) = U$, $E(f) = V$. Certainly, the set of local diffeomorphisms is not a group of transformations because the composition is not well-defined.

A generalization of the notion of transformation group is a *pseudogroup*.

Definition 1. A *pseudogroup* Γ of transformations of a smooth manifold M is a collection of local diffeomorphisms such that

- For any $f \in \Gamma$ and open $U' \subset D(f)$, we have $f|_{U'} \in \Gamma$;
- Let $U = \cup U_i$, where U_i are open. Let local diffeomorphism $f : U \to V$ be such that $f|_{U_i} \in \Gamma$, then $f \in \Gamma$.
- For any $f \in \Gamma$, we have $f^{-1} \in \Gamma$.
- For $f, g \in \Gamma$ such that $E(f) \subset D(g)$ we have $g \circ f \in \Gamma$;

Example 1.1. All local diffeomorphisms of a manifold.

Example 1.2. Pseudogroup of automorphisms of a tensor field: a) pseudogroup of analytical diffeomorphisms b) pseudogroup of symplectic diffeomorphisms.

Example 1.3. The pseudogroup $\Gamma(G)$ generated by a group G of transformations of a manifold M. The elements of $\Gamma(G)$ are the restrictions of elements of G to open sets in M.

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We can say informally that a subset Γ of local diffeomorphism of M is a pseudogroup if and only if (i) the composition and inverse elements are well-defined, (ii) the elements of Γ are defined by a local condition (in fact, by a condition on germs).

1.2. **Sheaves.** Here we give only main definitions and results of the sheaf theory. For a detailed exposition see [11], [12].

Definition 2. Let M be a topological space. Suppose that to each open U we assign a group $\mathcal{F}(U)$ and if $V \subset U$ we have group homomorphism $r^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ such that

$$r^U_V r^V_W = r^U_W \tag{1}$$

$$r^U_U = \mathrm{Id} \tag{2}$$

Then we get a *presheaf* $\mathcal{F}$ of groups.

In the same way we can define presheaves of rings, vector spaces, etc.

We will call the elements of $\mathcal{A}(U)$ *the sections of* $\mathcal{A}$ *over* U.

Example 1.4.

- The presheaf of smooth sections of a smooth bundle ξ: to each open set U we put in correspondence the set of sections of ξ over U, and the map r^U_V is the restriction map.
- The presheaf of constant functions on a manifold: to each open set U we put in correspondence the set of constant functions on U, and the map r^U_V is again the restriction map.

Let $\mathcal{A}$, $\mathcal{B}$ be preasheaves. A presheaf morphism $\phi : \mathcal{A} \to \mathcal{B}$ is the collection of homomorphisms $\phi_U : \mathcal{A}(U) \to \mathcal{B}(U)$ given for each open U such that, for every $V \subset U$, we have $r^U_V \phi_U = \phi_V r^U_V$.

If $\mathcal{A}$, $\mathcal{B}$ are preasheaves of abelian groups, then we can define the kernel and the image of ϕ: the kernel of ϕ is the presheaf given by

$$U \to \mathrm{Ker}(\phi : \mathcal{A}(U) \to \mathcal{B}(U)),$$

and the image of ϕ is the presheaf given by

$$U \to \mathrm{Im}(\phi : \mathcal{A}(U) \to \mathcal{B}(U)).$$

Definition 3. A presheaf $\mathcal{A}$ is called a *sheaf* if

(1) If $U = \cup U_i$ and $s, t \in \mathcal{A}(U)$ and $r^U_{U_i}(s) = r^U_{U_i}(t)$, then $s = t$;

(2) Let $U = \cup U_i$. Then for any collection $s_i \in \mathcal{A}(U_i)$ such that $r^{U_i}_{U_i \cap U_j}(s_i) = r^{U_j}_{U_i \cap U_j}(s_j)$, there exists a (unique) $s \in \mathcal{A}(U)$ such that $r^U_{U_i}(s) = s_i$.

Example 1.5.

(1) The presheaf of local sections of a bundle is a sheaf.
(2) The presheaf of constant functions on a manifold is not a sheaf.

1.2.1. *Passing from a presheaf to a sheaf.* Let $\mathcal{A}$ be a presheaf on a manifold M. Then, for each $x \in M$, we can define a germ of a section $s \in \mathcal{A}(U)$ in the standard way. Namely, we consider the set of pairs $(W, s \in \mathcal{A}(W))$ such that $x \in W$ and take the equivalence relation on this set defined as follows: $(U, s \in \mathcal{A}(U)) \sim (V, t \in \mathcal{A}(V))$ if and only if there exists $x \ni W \subset U \cap V$ such that $r^U_W(s) = r^V_W(t)$. Denote by $\mathcal{A}_x$ the set of germs of sections of $\mathcal{A}$ at x.

Now we consider the set $E(\mathcal{A}) = \cup_{x \in M} \mathcal{A}_x$ and introduce a topology on $E(\mathcal{A})$ in the following way. The idea is to set up a minimal topology such that the sections of the presheaf are continuous maps. The prebase of the topology on $E(\mathcal{A})$ consists of sets

$$\Omega = \{< s >_x |\ s \in \mathcal{A}(U)\} \tag{3}$$

Note that this topology has a lot of open sets, and, in general, is not Hausdorff.

We have the natural projection $\pi : E(\mathcal{A}) \to M$, $< s >_x \to x$ and thus we obtain a covering $(E(\mathcal{A}), \pi, M)$, in the sense that for each point $y \in E(\mathcal{A})$ there exists a neighborhood in $E(\mathcal{A})$ homeomorphic to a neighborhood of $x = \pi(y)$. Then the sheaf $\widehat{\mathcal{A}}$ associated with $\mathcal{A}$ is the sheaf of continuous sections of $\pi : E(\mathcal{A}) \to M$. We can describe the sections of this sheaf in the following way: a section $s : U \to E(\mathcal{A})$, $x \to < \sigma_x >_x$ is continuous if for each $x \in U$ there exist a neighborhood $x \ni W \subset U$ and a section $\sigma \in \mathcal{A}(U)$ such that $< s_x >_x = < \sigma >_x$.

Example 1.6. The sheaf associated with the presheaf of constant functions is the sheaf of locally constant functions.

Let $\mathcal{A}$ and $\mathcal{B}$ be sheaves of abelian groups. If $\phi : \mathcal{A} \to \mathcal{B}$ is a sheaf morphism, then we have the image presheaf $U \to \phi(\mathcal{A}(U))$ which is not a sheaf in general. However we denote by $\mathrm{Im}\phi$ the sheaf associated with the image sheaf.

Also, for sheaves $\mathcal{B} \subset \mathcal{A}$ we define the presheaf $U \to \mathcal{A}(U)/\mathcal{B}(U)$, and we denote by $\mathcal{A}/\mathcal{B}$ the sheaf associated to this presheaf.

1.2.2. *Sheaf cohomology.* Let $\mathcal{G}$ be a sheaf of abelian groups on a manifold M. Take at most countable open covering $\mathcal{U} = \{U_i\}$ of M. Define the *nerve* of $\mathcal{U}$ be $N(\mathcal{U}) = \sqcup N^m(\mathcal{U})$, where

$$N^m(\mathcal{U}) = \{\{i_0 \dots i_m\} \mid U_{i_0} \cap U_{i_2} \cap \dots U_{i_m} \neq \emptyset\} \tag{4}$$

Now an *m-cochain* on M is a family of sections $\{c_{i_0 \dots i_m} \in \mathcal{G}(U_{i_0} \cap U_{i_2} \cap \dots U_{i_m}) \mid \{i_0 \dots i_m\} \in N^m(\mathcal{U})\}$. The order of indices is unessential, so we assume that $c_{i_0 i_1 \dots i_m}$ is skew-symmetric with respect to i_0, i_1, ..., i_m.

The set $\check{C}^m(\mathcal{U};\mathcal{G})$ of m-cochains is evidently an abelian group. We have the group homomorphism:

$$d : \check{C}^m(\mathcal{U};\mathcal{G}) \to \check{C}^{m+1}(\mathcal{U};\mathcal{G}) \tag{5}$$

$$(dc)_{i_0 \dots i_{m+1}} = \sum_{k=0}^{m+1} (-1)^k r^{U_{i_0 \dots \widehat{i_k} \dots i_{m+1}}}_{U_{i_0 \dots i_{m+1}}} (c_{i_0 \dots \widehat{i_k} \dots i_{m+1}})$$

For example,

$$d : C^0(\mathcal{U};\mathcal{G}) \to C^1(\mathcal{U};\mathcal{G}), \qquad dc_{\beta\alpha} = c_\beta - c_\alpha \tag{6}$$

$$d : C^1(\mathcal{U};\mathcal{G}) \to C^2(\mathcal{U};\mathcal{G}), \qquad dc_{\alpha\beta\gamma} = c_{\beta\gamma} - c_{\alpha\gamma} + c_{\alpha\beta} \tag{7}$$

One can prove that $d \circ d = 0$ by direct calculations, hence we obtain a complex $(\check{C}^*(\mathcal{U};\mathcal{G}), d)$ called the Cech complex of the covering $\mathcal{U}$ with coefficients in the sheaf $\mathcal{G}$. Certainly, the cohomology of this complex depends on the covering.

Example 1.7. $\check{H}^0(\mathcal{U};\mathcal{G}) = \mathcal{G}(M)$.

Now the set of open coverings is a partially ordered set with respect to the inscribing relation $\leq$, and, if $\mathcal{V} \leq \mathcal{U}$, we have the natural map $\check{H}(\mathcal{U};\mathcal{G}) \to \check{H}(\mathcal{V};\mathcal{G})$. Thus we can define the Cech cohomology of M with coefficients in $\mathcal{G}$:

$$\check{H}(M;\mathcal{G}) = \varinjlim \check{H}(\mathcal{U};\mathcal{G}) \tag{8}$$

The following theorems make it possible to calculate the Cech cohomology without taking the direct limit.

Let $\mathcal{G}$ be a sheaf of abelian groups on M. An open covering is said to be 'fine' if $\check{H}^q(U_I;\mathcal{G}) \cong 0$, $q > 0$, for each $I \in N(\mathcal{U})$.

Theorem 1 (Leray)**.** *If $\mathcal{U}$ is a fine covering on M, then $\check{H}(M;\mathcal{G}) \cong \check{H}(\mathcal{U};\mathcal{G})$.*

An exact sequence of sheaves

$$0 \to \mathcal{G} \xrightarrow{i} \mathcal{F}_0 \xrightarrow{d} \mathcal{F}_1 \xrightarrow{d} \dots \tag{9}$$

is called a *resolution* of $\mathcal{G}$. If all the sheaves $\mathcal{F}$ are fine, then we say that (9) is a fine resolution of $\mathcal{G}$. From the fine resolution 9 we get the complex $(F_k(M), d)$.

Theorem 2 (Abstract de Rham Theorem)**.** *Given a fine resolution $\mathcal{F}_k$ of a sheaf $\mathcal{G}$, we have $\check{H}(M;\mathcal{G}) \cong H(\mathcal{F}_*(M), d)$.*

Example 1.8. The sheaf $\mathbb{R}_M$ of locally constant functions on a manifold M admits the fine resolution

$$0 \to \mathbb{R}_M \to \Omega^0_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \dots \xrightarrow{d} \Omega^n_M \to 0,$$

where Ω^q_M is the sheaf of differential q-forms on M, therefore the cohomology of $H(M;\mathbb{R}_M)$ with coefficients in the sheaf of locally constant functions is isomorphic to the de Rham cohomology.

Theorem 3. *If $0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{p} \mathcal{C} \to 0$ is a exact sequence of sheaves, we have a long exact cohomology sequence*

(10)

$$\dots \to \check{H}^{m-1}(M;\mathcal{C}) \xrightarrow{\delta} \check{H}^m(M;\mathcal{A}) \xrightarrow{i_*} \check{H}^m(M;\mathcal{B}) \xrightarrow{p_*} \check{H}^m(M;\mathcal{C}) \xrightarrow{\delta} \check{H}^{m+1}(M;\mathcal{A}) \to \dots$$

2. Deformations of pseudogroup structures associated with integrable first order G-structures

We will apply deformation theory to the integrable first order G-structures (for the general theory of G-structures we refer the reader to e. g. [14], [15], [13]).

2.1. **Integrable G-structures.** Let M be a smooth n-dimensional manifold, $L(M) \to M$ the frame bundle of M, and $\mathfrak{X}(M)$ the Lie algebra of vector fields on M. For a Lie subgroup $G \subset GL(n)$, consider the bundle $E_G(M) = L(M)/G \to M$.

Remark 2.1. E_G is a natural bundle: this means that E_G is the functor $\mathcal{M}an \to \mathcal{B}undles$, where $\mathcal{M}an$ is the category of smooth manifolds whose morphisms are diffeomorphisms, and $\mathcal{B}undles$ is the category of fiber bundles. This functor sends a manifold M to the fiber bundle $E_G(M)$, and a diffeomorphism $f : M \to M'$ to the natural bundle morphism $f^c : E_G(M) \to E_G(M')$.

Definition 4. *A first order G-structure* is a section $s\colon M \to E_G(M)$.

Example 2.1 (**C**). *An almost complex structure* is a section $s\colon M \to E_G(M)$, where $G = \mathrm{GL}(n,\mathbb{C}) \subset \mathrm{GL}(2n,\mathbb{R})$. One can easily prove that the sections of $E_G(M)$ are in 1-1-correspondence with the linear operator fields J such that $J^2 = -I$.

Example 2.2 (**F**). *A distribution* of codimension q can be identified with a section $s\colon M \to E_G(M)$, where $G = \{\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}\} \subset \mathrm{GL}(n,\mathbb{R})$, where $A \in GL(q,q)$, $B \in Mat(n-q,q)$, $C \in GL(n-q,n-q)$.

Example 2.3 (**S**). *An almost symplectic structure* is a section of $s\colon M \to E_G(M)$, where $G = \mathrm{Sp}(n) \subset \mathrm{GL}(2n, \mathbb{R})$. One can easily prove that the sections of $E_G(M)$ are in 1-1-correspondence with non-degenerate differential 2-forms on M.

Definition 5. We say that a *first order G-structure s is integrable* iff s is locally equivalent to a given first order G-structure s_0 on $\mathbb{R}^n$.

Example 2.4 (**C**). *A complex structure* is a section $s\colon M \to E_G(M)$ locally equivalent to $J \in T^1_1(\mathbb{R}^n)$ such that $J\partial_k = \partial_{n+k}$, $J\partial_{n+k} = -\partial_k$, $k = \overline{1, n}$.

Example 2.5 (**F**). *A foliation* of codimension q is a section $s\colon M \to E_G(M)$ locally equivalent to the distribution Δ given by equations $dx^1 = 0$, $dx^2 = 0$, $\dots$, $dx^q = 0$.

Example 2.6 (**S**). *A symplectic structure* is a section $s\colon M \to E_G(M)$ which is locally equivalent to $\omega_0 = dx^1 \wedge dx^2 + \dots + dx^{2n-1} \wedge dx^{2n}$.

Remark 2.2. For all the classical structures, like the complex structure, the foliation structure, etc., the section s_0 is invariant under the translation group, so s_0 is determined by its value at a point $0 \in \mathbb{R}^n$.

2.2. **Γ-structure associated with integrable G-structure.** Let Γ_0 be a pseudogroup of transformations of $\mathbb{R}^n$. A Γ_0*-structure on a manifold M* is a maximal atlas $\{(U_\alpha, \phi_\alpha)\}$ on M whose transition functions $\phi_\beta\phi_\alpha^{-1}$ lie in Γ_0.

Given an integrable G-structure $s : M \to L(M)/G$, and so the model section $s_0 : \mathbb{R}^n \to L(\mathbb{R}^n)/G$, we obtain the pseudogroup $\Gamma_0 = \Gamma(s_0)$ of automorphisms of s_0. For classical G-structures, this pseudogroup is transitive (see the remark above).

It is clear that a *manifold M is endowed with an integrable G-structure if and only if M is endowed with $\Gamma(s_0)$-structure.*

For a manifold M endowed with a Γ_0-structure $\{(U_\alpha, \phi_\alpha)\}$, we have the pseudogroup Γ of diffeomorphisms of $f : M \to M$ such that the local representations $\phi_\beta f \phi_\alpha^{-1}$ lie in Γ_0. In case the Γ_0-structure corresponds to an integrable G-structure $s : M \to L(M)/G$, the pseudogroup Γ consists of local automorphims of s: $f \in \Gamma$ if and only if $f^*(s) = s$.

A vector field X is called a Γ*-vector field* if the flow ϕ_t consists of elements of Γ. On each manifold M endowed with a Γ_0-structure we have the sheaf $\mathfrak{X}_\Gamma$ of Γ-vector fields. In case the Γ_0-structure corresponds to an integrable G-structure $s : M \to L(M)/G$, a vector field X lies in $\mathfrak{X}_\Gamma(U)$ if and only if $\mathcal{L}_X s = 0$.

Now we continue considering the above examples and present the corresponding pseudogroups Γ_0, as well as the sheaves of Γ_0-vector fields.

Example 2.7 (**C**). *A complex structure.* Γ_0 consists of local diffeomorphisms $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ which satisfy the Cauchy-Riemann equations, i.e. *df* commutes with J_0, where

$$J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

The sheaf $\mathfrak{X}_\Gamma$ of Γ-vector fields on M consists of holomorphic vectors fields of (M, J).

Example 2.8 (**F**). *A foliation* of codimension q. Let (x^a, x^α), $a = \overline{1, q}$, $\alpha = \overline{q+1, n}$, be coordinates on $\mathbb{R}^n$. The pseudogroup Γ_0 consists of local diffeomorphisms $f : \mathbb{R}^n \to \mathbb{R}^n$ whose germs are of the form $y^a = f^a(x^b)$, $y^\alpha = f^\alpha(x^b, x^\beta)$. In fact, a local diffeomorphism lies in Γ if and only if $\frac{\partial f^a}{\partial x^\alpha} = 0$. And the sheaf $\mathfrak{X}_\Gamma$ consists of vector fields V with local expression

$$V = V^a(x^b)\partial_a + V^\alpha(x^b, x^\beta)\partial_\alpha$$

Example 2.9 (**S**). *A symplectic structure.* A local diffeomorphism lies in Γ_0 if and only if $f^*\omega_0 = \omega_0$. The sheaf $\mathfrak{X}_\Gamma$ consists of vector fields V which are locally Hamiltonian vector fields: $V \in \mathfrak{X}(U)$ if and only if $\mathcal{L}_V\omega = 0$ if and only if $di_V\omega = 0$ if and only if V has local expression $V^k = \omega^{ks}\partial_k f$.

2.3. **Deformations of pseudogroup structure.** Let us recall definitions and results of the theory of deformations of pseudogroup structures (see [5]–[7] and also [8], [9]).

Let Γ be a pseudogroup of local diffeomorphisms of a manifold M.

Definition 6. A *one-parameter family ϕ_s of elements of* Γ *is said to be smooth* if $U = \{(x, s) \in M \times \mathbb{R} \mid x \in \mathcal{D}(\phi_s)\}$, where $\mathcal{D}(g)$ is the domain of $g \in \Gamma$, is open in $M \times \mathbb{R}$ and $\Phi : U \to \mathbb{R}^n$, $\Phi(x, s) = \phi_s(x)$, is smooth. *A smooth one-parameter family of elements of* Γ *will be called a curve in* Γ.

Let ϕ_s, $|s| < \varepsilon$, be a curve in Γ such that ϕ_0 is the identity map of an open set $U \subset M$. Then, on U we have the vector field $V(x) = \frac{d}{ds}\big|_{s=0}\phi_s(x)$, which is called *the vector field tangent to ϕ_s at $s = 0$*. Let us recall that a vector field V on an open set $U \subset M$ is called a Γ-vector field if its flow consists of elements of Γ.

Definition 7. If any vector field tangent to a curve in Γ is a Γ-vector field, then *the pseudogroup is said to have the Lie pseudoalgebra* [5]. If a pseudogroup Γ has Lie pseudoalgebra, then the set $\mathfrak{X}_\Gamma(U)$ of Γ-vector fields on U is a Lie subalgebra in the Lie algebra $\mathfrak{X}(U)$ of vector fields on U, and the correspondence $U \to \mathfrak{X}_\Gamma(U)$ determines a sheaf $\mathfrak{X}_\Gamma$ of Lie algebras on M.

Remark 2.3. Let Γ be the pseudogroup associated with an integrable first order structure $s : M \to L(M)/G$. Then Γ has the Lie pseudoalgebra, and this pseudoalgebra coincides with $\mathfrak{X}_\Gamma$. The reason is that in the definition of the Lie derivative: $\mathcal{L}_X s = \frac{d}{dt}|_{t=0}\phi_t^*(s)$ one can take any smooth curve of diffeomorphisms tangent to X.

A *deformation of a* Γ_0*-atlas* $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ is a family of Γ_0-atlases $\mathcal{A}(s) = \{(U_\alpha, \phi_\alpha(s))\}$, where $|s| < \varepsilon$, such that

1) $U_\alpha \times (-\varepsilon, \varepsilon) \to \mathbb{R}^n$, $(x, s) \to \phi_\alpha(s)(x)$ is smooth for each α;

2) $\phi_{\beta\alpha}(s) = \phi_\beta(s)\phi_\alpha^{-1}(s)$ is a curve in Γ_0 for every α, β;

3) $\mathcal{A}(0) = \mathcal{A}$.

A deformation $\mathcal{A}(s)$ of a Γ_0-atlas $\mathcal{A}$ is said to be *nonessential* if $\phi_\alpha(s)\phi_\alpha^{-1}$ lie in Γ_0, i. e. for every s, $\mathcal{A}(s)$ is compatible with the maximal Γ_0-atlas determined by $\mathcal{A}$.

Let $\mathcal{A}(s) = \{(U_\alpha, \phi_\alpha(s))\}$ be a deformation of a Γ_0-atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$. Then $\gamma_{\beta\alpha}(s) = \phi_\beta^{-1}\phi_{\beta\alpha}(s)\phi_\alpha$ is a curve in Γ and $\gamma_{\beta\alpha}(0) = \mathrm{Id}$.

$\frac{d}{ds}\big|_{s=0}\phi_{\beta\alpha}\phi_{\alpha\beta}(s)$ is a Γ_0-vector field on $\phi_{\beta\alpha}(\phi_\alpha(U_\alpha \cap U_\beta)) \subset \phi_\beta(U_\beta)$.

We define the vector field $V_{\beta\alpha}$ on $U_\alpha \cap U_\beta$,

$$V_{\beta\alpha}(p) = d\phi_\beta^{-1}\left(\frac{d}{ds}\Big|_{s=0}\phi_{\beta\alpha}(s)(\phi_\alpha(p))\right), \tag{11}$$

and the vector field W_α on U_α,

$$W_\alpha(p) = d\phi_\alpha^{-1}\left(\frac{d}{ds}\Big|_{s=0}\phi_\alpha(s)(p)\right). \tag{12}$$

Statement 1. 1) $V_{\beta\alpha} = W_\beta|_{U_\alpha \cap U_\beta} - W_\alpha|_{U_\alpha \cap U_\beta}$;

2) $\{V_{\beta\alpha}\}$ *is a* 1*-cocycle on the covering* $\mathcal{U} = \{U_\alpha\}$ *with coefficients in* $\mathfrak{X}_\Gamma$;

3) *if the deformation* $\mathcal{A}(s)$ *is nonessential, the cohomology class* $[V_{\beta\alpha}] \in H^1(M; \mathfrak{X}_\Gamma)$ *is trivial.*

The cocycle $\{V_{\beta\alpha}\}$ is called an *infinitesimal deformation of the* Γ_0*-structure*. The class $[V_{\beta\alpha}] \in H^1(M; \mathfrak{X}_\Gamma)$ is called an *essential infinitesimal deformation* of the Γ_0-structure.

In general, not every infinitesimal deformation is generated by a deformation of Γ_0-structure. In [9] it is proved that if an essential infinitesimal deformation represented by a cocycle $\{V_{\beta\alpha}\}$ is generated by a deformation of Γ_0-structure, then the cocycle $\{W_{\alpha\beta\gamma} = [V_{\alpha\beta}, V_{\beta\gamma}]\}$ of Γ-vector fields, where [,] is the Lie bracket of vector fields, represents the trivial cohomology class in $H^2(M; \mathfrak{X}_\Gamma)$. The cohomology class $[W_{\alpha\beta\gamma}] \in H^2(M, \mathfrak{X}_\Gamma)$ is called the *obstruction to integrability of the infinitesimal deformation* $\{V_{\beta\alpha}\}$.

3. Fine resolution for the sheaf of Γ-vector fields.

We know that the space of infinitesimal essential deformations of a Γ-structure is $H^1(M;\mathfrak{X}_\Gamma)$, which is defined in terms of the Čech complex. However, for calculations and for establishing relations with other geometrical structures we often need to have a differential complex whose cohomology is isomorphic to $H^*(M;\mathfrak{X}_\Gamma)$. Using the Spencer P-complex, we will construct such a complex for integrable first order G-structures.

3.1. **Lie derivative of a section of a natural bundle.** Let M be a manifold and $E : \mathcal{M}an \to \mathcal{B}undles$ be a natural bundle. Let us take a section $s : M \to E(M)$ and a vector field $X \in \mathfrak{X}(M)$. Denote by ϕ_t the flow of X, and since the bundle $E(M)$ is natural, we have the one-parameter group $\phi_t^c : E(M) \to E(M)$, and hence the complete lift $X^c \in \mathfrak{X}(E(M))$. Now denote by $VE(M) \to E(M)$ the vertical bundle of $E(M)$, then the pullback bundle $s^*(VE(M))$ is a vector bundle over M, and for each $p \in M$ there is defined the isomorphism $\Pi_p\colon (VE(M))_{s(p)} \to (s^*(VE(M))_p$. Then

$$(\mathcal{L}_X s)_p \overset{def}{=} \Pi(X^c(s(p)) - ds_p(X(p))),$$

is the Lie derivative of s with respect to X at a point $p \in M$.

From the definition it immediately follows that $\mathcal{L}_X s = 0$ if and only if X is an infinitesimal automorphism of s, i.e., if the flow ϕ_t^c preserves s.

Now let $s\colon M \to E_G(M)$ be an integrable first order structure. We consider the Lie derivative $\mathcal{L}_X s$ as a first-order differential operator $\mathcal{D}_{\mathcal{L}}\colon TM \to s^*(VE)$. In [1], for a differential operator $\mathcal{D}\colon \Gamma(\xi) \to \xi'$, where ξ, ξ' are vector bundles and $\Gamma(\xi)$ is the sheaf of sections of ξ, a differential complex was constructed (the Spencer P-complex)

$$0 \to \Theta \to \Gamma(\xi) \xrightarrow{\mathcal{D}} F^0 \to F^1 \to \dots,$$

where Θ is the kernel of $\mathcal{D}$. This construction, applied to a Lie derivative associated with a general pseudogroup structure, gives the deformation complex of this structure [1]. Then, specializing this complex to the case of the pseudogroup associated to an integrable G-structure, we obtain the fine resolution for the sheaf $\mathfrak{X}_\Gamma$ (for details, see [29]). In the next subsection we present this complex in terms of tensor fields and covariant derivatives.

3.2. ***P*-complex for the Lie derivative.** Let $s : M \to E_G(M) = L(M)/G$ be an integrable first order G-structure. Let us denote by $E_\mathfrak{g}$ the subbundle of the bundle $T_1^1(M)$ consisting of linear operators whose matrices written with respect to the corresponding Γ_0-atlas lie in the Lie algebra $\mathfrak{g}$ of G, and let $F_\mathfrak{g} = T_1^1(M)/E_\mathfrak{g}$.

The P-complex of the Lie derivative is isomorphic to the complex $(C^q(P), d)$, where

$$C^q(P) = \frac{\Omega^k(M) \otimes TM}{\mathrm{Alt}(\Omega^{k-1}(M) \otimes E_{\mathfrak{g}})},$$

where $\mathrm{Alt}(t^j_{i_1 \dots t_{k-1} t_k}) = t^j_{[i_1 \dots t_{k-1} t_k]}$ and the differential $d : C^q(P) \to C^{q+1}(P)$ is induced by the differential operator $D = \mathrm{Alt} \circ \nabla$, where Alt is the alternation and ∇ is the covariant derivative of a torsion-free connection adapted to the G-structure s: $\nabla s = 0$. This means that, with respect to local coordinates adapted to s (the charts of the Γ_0-atlas), we have $(D\omega)_{i_1 \dots i_{q+1}}{}^j = \nabla_{[i_1} \omega_{i_2 \dots i_{q+1}]}{}^j$.

The definition of d does not depend on an adapted connection choice because, if $\nabla(s) = \overline{\nabla}(s) = 0$, we have $T^k_{ij} = \Gamma^k_{ij} - \overline{\Gamma}^k_{ij}$ in $E_{\mathfrak{g}} \otimes T^*M$, so $\mathrm{Alt}(\nabla\omega) - \mathrm{Alt}(\overline{\nabla}\omega)$ lies in $\mathrm{Alt}(\Omega^{k-1}(M) \otimes E_{\mathfrak{g}})$.

Example 3.1 (**C**). For a complex structure, the complex (C^q, d) is the Dolbeaut complex of vector-valued forms (see, e.g., [25]).

Example 3.2 (**F**). Let a foliation structure on a smooth manifold M be given by the integrable distribution Δ. Then $E_{\mathfrak{g}} = \{A \in T^1_1(M) \mid A(\Delta) \subset \Delta\}$. Therefore $C^p = \Omega^p(\Delta) \otimes (TM/\Delta)$. If (x^i, x^α) are adapted local coordinates, i.e., if Δ is given by the equations $dx^i = 0$, then d can be written locally as $(d\omega)_{\alpha_1 \dots \alpha_{q+1}}{}^i = \partial_{[\alpha_1} \omega_{\alpha_2 \dots \alpha_{q+1}]}{}^j$. Thus we arrive at Vaisman's foliated cohomology [2].

Example 3.3 (**S**). Let us consider a symplectic manifold (M, θ). Then the subbundle $E_{\mathfrak{g}}$ consists of those linear operators that are skew-symmetric with respect to θ: $\theta(AX, Y) + \theta(X, AY) = 0$, and using the isomorphism $T^1_1(M) \to T^2(M)$ determined by θ, we obtain that $C^q \cong \Omega^{q+1}(M)$ and . The adapted connection ∇ satisfies $\nabla\omega = 0$ (a symplectic connection), and the differential $d \colon C^q \to C^{q+1}$ is the exterior differential. Thus we find that the kernel of d is the Lie algebra of (locally) Hamiltonian vector fields.

3.3. **Fine resolution for $\mathfrak{X}_\Gamma$.** It is clear that the construction of (C^*, d) transforms to the sheaf level. So we have the sequence of sheaves

$$0 \to \mathfrak{X}_\Gamma \xrightarrow{i} \mathfrak{X} \xrightarrow{\mathcal{L}(s)} C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} \dots \tag{13}$$

From the construction, it follows that these sheaves are fine, since these ones are sheaves of modules over the fine sheaf C^∞.

So we arrive at

Theorem 4. *If the sheaf sequence (13) is exact (the "Poincare lemma" holds), then the space of essential deformations $H^1(M; \mathfrak{X}_\Gamma)$ is isomorphic to $H^1(C^*(M), d)$.*

4. Deformation of symplectic structure with Martinet singularities

4.1. **Symplectic structure with singularities.** Let ω_0 be a closed differential 2-form on $\mathbb{R}^{2n}$ such that $\Sigma_0 = \{x \in \mathbb{R}^{2n} \mid \det \omega_0(x) = 0\}$ is an embedded manifold, and Γ_0 be the pseudogroup consisting of local diffeomorphisms f of $\mathbb{R}^{2n}$ such that $f^*\omega_0 = \omega_0$.

A symplectic structure with singularities of type ω_0 on a $2n$-dimensional manifold M can be defined in two *equivalent* ways.

Definition 8. *A symplectic structure with singularities of type* ω_0 on a $2n$-dimensional manifold M is a Γ_0-structure on M, i. e. a maximal atlas whose transition functions lie in Γ_0.

Definition 9. A 2-form ω on a $2n$-dimensional smooth manifold M is called a *symplectic structure with singularities of type* ω_0 if for any point $p \in M$ there exist a neighborhood $U(p)$ and a diffeomorphism $\phi_p : U(p) \to \mathcal{O} \subset \mathbb{R}^{2n}$, where $\mathcal{O}$ is an open subset in $\mathbb{R}^{2n}$, such that $\omega|_{U(p)} = \phi_p^*(\omega_0)$.

The singular submanifold of a symplectic structure with singularities. Let a 2-form ω be a symplectic structure with singularities of type ω_0 on a $2n$-dimensional manifold M.

Lemma 1. *The set* $\Sigma = \{p \in M \mid \det \omega(p) = 0\}$ *is an embedded closed submanifold in* M*, and* $\dim \Sigma = \dim \Sigma_0$.

The submanifold Σ will be called *the singular submanifold* of a symplectic structure with singularities.

Lemma 2. *Let* ω *be a symplectic structure with singularities of type* ω_0 *on a manifold* M*, and* $U \subset M$ *be an open set. For* $V, W \in \mathfrak{X}(U)$*, from* $i_V\omega = i_W\omega$ *it follows that* $V = W$.

4.2. **Infinitesimal deformations of symplectic structure with singularities.** We have defined the symplectic structure with singularities to be a pseudogroup structure (Definition 8), and in the equivalent way, to be a closed 2-form (Definition 9). Therefore, there are defined two spaces of essential infinitesimal deformations of symplectic structure with singularities. The first one is the space of essential infinitesimal deformations of pseudogroup structure, which is isomorphic to the Čech cohomology group $\check{H}^1(M; \mathfrak{X}_\Gamma)$ with coefficients in the sheaf of Γ-vector fields, where Γ is the pseudogroup of automorphims of symplectic structures with singularities. The second one is the space $\mathcal{D}_{\text{ess}}(\omega)$ of essential infinitesimal deformations of the closed 2-form ω. In this section, for a compact manifold M we construct a homomorphism $\phi : \check{H}^1(M; \mathfrak{X}_\Gamma) \to \mathcal{D}_{\text{ess}}(\omega)$ and study its properties.

In what follows we assume that M is compact.

4.2.1. *Infinitesimal deformations of a closed differential form.* In 2.3 we have defined the deformations of pseudogroup structure and now we define the deformations of a closed differential form. Let η be a closed differential form on a manifold M. A *deformation* of η is a smooth family of closed forms $\eta(s)$, $|s| < \varepsilon$, where $\eta(0) = \eta$. Let $\phi(s)$ be a flow of local diffeomorphisms of M. Then the family of forms $\eta(s) = \phi(s)^*(\eta)$ is called a *nonessential deformation* of η. A closed form θ is called a (nonessential) infinitesimal deformation of η, if there exists a (nonessential) deformation $\eta(s)$ of η such that $\theta = \frac{d}{ds}\big|_{s=0}\eta(s)$. Note that a closed form θ is a nonessential infinitesimal deformation of η if and only if there exists a vector field $V \in \mathfrak{X}(M)$ such that $\theta = L_V\eta = d(i_V\eta)$.

For any closed p-form θ, the family $\eta(s) = \eta + s\theta$ is a family of closed forms, and $\theta = \frac{d}{ds}\big|_{s=0}\eta(s)$, therefore the set of infinitesimal deformations $\mathcal{D}(\eta)$ of $\eta \in \Omega^p(M)$ coincides with the space of closed p-forms. The set of nonessential infinitesimal deformations is the subspace $\mathcal{D}_0(\eta) \subset \mathcal{D}(\eta)$ consisting of exact p-forms $d(i_V\eta)$, where $V \in \mathfrak{X}(M)$. Then the quotient space $\mathcal{D}_{\text{ess}}(\eta) = \mathcal{D}(\eta)/\mathcal{D}_0(\eta)$ is called the *space of essential infinitesimal deformations of the closed form* η.

From the definition of the space of essential infinitesimal deformations it follows that for any closed p-form η we have the surjective homomorphism $\psi : \mathcal{D}_{\text{ess}}(\eta) \to H^p(M)$ which maps each essential infinitesimal deformation of η to the corresponding de Rham cohomology class.

4.3. **Infinitesimal deformations of symplectic structure with singularities.** Let us establish relationship between infinitesimal deformations of symplectic structure with singularities of type ω_0 considered as a pseudogroup structure (see 2.3) and infinitesimal deformations of the corresponding closed 2-form ω (see 4.2.1).

Let a symplectic structure with singularities of type ω_0 be given on a manifold M by a Γ_0-atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$. Let $\omega \in \Omega^2(M)$ be the corresponding closed 2-form (see Definitions 8 and 9). Let us denote by Γ the pseudogroup consisting of local diffeomorphisms f of M such that $f^*\omega = \omega$. Hence follows that a vector field V is a section of the sheaf $\mathfrak{X}_\Gamma$ of Γ-vector fields if and only if $L_V\omega = 0$.

Each deformation $\mathcal{A}(s) = \{(U_\alpha, \phi_\alpha(s))\}$ of the Γ_0-atlas determines a deformation $\omega(s)$ of the form ω. Namely, the family of closed 2-forms $\omega(s)$ is given by $\omega(s)|_{U_\alpha} = \phi_\alpha^*(s)\omega_0$. Let $\eta = \frac{d}{ds}\big|_{s=0}\omega(s)$ be the corresponding infinitesimal deformation of ω, then $\eta|_{U_\alpha} = L_{W_\alpha}\omega$, where the vector field $W_\alpha \in \mathfrak{X}(U_\alpha)$ is defined by (12). And $V_{\alpha\beta} = W_\alpha - W_\beta$ is an infinitesimal deformation of $\mathcal{A}$ determined by the deformation of $\mathcal{A}(s)$ (see Statement 1).

Now we define a map ϕ from the space $\check{H}^1(M, \mathfrak{X}_\Gamma)$ of essential infinitesimal deformations of the Γ_0-atlas to the space $\mathcal{D}_{\text{ess}}(\omega)$ of essential infinitesimal deformations of the closed

2-form determined by this atlas. We proceed as follows. Let an essential infinitesimal deformation be given by a cocycle of Γ-vector fields $V_{\beta\alpha}$ on the covering $\mathcal{U} = \{U_\alpha\}$ of M. Since the sheaf $\mathfrak{X}$ of vector fields on M is fine, there exist vector fields $W_\alpha \in \mathfrak{X}(U_\alpha)$ such that $V_{\beta\alpha} = W_\beta - W_\alpha$. We set $\eta_\alpha = L_{W_\alpha}\omega$, then, since $L_{V_{\beta\alpha}}\omega = 0$, we get $\eta_\alpha|_{U_{\alpha\beta}} = \eta_\beta|_{U_{\alpha\beta}}$. Therefore, we get the form η such that $\eta|_{U_\alpha} = \eta_\alpha$, and $d\eta = 0$.

If $V_{\beta\alpha} = W'_\beta - W'_\alpha$, then on $U_\alpha \cap U_\beta$ we have $W'_\beta - W'_\alpha = W_\beta - W_\alpha$. Therefore, we obtain the vector field W such that $W|_{U_\alpha} = W'_\alpha - W_\alpha$, and

$$\eta' - \eta = L_W\omega = d(i_W\omega).$$

Thus, η and η' determine the same class in the space $\mathcal{D}_{\mathrm{ess}}(\omega)$.

Thus we obtain the map

$$\widetilde{\phi} : \check{Z}^1(\mathcal{U}; \mathfrak{X}_\Gamma) \to \mathcal{D}_{\mathrm{ess}}(\omega), \quad \widetilde{\phi}(V_{\beta\alpha}) = [\eta].$$

From our construction follows that, if cocycles $V'_{\beta\alpha}$ and $V_{\beta\alpha}$ are cohomologous, they define the same closed form η, and, therefore, $\widetilde{\phi}(V'_{\beta\alpha}) = \widetilde{\phi}(V_{\beta\alpha})$, i.e. we obtain the linear map

$$\phi : \check{H}^1(M; \mathcal{X}_\Gamma) \to \mathcal{D}_{\mathrm{ess}}(\omega). \tag{14}$$

Statement 2. *Let us consider a symplectic structure ω with singularities of type ω_0 given by a Γ_0-atlas $\mathcal{A}$. Given a deformation $\mathcal{A}(s)$ of $\mathcal{A}$, we denote by $\omega(s)$ the corresponding deformation of ω. If $[V_{\beta\alpha}] \in H^1(M, \mathfrak{X}_\Gamma)$ is an essential infinitesimal deformation determined by $\mathcal{A}(s)$, and $\eta = \frac{d}{ds}\big|_{s=0}\omega(s)$, then $\phi([V_{\beta\alpha}]) = [\eta]$.*

Properties of the map ϕ.

Statement 3. *ϕ is a monomorphism.*

Let us define by Ω^p the sheaf of p-forms on M. Let $\iota : \mathfrak{X} \to \Omega^1$ be the sheaf morphism given by $\iota_U : \mathfrak{X}(U) \to \Omega^1(U)$, $\iota(V) = i_V\omega$.

Statement 4. *ϕ is surjective if and only if $\Omega^1 = \iota(\mathfrak{X}) + d\Omega^0$.*

4.4. **The sheaf of local Hamiltonians and the map ϕ.** *The sheaf of local Hamiltonians* is a subsheaf $\mathcal{T}$ of the sheaf Ω^0 of smooth functions on M. For any open set U a smooth function f lies in $\mathcal{T}(U)$ if and only if for any point $p \in U$ there exist a neighborhood $U' \subset U$ of p and $V \in \mathfrak{X}(U')$ such that $df = i_V\omega$. We will denote by i the inclusion $\mathbb{R}_M \to \mathcal{T}$.

Let us construct a sheaf morphism $\pi : \mathcal{T} \to \mathfrak{X}_\Gamma$. For $f \in \mathcal{T}(U)$ there exists a covering $\mathcal{U} = \{U_\alpha\}$ of U such that on each U_α there exists a vector field V_α with the property that $df = i_{V_\alpha}\omega$, which is uniquely defined by virtue of Lemma 2. Then, on $U_\alpha \cap U_\beta$ we have $V_\alpha = V_\beta$. Thus, on U we obtain the vector field V such that $df = i_V\omega$, which is uniquely defined. Furthermore, $L_V\omega = d(i_V\omega) = d(df) = 0$, therefore $V \in \mathfrak{X}_\Gamma(U)$. Thus we have obtained the map $\pi_U : \mathcal{T}(U) \to \mathfrak{X}_\Gamma(U)$, which is evidently linear. All the maps π_U determine a sheaf morphism $\pi : \mathcal{T} \to \mathfrak{X}_\Gamma$.

In fact $\mathcal{T}$ is the sheaf of algebras. The *Poisson bracket* $\{\ ,\ \}$ of sections of $\mathcal{T}$ is defined in the following way. Let $f, g \in \mathcal{T}(U)$, $df = i_V\omega$, and $dg = i_W\omega$, where $V, W \in \mathfrak{X}_\Gamma(U)$ are uniquely defined. Set

$$\{f, g\} = dg(V) - df(W) = 2\omega(W, V). \tag{15}$$

Statement 5.

$$0 \to \mathbb{R}_M \xrightarrow{i} \mathcal{T} \xrightarrow{\pi} \mathfrak{X}_\Gamma \to 0$$

is an exact sequence of sheaves.

Corollary 1. *1. The sequence*

$$\cdots \to \check{H}^0(M; \mathfrak{X}_\Gamma) \xrightarrow{\delta} \check{H}^1(M; \mathbb{R}_M) \xrightarrow{i_*} \check{H}^1(M; \mathcal{T}) \xrightarrow{\pi_*} \tag{16}$$
$$\xrightarrow{\pi_*} \check{H}^1(M; \mathfrak{X}_\Gamma) \xrightarrow{\delta} \check{H}^2(M; \mathbb{R}_M) \xrightarrow{i_*} \dots$$

is exact.

2) The diagram

$$\begin{array}{ccc} \check{H}^1(M; \mathfrak{X}_\Gamma) & \xrightarrow{\phi} & \mathcal{D}_{\mathrm{ess}}(\omega) \\ {\scriptstyle\delta}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ \check{H}^2(M; \mathbb{R}_M) & \xrightarrow[\cong]{\eta} & H^2(M), \end{array} \tag{17}$$

is commutative. Here ψ is defined in 4.2.1, δ is the connecting homomorphism of exact sequence (16), and η is the standard isomorphism between the Čech cohomology with coefficients in $\mathbb{R}_M$ and the de Rham cohomology.

4.4.1. *Obstructions to integrability of infinitesimal deformations.*

Statement 6. *Let $[v = \{V_{\beta\alpha}\}] \in H^1(M; \mathfrak{X}_\Gamma)$ be an essential infinitesimal deformation of a symplectic structure with singularities of type ω_0, and $[w = \{W_{\alpha\beta\gamma}\}] \in H^2(M, \mathfrak{X}_\Gamma)$ be the obstruction to integrability of this infinitesimal deformation* (*see* 1.1)*. Then $[w]$ lies in the image of the map $\pi_* : H^2(M; \mathcal{T}) \to H^2(M; \mathfrak{X}_\Gamma)$ (see* (16)*).*

Let a 1-cocycle $v \in Z^1(\mathcal{U}; \mathfrak{X}_\Gamma)$, where $\mathcal{U}$ is a covering of M consisting of contractible open sets, represent an infinitesimal deformation of a symplectic structure with singularities of type ω_0. Then the 2-cocycle $w = \{W_{\alpha\beta\gamma} = [V_{\alpha\beta}, V_{\beta\gamma}]\} \in Z^2(\mathcal{U}; \mathfrak{X}_\Gamma)$ represents the obstruction to integrability of v. Given v, we have the cochain $f = \{f_{\beta\alpha}\} \in C^1(\mathcal{U};\mathcal{T})$ such that $\pi(f) = v$ (see Statement 5). Take the cochain $g_{\alpha\beta\gamma} = \{f_{\alpha\beta}, f_{\beta\gamma}\} \in C^2(\mathcal{U};\mathcal{T})$, where $\{\ ,\ \}$ is the Poisson bracket (15). Since π is a morphism between sheaves of Lie algebras, we obtain that $\pi(g) = w$.

4.5. **Infinitesimal deformations of symplectic structure with Martinet singularities.**

4.5.1. *Martinet singularities.* In [16]–[19] it was proved that on $\mathbb{R}^4$ five generic types of germs of closed 2-forms exist, among them the following four types are stable:

Type 0

$$\omega_0 = dx^1 \wedge dx^2 + dx^3 \wedge dx^4,$$

Type I

$$\omega_0 = x^1 dx^1 \wedge dx^2 + dx^3 \wedge dx^4,$$

Type II-e (the elliptic type)

$$\omega_0 = dx^1 \wedge dx^2 + x^3 dx^1 \wedge dx^4 + x^3 dx^2 \wedge dx^3 + x^4 dx^2 \wedge dx^4 + (x^1 - (x^3)^2) dx^3 \wedge dx^4, \tag{18}$$

Type II-h (the hyperbolic type)

$$\omega_0 = dx^1 \wedge dx^2 + x^3 dx^1 \wedge dx^4 + x^3 dx^2 \wedge dx^3 - x^4 dx^2 \wedge dx^4 + (x^1 - (x^3)^2) dx^3 \wedge dx^4. \tag{19}$$

One can characterize these types in the following way (see [16] for details). Let ω be a closed 2-form on a four-dimensional manifold M. Let us denote by Σ the set of points, where ω is degenerate. In what follows we assume that Σ is a three-dimensional submanifold of M.

If a point p does not lie in Σ, then, by the Darboux theorem, ω has type 0 at p.

Now let $p \in \Sigma$ and $\omega(p) \neq 0$. If the kernel E_p of $\omega(p)$ is transversal to Σ, then ω has type I at p. The set $V = U \cap \Sigma$ is open in Σ and consists of points of type I, and $U \setminus V$ is everywhere dense in U and consists of points of type 0.

Let $\Sigma' \subset \Sigma$ be the set of points p such that $E_p \subset T_p\Sigma$. If Σ' is a submanifold in a neighborhood of p, and E_p is transversal to Σ', then ω has type II-e, or II-h, at p. Let $V = U \cap \Sigma$, $W = U \cap \Sigma'$. Then $U \setminus \Sigma$ is everywhere dense in U and consists of points of type 0, $V \setminus W$ is everywhere dense in V and consists of points of type I, and W consists of points of type II.

Let us consider a closed 2-form ω on a four-dimensional manifold M. If all points of M has type 0, then (M,ω) is a symplectic manifold. If any point of M has type 0 or I, then (M,ω) will be called a *symplectic manifold with Martinet singularities of type* I. If any point of M has type 0, I, or II, then (M, ω) will be called a *symplectic manifold with Martinet singularities of type* II.

Note that a symplectic manifold (M, ω) with Martinet singularities is a symplectic manifold with singularities of type ω_0, where ω_0 is given by (18) or (19). Indeed, let $\phi(p) = (x_0^1, x_0^2, x_0^3, x_0^4)$. Then p has type 0 if and only if $x_0^1 \neq 0$; p has type I if and only if $x_0^1 = 0$ and $(x_0^3)^2 + (x_0^4)^2 \neq 0$; and p has type II if and only if $x_0^1 = 0$, $x_0^3 = 0$, and $x_0^4 = 0$. Thus the symplectic structure is a partial case of the symplectic structure with Martinet singularities of type I, which in turn is a partial case of the symplectic structure with Martinet singularities of type II.

Example of compact manifold with symplectic structure having Martinet singularity. Let us take the four-dimensional torus $\mathbb{T}^4$, and let $\pi : \mathbb{R}^4 \to \mathbb{T}^4$ be the standard covering. Let us denote by θ^a, $a = \overline{1,4}$, the 1-forms on $\mathbb{T}^4$ such that $\pi^*(\theta^a) = dx^a$. Set $\omega = f\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4$, where $\pi^* f = \cos(x^1)$. It is clear that $d\omega = 0$. Let us denote by $p_1, p_2 \in \mathbb{S}^1$ the zeroes of f. Then $\Sigma = p_1 \times \mathbb{T}^3 \cup p_2 \times \mathbb{T}^3$, and the restriction of ω to each connected component of Σ is $\theta^3 \wedge \theta^4$, i.e. it has rank 2. Then ω determines a symplectic structure with Martinet singularity on $\mathbb{T}^4$.

Type 0

If ω is a symplectic structure without singularities, then the sheaf $\mathcal{T}$ coincides with the sheaf of smooth functions Ω^0, therefore $H^k(M; \mathcal{T}) = 0$ for $k > 0$. Furthermore, $\iota : \mathfrak{X} \to \Omega^1$ is a sheaf isomorphism (see 1.4). Hence, Corollary 1 and Statement 4 imply the well-known fact that for a symplectic structure $H^1(M; \mathfrak{X}_\Gamma) \cong \mathcal{D}_{\text{ess}}(\omega) \cong H^2(M)$.

Type I

Statement 7. *For a symplectic structure ω with Martinet singularities, $\phi : H^1(M; \mathfrak{X}_\Gamma) \to \mathcal{D}_{\text{ess}}(\omega)$ is an isomorphism.*

Type II

In this case the homomorphism ϕ is not surjective. Let ω be a symplectic structure with singularities of type

$$\omega_0 = dx^1 \wedge dx^2 + x^3 dx^1 \wedge dx^4 + x^3 dx^2 \wedge dx^3 + x^4 dx^2 \wedge dx^4 + (x^1 - (x^3)^2) dx^3 \wedge dx^4$$

Then the form $\alpha = (x^3)^2 dx^4$ cannot be represented as $i_V \omega + df$, therefore, by Statement 4, ϕ is not surjective.

4.5.2. *Calculation of $H^*(M; \mathcal{T})$ for Martinet singularities.* On the submanifold Σ of singular points we have the foliation $\mathcal{F}$ with singularities whose regular leaves are integral curves of the one-dimensional distribution on $\Sigma \setminus \Sigma'$ obtained by the intersection of the kernel of ω with the spaces tangent to Σ, and whose singular leaf is Σ'. Let us denote by $\mathcal{F}_b$ the sheaf of basic functions of the foliation $\mathcal{F}$, i. e. the sheaf of functions which are locally constant along the leaves of $\mathcal{F}$.

Let μ be a volume form on M. Then $\omega \wedge \omega = \mathrm{Pf}_\mu(\omega)\mu$, where $\mathrm{Pf}_\mu(\omega)$ is called the *Pfaffian of ω with respect to μ*. It is evident that, if $\mu' = \lambda\mu$ is another volume form, where λ is nonvanishing function, then $\mathrm{Pf}_\mu(\omega) = \lambda \mathrm{Pf}_{\mu'}(\omega)$. Hence, in the ring sheaf C^∞ of smooth functions we have the *subsheaf $\mathcal{I}$ of principal ideals generated by* $\mathrm{Pf}_\mu(\omega)$:

$$\mathcal{I}(U) = \{\, \mathrm{Pf}_\mu(\omega)|_U \cdot f \mid f \in C^\infty(U)\},$$

which does not depend on the choice of μ.

Note that $\mathcal{I}$ *is the ideal sheaf whose sections are the functions vanishing on Σ, i. e. $C^\infty/\mathcal{I}$ is the sheaf of smooth functions on Σ.*

Let X be a topological space, $A \subset X$ be a closed subset, and $\mathcal{G}$ be a ring sheaf on A. Then we have the sheaf $\mathcal{G}^X$ on X generated by the presheaf: $U \to 0$ if $U \cap A = \emptyset$, otherwise $U \to \mathcal{G}(U \cap A)$ (see [24]).

Let $r : C^\infty_M \to (C^\infty_\Sigma)^M$ be the sheaf morphism determined by the restriction of functions to Σ. This means that, if $U \cap \Sigma = \emptyset$, then $(C^\infty_\Sigma)^M(U) = 0$ and $r_U : C^\infty_M(U) \to (C^\infty_\Sigma)^M(U)$ is the zero homomorphism; if $U \cap \Sigma = U' \neq \emptyset$, then $r_U : C^\infty_M(U) \to C^\infty_\Sigma(U')$ is the restriction of $f \in C^\infty(U)$ to U'. Now, let us denote by i the inclusion $\mathcal{I}^2 \to C^\infty_M$, and set $\widetilde{\mathcal{F}}_b = \mathcal{F}_b^M$.

Statement 8.

$$0 \to \mathcal{I}^2 \xrightarrow{i} \mathcal{T} \xrightarrow{r} \widetilde{\mathcal{F}}_b \to 0. \tag{20}$$

is the exact sequence of sheaves on M.

Corollary 2. *For a symplectic manifold (M, ω) with Martinet singularities,*

$$H^q(M; \mathcal{T}) \cong H^q(\Sigma; \mathcal{F}_b), \quad q > 0,$$

where $\mathcal{T}$ is the sheaf of local Hamiltonians on M, and $\mathcal{F}_b$ is the sheaf of basic functions of the foliation with singularities induced by the kernel of ω on the singular submanifold Σ.

Corollary 3. *For a symplectic manifold (M, ω) with Martinet singularities such that $\Sigma' = \emptyset$,*

$$H^q(M; \mathfrak{X}_h) \cong H^{q+1}_{DR}(M), \quad q > 2,$$

where $\mathfrak{X}_h$ is the sheaf of Hamiltonian vector fields on M, $H^k_{DR}(M)$ is the de Rham cohomology of M.

Corollary 4. *If the foliation $\mathcal{F}$ is a fiber bundle, then $H^q(M;\mathfrak{X}_\Gamma) \cong H^{q+1}_{DR}(M)$ for $q > 0$. In particular, if $H^2_{DR}(M) \cong 0$, then ω is infinitesimally rigid.*

Example 4.1. Let $(\widehat{M}, \widehat{\omega})$ be a symplectic $2n$-dimensional manifold, M be a $2n$-dimensional manifold, and $\pi : M \to \widehat{M}$ be a smooth map. Let $\omega = \pi^*\omega$. Then ω is a closed 2-form on M, and the following diagram

$$\begin{array}{ccc} TM & \xrightarrow{\alpha_\omega} & T^*M \\ {\scriptstyle d\pi}\big\downarrow & & \big\uparrow{\scriptstyle d\pi^*} \\ T\widehat{M} & \xrightarrow{\alpha_{\hat{\omega}}} & T^*\widehat{M}\,. \end{array}$$

is commutative. Then

$$\operatorname{Ann}(\omega) = \ker \alpha_\omega = d\pi^{-1}(\alpha^{-1}_{\hat{\omega}} \ker d\pi^*). \tag{21}$$

Now let $\widehat{M}$ be $\mathbb{R}^4$ (with coordinates y^α, $\alpha = \overline{1,4}$) endowed with the standard symplectic structure

$$\widehat{\omega} = dy^1 \wedge dy^2 + dy^3 \wedge dy^4,$$

$M = \mathbb{S}^4 \subset \mathbb{R}^5$ be the standard sphere given by $\sum\limits_{i=1}^{5}(x^i)^2 = 1$, and $\pi : \mathbb{S}^4 \to \mathbb{R}^4$ be the restriction of the projection $\mathbb{R}^5 \to \mathbb{R}^4$, $y^\alpha = x^\alpha$, $\alpha = \overline{1,4}$.

Let Π be the hyperplane in $\mathbb{R}^5$ given by $x^5 = 0$, and $\mathbb{S}^3_0 = \mathbb{S}^4 \cap \Pi$ be the equator of $\mathbb{S}^4$. It is clear that $d\pi_p : T_p\mathbb{S}^4 \to T_p\mathbb{R}^4$ is an isomorphism for any $p \in \mathbb{S}^4 \setminus \mathbb{S}^3_0$, hence ω is nondegenerate on $\mathbb{S}^4 \setminus \mathbb{S}^3_0$. Further, by a coordinate calculation with the use of (21), we get that at points of $\mathbb{S}^3_0$ the kernel of ω is spanned by ∂_5 and $-x^2\partial_1 + x^1\partial_2 - x^4\partial_3 + x^3\partial_4$. Hence $\operatorname{Ann}(\omega) \pitchfork \mathbb{S}^3_0$, and the one-dimensional foliation $\mathcal{F}$ corresponding to the distribution $\ker(\omega) \cap T\mathbb{S}^3_0$ on $\mathbb{S}^3_0$ is the Hopf bundle. Then ω is a symplectic structure with Martinet singularities of type I, which satisfies the assumptions of Corollary 4. Hence ω is rigid.

4.5.3. *Differential complex associated to symplectic form with Martinet singularities.*

Here we expose results of [30].

We start with the following standard algebraic construction. Let K be a ring and $d : K \to K$ be a differentiation such that $d^2 = 0$. Let I be an ideal in K, then

$$I' = \{a + k_i\, db_i \mid a, b_i \in I\}$$

also is an ideal in K such that $d : I' \to I'$. Then we have the following exact sequence of differential rings:

$$0 \to I' \to K \to K/I' \to 0$$

The same construction can be done for sheaves of rings over a smooth manifold M. Then, for a sheaf of rings $\mathcal{K}$ over M endowed with a differential d such that $d^2 = 0$, and a subsheaf $\iota : \mathcal{I} \hookrightarrow \mathcal{K}$ of ideals, we get the sheaf $\mathcal{I}'$ of ideals and the exact sequence of sheaves

$$0 \to \mathcal{I}' \to \mathcal{K} \to \mathcal{K}/\mathcal{I}' \to 0$$

and the corresponding cohomology exact sequence

$$\cdots \to H^k(M;\mathcal{I}') \to H^k(M;\mathcal{K}) \to H^k(M;\mathcal{K}/\mathcal{I}') \to H^{k+1}(M;\mathcal{I}') \to \ldots$$

For any vector bundles $\xi : E_\xi \to M$ and $\eta : E_\eta \to M$, each vector bundle morphism $Q : \xi \to \eta$ determines the morphism $\mathcal{Q} : \Gamma_\xi \to \Gamma_\eta$ of sheaves of vector spaces: for each $s \in \Gamma_\xi(U)$, the section $\mathcal{Q}(s)(p) = Q_p(s(p))$, $p \in U$, lies in $\Gamma_\eta(U)$. The kernel of $\mathcal{Q}$ is the sheaf $\mathcal{K}(U) = \{s \in \Gamma_\xi(U) \mid \mathcal{Q}(s) = 0\}$ of modules over the fine sheaf C^∞, therefore $\mathcal{K}$ is also fine. Now let us consider the presheaf $\mathcal{F}(U) = \{t \in \Gamma_\eta(U) \mid t = \mathcal{Q}(s)\}$.

Lemma 3. *The presheaf $\mathcal{F}$ is a sheaf.*

Let $\xi : E_\xi \to M$ be a vector bundle, and $A : \xi \to \Lambda^q M$ be a vector bundle morphism. Denote by $\mathcal{A} : \Gamma_\xi \to \Omega_M$ the sheaf morphism corresponding to A. Then the subsheaf $\mathcal{A}(\Gamma_\xi)$ generates the subsheaf $\mathcal{I}_M \subset \Omega_M$ of ideals. Let us denote by $\mathcal{F}_M = \oplus \mathcal{F}^q_M \subset \Omega_M$ the corresponding graded sheaf $\mathcal{I}'_M$ of differential ideals.

We take an open $U \subset M$ such that ξ and ΛM are trivial over U. Then, on U we get q-forms ω_a, $a = \overline{1, \mathrm{rank}\xi}$, which span $\mathcal{A}(U)$ over the ring $C^\infty(U)$ of functions on U. One can easily see that

$$\mathcal{F}^k(U) = \{\phi^a \wedge \omega_a + \psi^a \wedge d\omega_a \mid \phi^a \in \Omega^k(U), \psi^a \in \Omega^{k-1}(U)\} \tag{22}$$

Thus, to any morphism $A : \xi \to \Lambda^q M$ we associate the complex $(\mathcal{F}^*, d)$ of sheaves, which is a subcomplex of the de Rham complex (Ω_M, d) considered also as a complex of sheaves. Also, we have the exact sequence of sheaves

$$0 \to \mathcal{F} \to \Omega_M \to \mathcal{G} = \Omega_M/\mathcal{F} \to 0.$$

Remark 4.1. Let $q = 1$, and $A : \xi \to \Omega^1$ be a morphism. Then the sheaf $\mathcal{A}(\Gamma_\xi)$ is the sheaf of sections of a subbundle (with singularities) in T^*M. If $\mathrm{rank}A$ is constant, then A determines a distribution on M, and if, in addition, this distribution is integrable, $(\mathcal{F}^*(M), d)$ is the complex generated by the basic forms of the corresponding foliation, which is widely used in the foliation theory [26], [2].

Remark 4.2. If $A : \xi \to \Omega^q$ is surjective, then the associated complex $(\mathcal{F}^*(M), d)$ is the de Rham complex of M.

Hereafter we assume that A *is surjective on an open everywhere dense set* $M \setminus \Sigma$, where $\iota : \Sigma \hookrightarrow M$ is an embedded submanifold. Then, for each open U such that $U \cap \Sigma = \emptyset$ we have $\mathcal{F}^k(U) = \Omega^k(U)$, hence $\mathcal{G}(U) = 0$. From this follows that the sheaf $\mathcal{G}$ is supported on Σ.

For a closed $\omega \in \Omega^{q+1}(M)$, the Lie derivative $D(V) = L_V\omega = di_V\omega$ is a first order differential operator $TM \to \Lambda^{q+1}M$. The operator D can be included to the complex of sheaves associated to the vector bundle morphism $I_\omega : TM \to \Lambda^q$, $I(V) = i_V\omega$:

$$0 \to \mathfrak{X}_\Gamma \to \mathfrak{X}_M \xrightarrow{D} \mathcal{F}^1 \xrightarrow{d} \mathcal{F}^2 \xrightarrow{d} \dots, \tag{23}$$

where $\mathfrak{X}_\Gamma$ is the sheaf of infinitesimal automorphisms of ω. Evidently, all the sheaves in (23), except for $\mathfrak{X}_\Gamma$ are fine. Therefore, if (23) is locally exact, it gives a fine resolution for $\mathfrak{X}_\Gamma$. However, in general, (23) fails to be locally exact.

Let ω be a closed 2-form on a smooth manifold M such that $\det \omega = 0$ on a closed submanifold $i : \Sigma \hookrightarrow M$ and $\mathrm{rank}\omega = 2m$ is constant on Σ. Then ω determines the vector bundle morphism $I_\omega : TM \to \Lambda^1 M$, $I_\omega(V) = i_V\omega$, which is a vector bundle isomorphism over $M \setminus \Sigma$. The kernel of I_ω is a vector bundle over Σ, call it $\epsilon : E \to \Sigma$. Denote by $\mathcal{I}_\omega$ the corresponding sheaf morphism $\mathfrak{X}_M \to \Omega^1_M$. From Lemma proved in [28] it follows that $\tau \in \mathcal{I}_\omega(\mathfrak{X}_M)$ if and only if $\tau|_E = 0$.

We will consider the symplectic structures with Martinet singularities.

Let ω be a closed 2-form on a $2n$-dimensional manifold. Assume that for each point $p \in M$ one can take a chart (U, u^i) such that

$$\omega = u^1 du^1 \wedge du^2 + du^3 \wedge du^4 \dots + du^{2n-1} \wedge du^{2n}. \tag{24}$$

Statement 9. *Let* $(\mathcal{F}^*, d)$ *be the complex of sheaves associated to the symplectic form* ω *with Martinet singularities locally given by* (24)*. Then* $\mathcal{F}^1$ *is the subsheaf of* Ω^1_M *consisting of forms which vanish on the subbundle* $E \subset TM|_\Sigma$*, and* $\mathcal{F}^k = \Omega^k_M$ *for* $k \geq 2$.

Statement 10. *For* $\mathcal{D} : \mathfrak{X}_M \to \Omega^2_M$, $D(V) = L_V\omega$*, the sequence of sheaves* (*see* (23))

$$0 \to \mathfrak{X}_\Gamma \xrightarrow{i} \mathfrak{X}_M \xrightarrow{D} \Omega^2 \xrightarrow{d} \Omega^3 \xrightarrow{d} \dots \tag{25}$$

is a fine resolution for the sheaf $\mathfrak{X}_\Gamma$.

Corollary 5. *For* ω *with Martinet singularities locally given by* (24)*,* $H^q(M; \mathfrak{X}_\Gamma) \cong H^{q+1}_{DR}(M)$*,* $q \geq 1$*, where* $H_{DR}(M)$ *is the de Rham cohomology.*

Let us consider another type of Martinet singularities. Let ω be a closed 2-form on a four-dimensional manifold, and assume that for each point $p \in M$ one can take a chart

(U, u^i) such that

$$\begin{aligned} \omega_0 =& du^1 \wedge du^2 + u^3 du^1 \wedge du^4 \\ &+ u^3 du^2 \wedge du^3 + u^4 du^2 \wedge du^4 + (u^1 - (u^3)^2) du^3 \wedge du^4, \end{aligned} \tag{26}$$

Statement 11. *Let $(\mathcal{F}^*, d)$ be the complex of sheaves associated to the symplectic form ω with Martinet singularities locally given by (26). Then $\mathcal{F}^1$ is the subsheaf of Ω^1_M consisting of forms vanishing on the subbundle $E \subset TM|_\Sigma$, and $\mathcal{F}^k = \Omega^k_M$ for $k \geq 2$.*

Thus, in this case we also get the complex of sheaves (25). However, in this case the complex (25) is not locally exact.

References

[1] Pommaret, J.F. *Systems of partial differential equations and Lie pseudogroups*, Math. and Appl., **14** (1978).

[2] Vaisman I. d_f *-cohomologies of Lagrangian foliations*, Monatshefte für Math., **106** (1988), pp. 221–244.

[3] Martinet J. *Sur les singularités des formes différetielles* // Ann. Inst. Fourier. Grenoble. – 1970. – V. 20. – No 1. – P. 95–178.

[4] Arnol'd, V.I. *Singularities of caustics and wave fronts.* (Osobennosti kaustik i volnovykh frontov). (Russian) Biblioteka Matematika. 1. Moskva: Fazis. x+ 334 p. (1996). [ISBN 5-7036-0021-9]

[5] Spencer D.C. *Deformations of structures on manifolds defined by transitive continuous pseudogroups.* I. *Infinitesimal deformations of structure* // Ann. Math. – 1962. – V. 76. – No 2. – P. 306–398.

[6] Spencer D.C. *Deformations of structures on manifolds defined by transitive continuous pseudogroups.* II. *Deformations of structure* // Ann. of Math. – 1962. – V. 76. – No 3. – P. 399–445.

[7] Spencer D.C. *Deformations of structures on manifolds defined by transitive continuous pseudogroups.* III. *Structures defined by elliptic pseudogroup* // Ann. Math. – 1965. – V. 81. – No 2. – P. 389–450.

[8] Guillemin V., Sternberg S. *Deformation theory of pseudogroup structures* // Mem. Am. Math. Soc. – 1968. – V. 64. – 80 p.

[9] Kodaira K. *Complex manifolds and deformations of complex structures.* – Springer, 1986. – 400 p.

[10] Fomenko, A.T. *Symplectic geometry. Methods and applications.* (Simplekticheskaya geometriya. Metody i prilozheniya). (Russian) Moskva: Izdatel'stvo Moskovskogo Universiteta. 414 p. (1988). [ISBN 5-211-00083-8]

[11] Warner, Frank W. Foundations of differentiable manifolds and Lie groups. (1987).

[12] , Wells R.O.jun. Differential analysis on complex manifolds. 2nd ed. (English) Graduate Texts in Mathematics. 65. New York - Heidelberg - Berlin: Springer-Verlag. X, 260 p. (1980).

[13] Sternberg, Shlomo Lectures on differential geometry. 2nd ed. (English) New York: Chelsea Publishing Company. XV, 442 p. (1983).

[14] Kobayashi, Shoshichi Transformation groups in differential geometry. Reprint of the 1972 ed. (English) Classics in Mathematics. Berlin: Springer-Verlag. viii, 182 p.

[15] Albert, C.; Molino, P. Pseudogroupes de Lie transitifs. II: Th'eor'ems d'int'egrabilitè. (Transitive Lie pseudogroups. II: Integrability theorems). (English) Travaux en Cours, 19. Paris: Hermann. VIII, 131 p.

[16] Martinet J. *Sur les singularités des formes différentielles* // Ann. Inst. Fourier (Grenoble). – 1970. – V. 20. – No 1. – P. 95–178.

[17] Martinet J. *Singularities of smooth functions and maps* // London Math. Soc. Lecture Note Ser. –1982. – No 58. – 256 p.

[18] Roussarie R. *Modèles locaux de champs et de formes* // Astérisque. – 1975. – No 30. – 181 p.

[19] Golubitsky M., Tischler D. *An example of moduli for singular symplectic forms* // Invent. Math. – 1977. – V. 38. – No 3. – P. 219–225.

[20] Cannas da Silva A., Guillemin V., Woodwart C. *On the unfolding of folded symplectic structures* // Math. Res. Lett. – 2000. – V. 7. – No 1. – P. 35–53.

[21] Cannas da Silva A. *Fold-forms for four-folds* // Preprint. – 2003. – 16 p.

[22] Domitrz W. *Non-local invariants of Martinet's singular symplectic structure* // Banach Center Publ. – 2002. – V. 60. – P. 122–143.

[23] Fomenko, A.T. *Symplectic geometry. Methods and applications.* (Simplekticheskaya geometriya. Metody i prilozheniya). (Russian) Moskva: Izdatel'stvo Moskovskogo Universiteta. 414 p. (1988). [ISBN 5-211-00083-8]

[24] Bredon, G. E. *Sheaf theory.* McGraw-Hill New York 1967

[25] Griffiths, Phillip; Harris, Joseph *Principles of algebraic geometry.* 2nd ed. (English) Wiley Classics Library. New York, NY: John Wiley & Sons Ltd. xii, 813 p.

[26] Molino, Pierre *Riemannian foliations.* Progress in Mathematics, Vol. 73. Boston-Basel: Birkhäuser. XII, 339 p.

[27] Malakhaltsev, M.A. *Infinitesimal deformations of a symplectic structure with singularities.* Russ. Math. 47, No.11, 38-46 (2003); translation from Izv. Vyssh. Uchebn. Zaved., Mat. 2003, No.11, 42-50 (2003).

[28] Malakhaltsev, M.A. *Sheaf of local Hamiltonians of symplectic manifolds with Martinet singularities.* Russ. Math. No.11, 45-52 (2004)

[29] Malakhaltsev, M.A. *The Lie derivative and cohomology of G-structures.* Lobachevskii Journal of Mathematics, 1999, Vol.3, pp.215-220 (http://ljm.ksu.ru).

[30] Malakhaltsev, M.A. *Differential complex associated to closed differential forms of nonconstant rank* Lobachevskii Journal of Mathematics, 1999, Vol.23, pp.183-192 (http://ljm.ksu.ru).

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