

1. DISTRIBUTIONS ON MANIFOLDS

Let M be an n -dimensional smooth manifold. Assume that for each $p \in M$ a subspace $\Delta(p) \subset T_p M$ is assigned such that 1) $\dim \Delta(p) = k$ is constant, 2) for each $q \in M$ there exists a neighbourhood U with vector fields $E_1, E_2, \dots, E_k \in \mathfrak{X}(U)$ such that, for each $q \in U$, $\Delta(q) = \text{span}\{E_1(q), E_2(q), \dots, E_k(q)\}$. We say that the dimension of the distribution Δ equals k .

A distribution Δ on a manifold M is called *totally integrable* if for each $p \in M$ there exists a submanifold Σ passing through p such that $T_q \Sigma = \Delta(q)$, for any $q \in \Sigma$. This submanifold is called an integral submanifold of Δ . A totally integrable distribution is called a foliation, and the integral submanifolds are called the leaves.

Locally, a distribution Δ can be given either by vector fields E_a , $a = 1, \dots, k$, spanning the subspaces $\Delta(p) \subset T_p M$, or by differential forms $\omega^1, \omega^2, \dots, \omega^{n-k}$ such that, for any $V \in T_p M$, $V \in \Delta(p)$ if and only if $\omega^\alpha(V) = 0$, $\alpha = 1, \dots, n - k$

The following theorem gives necessary and sufficient conditions for a distribution be completely integrable.

Theorem 1 (Frobenius). *Let Δ be a distribution on a manifold M locally given by a frame field E_a , $a = 1, \dots, k$, or by the fields of 1-forms ω^α . The following conditions are equivalent:*

- a) *The distribution Δ is completely integrable;*
- b) *$[E_a, E_b] = Q_{ab}^c E_c$, where Q_{ab}^c are functions;*
- c) *$d\omega^\gamma = Q_\beta^\gamma \wedge \omega^\beta$, where Q_β^γ are 1-forms.*

Example 1.1. Let a distribution Δ on \mathbb{R}^3 be given by the 1-form $\omega = dz - xdx - ydy$. Then $d\omega = 0$, and Δ is completely integrable because this distribution consists of panes tangent to hyperbolic paraboloids of the family $z = \frac{1}{2}(x^2 + y^2)$.

Example 1.2. Let a distribution Δ on \mathbb{R}^3 be given by the 1-form $\omega = dy - zdx$. Then one can take the local frame spanning Δ as follows: $E_1 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}$, $E_2 = \frac{\partial}{\partial z}$. Then $[E_1, E_2] = -\frac{\partial}{\partial y}$, and this vector field does not lie in Δ , because $\omega(\frac{\partial}{\partial y}) = -1 \neq 0$. Thus this distribution is not completely integrable. This means that one cannot find a surface tangent to Δ .

2. NON-HOLONOMIC SURFACE IN E^3

Let E_3 be a three-dimensional Euclidean space, and Δ be a distribution on E_3 . We denote by (\cdot, \cdot) the scalar product in E_3 . This distribution can be given by a unit vector field \vec{n} normal to Δ . The integrability condition for Δ can be formulated in terms of \vec{n} .

Theorem 2. *Δ is integrable if and only if $(\vec{n}, \text{rot } \vec{n}) = 0$*

2.1. Geometric sense of nonholonomy. The nonholonomy of a distribution in E^3 can be visualized in the following manner. Take a small disk L in the distribution plane and draw a straight line $l(A)$ through any point A of the disk boundary ∂L in the direction of $\vec{n}(A)$. Thus we obtain a ruled surface near the circle, and take a curve γ passing through A_0 which is orthogonal to the straight lines $l(A)$ (see Fig. 1). This curve again meets $l(A_0)$

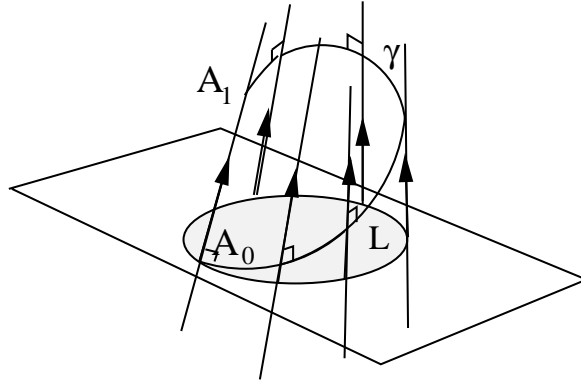


FIGURE 1. Geometric sense of nonholonomy. I

at a point A_1 , then $(\vec{n}, \text{rot } \vec{n})_{A_0} = \lim \frac{|A_0 A_1|}{\sigma}$, where σ is the area of L , as L contracts to A_0 . If Δ is holonomic (totally integrable), then γ lies on the integral surface passing through A_0 , hence $A_1 = A_0$, and $|A_0 A_1| = 0$.

Another way to visualize the nonholonomy is to take two unit orthogonal vector fields \vec{a}, \vec{b} spanning Δ . Let $\gamma(t)$ be an integral curve of the field \vec{b} , and for each t we take the integral curve $\delta_t(s)$ of the field \vec{a} passing through $\gamma(t)$. Thus we obtain a surface Σ (see Fig. 2). Now at a point $A(s) = \delta_t(s)$ we take the vector $\vec{v}(A(s))$ normal to the surface Σ at $A(s)$ and the vector $\vec{n}(A(s))$ normal to $\Delta(A(s))$. Denote by $\varphi(s)$ the angle between $\vec{v}(A(s))$ and $\vec{n}(A(s))$. Then

$$(1) \quad \frac{d\varphi}{ds} = -(\vec{n}, \text{rot } \vec{n})$$

If Δ is holonomic (totally integrable), then Σ is an integral manifold of Δ , hence the plane tangent to Σ at any $p \in \Sigma$ coincides with $\Delta(p)$, therefore $\vec{v}(p) = \vec{n}(p)$, and $\varphi = 0$.

2.2. Curvatures of nonholonomic surface.

2.2.1. Curvatures of holonomic surface. Let a surface Σ in E_3 be given by the parametric equation $\vec{r} = \vec{r}(u^1, u^2)$. Let us set $\vec{r}_i = \partial \vec{r}_i = \frac{\partial}{\partial u^i} \vec{r}$. The vectors \vec{r}_1, \vec{r}_2 give a frame of the tangent plane $T\Sigma$, and the coordinates of the metric tensor (the first fundamental form) with respect to this frame are $g_{ij} = (\vec{r}_i, \vec{r}_j)$. We denote by g^{ij} the tensor inverse to g_{ij} , this means that $g^{is} g_{js} = \delta_j^i$. Now let \vec{n} be the unit normal to $T\Sigma$. Then we obtain a

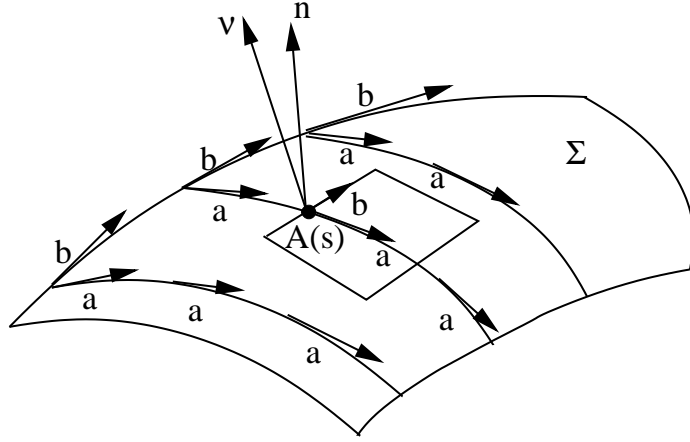


FIGURE 2. Geometric sense of nonholonomy. II

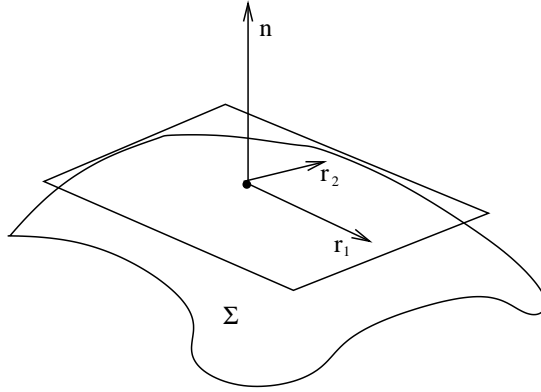


FIGURE 3. Moving frame

moving frame $\{\vec{r}_1, \vec{r}_2, \vec{n}\}$ and the derivation equations:

$$(2) \quad \partial_i \vec{r}_j = \Gamma_{ij}^k \vec{r}_k + h_{ij} \vec{n}$$

$$(3) \quad \partial_i \vec{n} = -h_i^j \vec{r}_j$$

Here

$$(4) \quad \Gamma_{ij}^k = \frac{1}{2} g^{ks} (\partial_i g_{js} + \partial_j g_{is} - \partial_s g_{ij})$$

are the connection coefficients (we will discuss them later), which depend only on the metric tensor, h_{ij} is the second fundamental form, and $h_j^i = g^{is} h_{js}$ is the shape operator.

The shape operator is a symmetric linear operator on $T\Sigma$, therefore it has real eigenvalues k_1, k_2 which are called principal curvatures of Σ . $H = \frac{1}{2}(k_1 + k_2)$ is called the mean curvature of Σ and $K = k_1 k_2$ the total (Gaussian) curvature of Σ .

Geometrically, the principal curvatures can be described as follows. Take a point $p \in \Sigma$. For any direction tangent to Σ given by a unit vector \vec{r} at p , we can take the plane

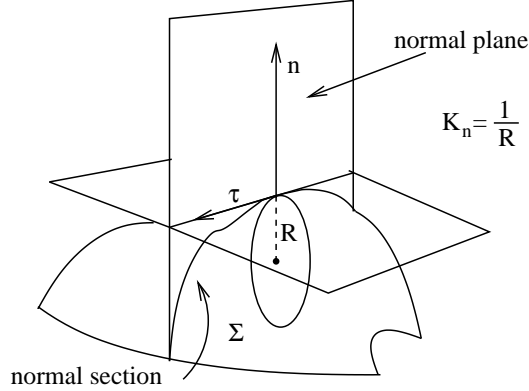


FIGURE 4. Normal curvature

passing through \vec{n} and $\vec{\tau}$ (a normal plane), and this plane meets the surface Σ along a normal section $\gamma(\vec{\tau})$. Let $\vec{\rho}(s)$ be the parametric equation of $\gamma(\vec{\tau})$ referred to the natural parameter s . Then, $k_n(\vec{\tau}) = (\frac{d^2}{ds^2}\vec{\rho}(s), \vec{n})$ is called the normal curvature of Σ with respect to the direction $\vec{\tau}$. As k_n is a continuous function on a circle, it attains its extremal values. One can easily prove that the principal curvatures k_1 and k_2 are just the extremal values of the normal curvature k_n . In addition, the normal curvature can be written in terms of the second fundamental form: $k_n(\vec{\tau}) = h(\vec{\tau}, \vec{\tau})$. Also, using (3), we get

$$(5) \quad k_n(\vec{\tau}) = -(\tau^i \partial_i \vec{n}, \vec{\tau}) = -(\nabla_{\vec{\tau}} \vec{n}, \vec{\tau}).$$

where ∇ stands for the directional derivative.

The total curvature can be described geometrically in another way. The unit normal \vec{n} determines the Gaussian map $G : \Sigma \rightarrow \mathbb{S}^2$, $p \in \Sigma \mapsto \vec{n}(p)$. Let U be a neighborhood of $p \in \Sigma$, and $V = G(U)$ be its image on \mathbb{S}^2 . Then the total curvature K at p equals $\lim \frac{\sigma(V)}{\sigma(U)}$ as U contracts to p . From this property one can obtain another formula for the total curvature

$$(6) \quad K = \frac{(\partial_1 \vec{n}, \partial_2 \vec{n}, \vec{n})}{\sqrt{g_{11}g_{22} - g_{12}^2}}$$

If we have a foliation in E^3 , whose leaves are given e. g. by implicit equations $\Phi(x^1, x^2, x^3) = \text{const}$, then the vector field \vec{n} is defined in E^3 . Then, from (6) it follows that

$$(7) \quad K = (\vec{P}, \vec{n}),$$

where \vec{P} is the *curvature vector* defined as follows:

$$(8) \quad \vec{P} = \left(\left(\frac{\partial}{\partial x^2} \vec{n}, \frac{\partial}{\partial x^3} \vec{n}, \vec{n} \right), \left(\frac{\partial}{\partial x^3} \vec{n}, \frac{\partial}{\partial x^1} \vec{n}, \vec{n} \right), \left(\frac{\partial}{\partial x^1} \vec{n}, \frac{\partial}{\partial x^2} \vec{n}, \vec{n} \right) \right),$$

where (\cdot, \cdot, \cdot) denotes the mixed product of three vectors. In particular, from (7) it follows the von Neumann formula:

$$K = \frac{\begin{vmatrix} \partial_{11}\Phi & \partial_{12}\Phi & \partial_{13}\Phi & \partial_1\Phi \\ \partial_{21}\Phi & \partial_{22}\Phi & \partial_{23}\Phi & \partial_2\Phi \\ \partial_{31}\Phi & \partial_{32}\Phi & \partial_{33}\Phi & \partial_3\Phi \\ \partial_1\Phi & \partial_2\Phi & \partial_3\Phi & 0 \end{vmatrix}}{(\partial_1\Phi)^2 + (\partial_2\Phi)^2 + (\partial_3\Phi)^2},$$

where the partial derivatives are taken with respect to x^1, x^2, x^3 .

2.2.2. Curvatures of nonholonomic surface. First type. Let a two-dimensional distribution (a nonholonomic surface) Δ be given in E^3 , and \vec{n} be the unit normal vector field of Δ . The formula (5) gives us the normal curvature of a surface in terms of the normal vector \vec{n} and a tangent vector $\vec{\tau}$, so it can be generalized to the nonholonomic case. We say that the *normal curvature of nonholonomic surface Δ in the direction of a unit vector $\tau \in \Delta$* is

$$(9) \quad k_n(\tau) = -(\nabla_{\vec{\tau}}\vec{n}, \vec{\tau}).$$

Let $A \in E^3$, and \vec{e}_1, \vec{e}_2 be an orthonormal frame of $\Delta(A)$. Then $\vec{\tau} = \cos \varphi \vec{e}_1 + \sin \varphi \vec{e}_2$, and one can easily see that k_n defined by (9) has the form

$$(10) \quad k_n(\varphi) = k_1 \cos \varphi + k_2 \sin \varphi$$

and it is clear that k_1 and k_2 are extremal values of $k_n(\varphi)$. k_1 and k_2 are called *the principal curvatures of Δ of first type*. Then we define the mean curvature $H_I = \frac{1}{2}(k_1 + k_2)$ and the total curvature $K_I = k_1 k_2$.

2.2.3. Curvatures of nonholonomic surface. Second type. For a non-holonomic distribution Δ we can define the total curvature of second type using (7), where \vec{P} is given by (8). We will denote it by K_{II} .

At the same time, this curvature can be obtained in the following way. Let us consider eigenvalues of the Jacobi matrix $J(\vec{n}) = \|\frac{\partial n^i}{\partial x^j}\|$, where n^i are Cartesian coordinates of the vector-function \vec{n} . Since \vec{n} maps a three-dimensional space into the two-dimensional sphere, the determinant of $J(\vec{n})$ is zero, hence one of the eigenvalues of $J(\vec{n})$ is zero. We will call the other two eigenvalues λ_1, λ_2 the principal curvatures of second type (they can be complex conjugate) and one can prove that $K_{II} = \lambda_1 \lambda_2$. At the same time, we can define the mean curvature of second type to be $H_{II} = \frac{1}{2}(\lambda_1 + \lambda_2)$.

Theorem 3. *For a nonholonomic surface Δ we have*

$$(11) \quad H_I = H_{II} \quad K_I - K_{II} = \frac{(\vec{n}, \text{rot } \vec{n})^2}{4}.$$

Example 2.1. An example of distribution which has $K_{II} = 0$, $H_{II} = 0$, and is nonholonomic can be constructed as follows. We take the family of concentric spheres in E^3 with center in the origin, and each sphere $S^2(r)$ of the family is uniquely determined by its radius $r > 0$. For each $r > 0$, we take a straight line $l(r)$ lying in the plane XOY such that the angle between $l(r)$ and OX is $\alpha(r)$, where $\alpha(r)$ is a function. Now, for each $S^2(r)$, we take the unit vector field \vec{n} tangent to this sphere which is generated by rotations about the axis $l(r)$ (certainly this field is not defined in the points where $l(r)$ meets $S^2(r)$). Thus we obtain a unit vector field \vec{n} which is defined everywhere in E^3 except for points where $l(r)$ meets $S^2(r)$. One can show that, if $\frac{d\alpha}{dr} \neq 0$, at points with $z \neq 0$ we have $(\vec{n}, \text{rot } \vec{n}) \neq 0$, so the distribution Δ orthogonal to \vec{n} is nonholonomic. At the same time, the second type mean and total curvature of Δ vanish.

Example 2.2. Another way to obtain a distribution Δ with zero total curvature of second type is to consider a surface Σ with a vector field \vec{a} along it (we assume that \vec{a} is not tangent to Σ). Then, for each point $p \in \Sigma$ we take a plane curve which is tangent to $\vec{a}(p)$ at p . Thus, a neighborhood of Σ is foliated by these curves, and for the vector field \vec{n} we take the unit vector field tangent to these curves. The distribution Δ orthogonal to \vec{n} has $K_{II} = 0$.

2.2.4. *Gauss-Bonnet formula.* For a closed surface Σ in E^3 , we have the Gauss-Bonnet formula

$$(12) \quad \int_{\Sigma} K dS = 2\pi\xi(\Sigma),$$

where K is the total curvature of Σ , and $\xi(\Sigma)$ is the Euler characteristic of Σ .

For a non-holonomic surface Δ , we have the following generalization of the Gauss-Bonnet formula.

Theorem 4. *Let Q be a closed oriented surface which lies in the domain of Δ . Then*

$$(13) \quad \int_Q (K_{II}\vec{n} + 2H_{II}\vec{k} + \nabla_{\vec{k}}\vec{n}, \vec{\nu})dS = 4\pi\theta,$$

where \vec{k} is the curvature vector of integral curves of \vec{n} , $\vec{\nu}$ is the unit normal to the surface Q , $\nabla_{\vec{k}}$ is the derivative in E^3 with respect to \vec{k} , and θ is the degree of the map $\vec{n}|_Q : Q \rightarrow S^2$.

Let us recall that the degree of a map $f : Q \rightarrow P$ between oriented surfaces is an integer which equals zero if f is homotopic to a constant map. Under assumptions of Theorem 4, Q is a boundary of a region $V \subset E^3$. If Δ is defined everywhere in V , and V is contractible, then \vec{n} is homotopic to a constant vector field, hence $\theta = 0$.

2.3. The shortest and the straightest curves on a nonholonomic surface. Let us recall that a geodesic on a surface Σ in the three-dimensional Euclidean space can be defined in two ways: a) a geodesic is a locally straightest line; b) a geodesic is a locally shortest curve. Consider these definitions in more details.

Let us consider a curve on a surface Σ given by a parametric equation $\vec{r} = \vec{\rho}(s)$, where s is the natural parameter. The length of projection of the curvature vector $\frac{d^2}{ds^2}\vec{\rho}$ onto the plane tangent to Σ is called the geodesic curvature. If the geodesic curvature is zero, or equivalently the curvature vector is collinear to the normal vector of the surface, we say that this curve is a straightest curve on Σ because it has no “acceleration”.

At the same time, if a curve γ joining sufficiently close points p and q on the surface Σ has the minimal length among all the curves joining these points, then γ is called the shortest curve.

For a surface $\Sigma \subset E^3$ the definition a) is equivalent to definition b). However, for the nonholonomic surface Δ the curves defined by a) and b) are different. We will call a curve γ in E^3 an admissible curve if its tangent vector $\frac{d}{dt}\gamma$ lies in $\Delta(\gamma(t))$.

2.3.1. Straightest lines on nonholonomic surfaces. Let a nonholonomic surface Δ be given. An admissible curve γ is called a straightest curve if its curvature vector is collinear to the normal vector \vec{n} of Δ .

Theorem 5. *Let $\vec{r} = \vec{\rho}(s)$ be parametric equation of an admissible curve γ . Then γ is a straightest line if and only if $\vec{\rho}(s)$ satisfies the following second order differential equation*

$$(14) \quad \frac{d^2\vec{\rho}}{ds^2} = - \left(\frac{d\vec{\rho}}{ds}, \nabla_{\frac{d\vec{\rho}}{ds}}\vec{n} \right) \vec{n},$$

Thus, given a point $p \in E^3M$ and a vector V_0 at p one can find a unique straightest curve γ which passes through p and whose tangent vector at p is V_0 .

2.4. Shortest lines on nonholonomic surfaces. Let us consider two points p and q in E^3 , and let $P_a(p, q)$ be the set of all admissible curves joining p and q . We say that a curve $\gamma \in P_a(p, q)$ such that $\gamma(a) = p$, $\gamma(b) = q$, is a shortest admissible curve if the length $L(\gamma) = \int_a^b |\frac{d\gamma}{dt}| dt$ is less than or equal to the length of any other curve in $P_a(p, q)$.

Theorem 6. *Let a curve $\gamma \in P_a(p, q)$ be given by a parametric equation $\vec{r} = \vec{\rho}(s)$, $s \in [a, b]$. Then, if γ is a shortest curve, then $\vec{\rho}(s)$ satisfies the third order ordinary differential equation*

$$(15) \quad \frac{d^2\vec{\rho}}{ds^2} + \frac{d}{ds} \left\{ \frac{(\frac{d^2\vec{\rho}}{ds^2}, \frac{d\vec{\rho}}{ds}, \vec{n})}{(\vec{n}, \text{rot } \vec{n})} \right\} \vec{n} + \frac{(\frac{d^2\vec{\rho}}{ds^2}, \frac{d\vec{\rho}}{ds}, \vec{n})}{(\vec{n}, \text{rot } \vec{n})} [\text{rot } \vec{n}, \frac{d\vec{\rho}}{ds}] = 0.$$

Therefore, given a point $p \in E^3M$ and a vector V_0 at p one can find an infinite number of shortest curves joining p and q which pass through p and whose tangent vector at p is V_0 .

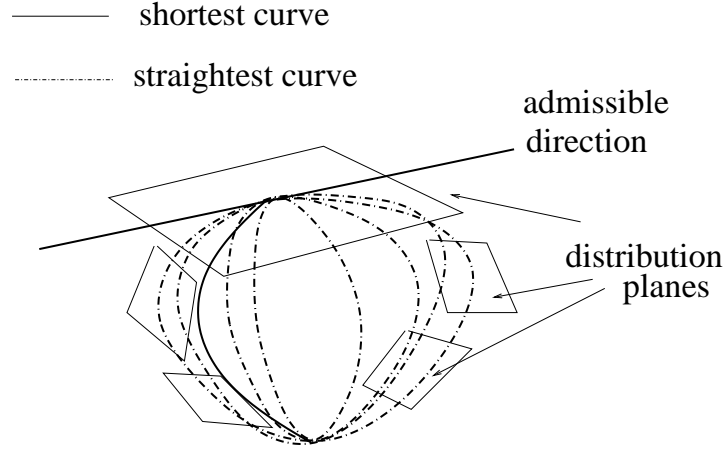


FIGURE 5. A unique straightest curve and a bundle of shortest curves

3. PROPERTIES OF METRIC OF SUB-RIEMANNIAN MANIFOLD

3.1. Accessibility of points. Let M be an n -dimensional differential manifold, and Δ be an m -dimensional distribution on M . We will call a curve γ in M admissible if its tangent vector lies in Δ . Let us denote by $P_a(p, q)$ the set of all admissible curves joining p and q (this set can be empty!). We say that a point q is accessible from a point p if $P_a(p, q)$ is not empty.

Theorem 7 (Sussman). *The set of points accessible from a given point p is an immersed submanifold in M .*

3.2. Completely non-integrable distributions. First let us define a distribution with singularities on an n -dimensional manifold M . Assume that for each $p \in M$ a subspace $\Delta(p) \subset T_p M$ is assigned such that for each $q \in M$ there exists a neighbourhood U with vector fields $E_1, E_2, \dots, E_k \in \mathfrak{X}(U)$, and for each $q \in U$, $\Delta(q) = \text{span}\{E_1(q), E_2(q), \dots, E_k(q)\}$. We see that this definition coincides with the definition of distribution except for the requirement that the subspaces $\Delta(p)$ have constant rank. Therefore, any distribution can be considered as a distribution with singularities.

Now let Δ be a distribution with singularities on M . Let $\mathfrak{X}_\Delta(U)$ be the set of vector fields X on an open set $U \subset M$ such that, for each $q \in U$, $X(q) \in \Delta(q)$. Now, for the distribution Δ we can construct another distribution Δ^1 with singularities in the following way. For any point $p \in M$ we set

$$(16) \quad \Delta^1(p) = \{X(p), [X, Y](p) \mid X, Y \in \mathfrak{X}_\Delta(U_p), \text{ where } U_p \text{ is a neighborhood of } p\}$$

From this definition it follows that $\Delta(p) \subset \Delta^1(p)$, for any $p \in M$.

It is clear that, if Δ is completely integrable, then $\Delta^1 = \Delta$.

For a distribution Δ we can sequentially construct the series Δ^k of distributions with singularities by setting $\Delta^{k+1} = (\Delta^k)^1$. Since $\Delta^k \subset \Delta^{k+1}$, and $\Delta^k = \Delta^{k+1}$ if and only if Δ^k is completely integrable, there exists N such that $\Delta^N = \Delta^{N+1} = \Delta^{N+2} = \dots$, and Δ^N is completely integrable. Thus, if the subspaces $\Delta^N(p)$ do not coincide with T_pM , then we get a foliation (with singularities) on M . And, if for each $p \in M$, $\Delta^N(p) = T_pM$, then we say that distribution Δ is completely non-integrable.

If a distribution Δ is completely non-integrable, then the following theorem implies that any point of M is accessible from any other point of M .

Theorem 8 (Caratheodory, Rashevskii, Chow). *Let a distribution Δ on a connected manifold M be completely non-integrable. Then, for any p, q in M , $P_a(p, q) \neq \emptyset$.*

3.3. Sub-Riemannian metric of completely non-integrable distributions. A sub-Riemannian manifold is a completely non-integrable distribution Δ endowed by a metric g . This means that, for any p in M , on $\Delta(p)$ we have the scalar product $g_p : \Delta(p) \times \Delta(p) \rightarrow \mathbb{R}$ such that for any $X, Y \in \mathfrak{X}_\Delta(M)$ the function $p \mapsto g_p(X(p), Y(p))$ is smooth. In other words, we can say that we have a metric on Δ considered as a vector subbundle of the vector bundle TM .

If $\gamma(s)$, $s \in [a, b]$, is an admissible curve in M , then we can define the length of $\gamma(s)$ as follows:

$$(17) \quad L(\gamma) = \int_a^b g_{\gamma(s)}\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right) ds$$

From theorem 8 it follows that we can define a distance d on M :

$$(18) \quad d(p, q) = \inf_{\gamma \in P_a(p, q)} L(\gamma).$$

This distance has some nice properties like the distance defined by a Riemannian metric. For example, we have the following theorem.

Theorem 9. *The topology defined by the distance d coincides with the original topology on M .*

A curve $\gamma(s)$, $s \in [a, b]$, is called a minimizing geodesic if $L(\gamma) = d(\gamma(a), \gamma(b))$.

Theorem 10 (Hopf-Rinow theorem for sub-Riemannian manifolds). a) *Sufficiently near points can be joined by a minimizing geodesic;*

b) *If (M, d) is a complete metric space, then any two points can be joined by a minimizing geodesic.*

4. REFERENCES

The detailed exposition of results on geometry of nonholonomic surface in the three-dimensional Euclidean space (stuff of section 2) can be found in the very interesting and profound book by Yu. Aminov:

Yu. Aminov: *The geometry of vector fields*. Amsterdam: Gordon and Breach Publishers. (2000).

For the introduction to the sub-Riemannian geometry (stuff of section 3), we can recommend the book:

R. Montgomery *A Tour of Subriemannian geometries, Their Geodesics and Applications*, Mathematical Surveys and Monographs, Vol. 91, AMS, 2002.