Introduction to Optimal Control Theory

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Abstract

These are lecture notes of the introductory course in Optimal Control theory treated from the geometric point of view. Optimal Control Problem is reduced to the study of controls (and corresponding trajectories) leading to the boundary of attainable sets. We discuss Pontryagin Maximum Principle, basic existence results, and apply these tools to concrete simple optimal control problems. Special sections are devoted to the general theory of linear time-optimal problems and linear-quadratic problems.

Contents

1 Optimal control problem

1.1 Problem statement

Consider a control system of the form

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{1}$$

where M is an open domain in $\mathbb{R}^n$ and U an arbitrary subset of $\mathbb{R}^m$. For the right-hand side of the control system, we suppose that:

$$q \mapsto f_u(q) \text{ is a smooth vector field on } M \text{ for any fixed } u \in U, \tag{2}$$

$$(q,u) \mapsto f_u(q) \text{ is a continuous mapping for } q \in M,\ u \in \overline{U},, \tag{3}$$

$$(q,u) \mapsto \frac{\partial f_u}{\partial q}(q) \text{ is a continuous mapping for } q \in M,\ u \in \overline{U}. \tag{4}$$

Admissible controls are vector-functions:

$$u \,:\, t \mapsto u(t) \in U, \quad t \in \mathbb{R}.$$

The set of all admissible controls is denoted by $\mathcal{U}$. In this lectures $\mathcal{U}$ is either the set of all piecewise smooth functions with values in U or the set of all bounded measurable functions with values in U. All results except those of Section 3 are valid for both classes of admissible controls. Substitute such a control $u = u(t)$ for control parameter into system (1), then we obtain a nonautonomous ODE $\dot{q} = f_u(q)$. By the classical Carathéodory's Theorem, for any point $q_0 \in M$, the Cauchy problem

$$\dot{q} = f_u(q), \qquad q(0) = q_0, \tag{5}$$

has a unique solution defined on an interval in $\mathbb{R}$. We will often fix the initial point q_0 and then denote the corresponding solution to problem (5) as $q_u(t)$.

In order to compare admissible controls one with another on a segment $[0, t_1]$, introduce a *cost functional*:

$$J(u) = \int_0^{t_1} \varphi(q_u(t), u(t))\, dt \tag{6}$$

with an integrand

$$\varphi \,:\, M \times U \to \mathbb{R}$$

satisfying the same regularity assumptions as the right-hand side f, see (2)–(4).

Take any pair of points $q_0, q_1 \in M$. We consider the following *optimal control problem*:

MINIMIZE THE FUNCTIONAL J AMONG ALL ADMISSIBLE CONTROLS $u = u(t)$, $t \in [0, t_1]$, FOR WHICH THE CORRESPONDING SOLUTION $q_u(t)$ OF CAUCHY PROBLEM (5) SATISFIES THE BOUNDARY CONDITION

$$q_u(t_1) = q_1. \tag{7}$$

We study two types of problems, with fixed t_1 and free t_1. A solution u of this problem is called an *optimal control*, and the corresponding curve $q_u(t)$ is the *optimal trajectory*.

So the optimal control problem is the minimization problem for $J(u)$ with constraints on u given by control system and the fixed endpoints conditions (5), (7). These constraints cannot usually be resolved w.r.t. u, thus solving optimal control problems requires special techniques.

1.2 Reduction to study of attainable sets

Fix an initial point $q_0 \in M$. *Attainable set* of control system (1) for time $t \geq 0$ from q_0 with measurable locally bounded controls is defined as follows:

$$\mathcal{A}_{q_0}(t) = \{q_u(t) \mid u \in \mathcal{U}\}.$$

Similarly, one can consider the attainable sets for time not greater than t:

$$\mathcal{A}_{q_0}^t = \bigcup_{0 \leq \tau \leq t} \mathcal{A}_{q_0}(\tau)$$

and for arbitrary nonnegative time:

$$\mathcal{A}_{q_0} = \bigcup_{0 \leq \tau < \infty} \mathcal{A}_{q_0}(\tau).$$

It turns out that optimal control problems on the state space M can be essentially reduced to the study of attainable sets of some auxiliary control systems on the extended state space

$$\widehat{M} = \mathbb{R} \times M = \{\widehat{q} = (y, q) \mid y \in \mathbb{R},\ q \in M\}.$$

Namely, consider the following extended control system on $\widehat{M}$:

$$\frac{d\,\widehat{q}}{d\,t} = \widehat{f}_u(\widehat{q}), \qquad \widehat{q} \in \widehat{M},\ u \in U, \tag{8}$$

with the right-hand side

$$\widehat{f}_u(\widehat{q}) = \begin{pmatrix} \varphi(q,u) \\ f_u(q) \end{pmatrix}, \qquad q \in M, \quad u \in U,$$

where φ is the integrand of the cost functional J, see (6). Then solutions $\widehat{q}_u(t)$ of the extended system (8) with the initial conditions

$$\widehat{q}_u(0) = \begin{pmatrix} y(0) \\ q(0) \end{pmatrix} = \begin{pmatrix} 0 \\ q_0 \end{pmatrix}$$

are expressed through solutions $q_u(t)$ of the original system (1) as

$$\widehat{q}_u(t) = \begin{pmatrix} J_t(u) \\ q_u(t) \end{pmatrix},$$

where

$$J_t(u) = \int_0^t \varphi(q_u(\tau), u(\tau))\, d\tau.$$

Thus attainable sets of the extended system (8) from the point $(0, q_0)$ have the form

$$\widehat{\mathcal{A}}_{(0,q_0)}(t) = \{(J_t(u), q_u(t)) \mid u \in \mathcal{U}\}.$$

Let $q(t)$, $t \in [0, t_1]$, be an optimal trajectory for the optimal control problem in M. Consider the corresponding trajectory

$$\widehat{q}(t) = \begin{pmatrix} J_t \\ q(t) \end{pmatrix}, \qquad t \in [0, t_1],$$

of the extended control system in $\widehat{M}$. The endpoint $\widehat{q}(t_1)$ must belong to the boundary of the attainable set $\widehat{\mathcal{A}}_{(0,q_0)}(t_1)$; moreover, this set should not intersect the ray

$$\{(y, q_1) \in \widehat{M} \mid y < J_{t_1}\}.$$

Indeed, if there exist points

$$(y, q_1) \in \widehat{\mathcal{A}}_{(0,q_0)}(t_1), \quad y < J_{t_1},$$

then the trajectory of the extended system

$$\widehat{q'}(t) = \begin{pmatrix} J'_t \\ q'(t) \end{pmatrix}$$

that steers $(0, q_0)$ to (y, q_1):

$$\widehat{q'}(0) = \begin{pmatrix} 0 \\ q_0 \end{pmatrix}, \qquad \widehat{q'}(t_1) = \begin{pmatrix} y \\ q_1 \end{pmatrix},$$

gives a trajectory $q'(t)$, $q'(0) = q_0$, $q'(t_1) = q_1$, with a smaller value of the cost functional:

$$J'_{t_1} = y < J_{t_1},$$

a contradiction with optimality of $q(\cdot)$.

So optimal trajectories (more precisely, their lift to the extended state space $\widehat{M}$) must come to the boundary of the attainable set $\widehat{\mathcal{A}}_{(0,q_0)}(t_1)$. In order to find optimal trajectories, we find those coming to the boundary of $\widehat{\mathcal{A}}_{(0,q_0)}(t_1)$, and then select optimal among them. The first step is much more important than the second one, so solving optimal control problems essentially reduces to the study of dynamics of boundary of attainable sets.

2 Pontryagin Maximum Principle

In this section we discuss the fundamental necessary condition of optimality for optimal control problems — Pontryagin Maximum Principle (PMP).

2.1 Geometric statement of PMP and discussion

Consider the optimal control problem stated in Sec. 1.1 for a control system

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{9}$$

with the initial condition

$$q(0) = q_0. \tag{10}$$

Define the following family of *Hamiltonians*:

$$h_u(p, q) = \langle p, f_u(q) \rangle, \qquad p \in \mathbb{R}^n,\ q \in M,\ u \in U,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product.

In Sec. 1.2 we reduced the optimal control problem to the study of boundary of attainable sets. Now we give a necessary optimality condition in this geometric setting.

Theorem 1 (PMP). *Let $\tilde{u}(t)$, $t \in [0, t_1]$, be an admissible control and $\tilde{q}(t) = q_{\tilde{u}}(t)$ the corresponding solution of* (9), (10). *If*

$$\tilde{q}(t_1) \in \partial \mathcal{A}_{q_0}(t_1),$$

then there exists a Lipschitz vector-function

$$p(t) \in \mathbb{R}^n, \qquad 0 \le t \le t_1,$$

such that

$$p(t) \neq 0, \tag{11}$$

$$\dot{p}(t) = -\frac{\partial h_{\tilde{u}(t)}}{\partial q}(p(t), \tilde{q}(t)), \tag{12}$$

$$h_{\tilde{u}(t)}(p(t), \tilde{q}(t)) = \max_{u \in U} h_u(p(t), \tilde{q}(t)) \tag{13}$$

for almost all $t \in [0, t_1]$.

If $u(t)$ is an admissible control, $q(t)$ the corresponding solution of (9), (10), and $p(t)$ a Lipschitz vector-function such that conditions (11)–(13) hold, then the triple $(u(t), p(t), q(t))$ is said to satisfy PMP. In this case $(p(t), (q(t))$ is often called an *extremal*, and $q(t)$ is called an *extremal trajectory*.

Remark. If $(u(t), p(t), q(t))$ satisfies PMP, then

$$h_{u(t)}(p(t), q(t)) = \text{const}, \qquad t \in [0, t_1]. \tag{14}$$

We skip a rather technical proof of the Pontryagin Maximum Principle but try to clarify its statement.

First we give an euristic explanation of the way the vector-function $p(t)$ appears naturally in the study of trajectories coming to boundary of the attainable set. Indeed, let

$$q_1 = \tilde{q}(t_1) \in \partial \mathcal{A}_{q_0}(t_1).$$

Consider a local convex approximation of the attainable set $\mathcal{A}_{q_0}(t_1)$ in the neighborhood of the point q_1, a convex cone with the vertix q_1. This convex cone has a hyperplane of support at q_1 determined by its normal vector $p(t_1)$ (the vector $p(t_1)$ is actually an analog of classical Lagrange multipliers).

In order to construct the whole curve $p(t)$, $t \in [0, t_1]$, consider the flow $P_\tau^{t_1} : M \to M$ generated by the control $\tilde{u}(\cdot)$:

$$P_\tau^{t_1} : q(\tau) \mapsto q(t_1), \qquad \tau \in [0, t_1],$$

where $\dot{q}(t) = f_{\tilde{u}(t)}(q(t)),\ t \in [\tau, t_1]$. It is easy to see that

$$P_\tau^{t_1}(\mathcal{A}_{q_0}(\tau)) \subset \mathcal{A}_{q_0}(t_1), \qquad \tau \in [0, t_1].$$

Indeed, if a point $q \in \mathcal{A}_{q_0}(\tau)$ is reachable from q_0 by a control $u(t)$, $t \in [0, \tau]$, then the point $P_\tau^{t_1}(q)$ is reachable from q_0 by the control

$$v(t) = \begin{cases} u(t), & t \in [0, \tau], \\ \tilde{u}(t), & t \in [\tau, t_1]. \end{cases}$$

Further, the flow $P_\tau^{t_1}$ satisfies the condition

$$P_\tau^{t_1}(\tilde{q}(\tau)) = \tilde{q}(t_1) = q_1, \qquad \tau \in [0, t_1].$$

Thus if $\tilde{q}(\tau) \in \operatorname{int} \mathcal{A}_{q_0}(\tau)$, then $q_1 \in \operatorname{int} \mathcal{A}_{q_0}(t_1)$. By contradiction, we obtain

$$\tilde{q}(\tau) \in \partial \mathcal{A}_{q_0}(\tau), \qquad \tau \in [0, t_1].$$

Consequently, we can find a hyperplane of support to the convex approximation of $\mathcal{A}_\tau(q_0)$ and the corresponding normal vector at any instant τ:

$$p(\tau) \in \mathbb{R}^{n} \setminus 0, \qquad \tau \in [0, t_1].$$

The normal vectors $p(t)$ are defined up to nonzero factors. They can be renormalized so that satisfy the equation $\dot{p}(t) = -\frac{\partial h_{\tilde{u}(t)}}{\partial q}(p(t), \tilde{q}(t))$

So the vector-function $p(t)$ in Pontryagin Maximum Principle appears naturally from hyperplanes of support to convex approximations of attainable sets.

Now we show the power of PMP by the following statement.

Proposition 1. *Assume that the maximized Hamiltonian of PMP*

$$H(p, q) = \max_{u \in U} h_u(p, q), \qquad p \in \mathbb{R}^{n},\ q \in M,$$

is defined and C^2-smooth on $(\mathbb{R}^n \setminus 0) \times M$.

If $(\tilde{u}(t), p(t), q(t))$, $t \in [0, t_1]$, *satisfies PMP, then*

$$\begin{cases} \dot{p}(t) = -\frac{\partial H}{\partial q}(p(t), q(t) \\ \dot{q}(t) = \frac{\partial H}{\partial p}(p(t), q(t)) \end{cases} \qquad t \in [0, t_1]. \tag{15}$$

Conversely, if a Lipshitzian vector-function $(p(t), q(t)) \in (\mathbb{R}^n \setminus 0) \times M$ *is a solution to the Hamiltonian system* (15), *then one can choose an admissible control* $\tilde{u}(t)$, $t \in [0, t_1]$, *such that* $(\tilde{u}(t), p(t), q(t))$ *satisfy PMP.*

That is, in the favorable case when the maximized Hamiltonian H is C^2-smooth, PMP reduces the problem to the study of solutions to just one Hamiltonian system (15). From the point of view of dimension, this reduction is the best one we can expect. Indeed, for a full-dimensional attainable set ($\dim \mathcal{A}_{q_0}(t_1) = n$) we have $\dim \partial\mathcal{A}_{q_0}(t_1) = n - 1$, i.e. we need an $(n-1)$-parameter family of curves to describe the boundary $\partial\mathcal{A}_{q_0}(t_1)$. On the other hand, the family of solutions to Hamiltonian system (15) with the initial condition $\pi(\lambda_0) = q_0$ is n-dimensional. Taking into account that the Hamiltonian H is homogeneous: $H(cp, q) = cH(p, q)$, $c > 0$; thus $(p(t), q(t))$ is a solution to Hamiltonian system (15) if and only if $(cp(t), q(t))$ is a solution to the same system and we obtain the required $(n-1)$-dimensional family of curves.

Now we prove Proposition 1.

Proof. Set $\lambda = (p, q)$, $\lambda_t = (p(t), q(t))$. We are going to show that if an admissible control $\tilde{u}(t)$ satisfies the maximality condition $h_{\tilde{u}(t)}(p(t), q(t)) = \max_{u \in U} h_u(p(t), q(t))$, then

$$\frac{\partial h_{\tilde{u}(t)}}{\partial \lambda}(\lambda_t) = \frac{dH}{d\lambda}(\lambda_t), \qquad t \in [0, t_1]. \tag{16}$$

In particular,

$$\frac{\partial H}{\partial p}(p(t), q(t)) = \frac{\partial h_{\tilde{u}(t)}}{\partial p}(p(t), q(t)) = f_{\tilde{u}(t)}(q(t)).$$

By definition of the maximized Hamiltonian H,

$$H(\lambda) - h_{\tilde{u}(t)}(\lambda) \geq 0 \qquad \lambda \in T^*M, \quad t \in [0, t_1].$$

On the other hand, by the maximality condition of PMP (13), along the extremal λ_t this inequality turns into equality:

$$H(\lambda_t) - h_{\tilde{u}(t)}(\lambda_t) = 0, \qquad t \in [0, t_1].$$

That is why

$$\frac{dH}{d\lambda}\lambda_t = \frac{\partial h_{\tilde{u}(t)}}{\partial \lambda}(\lambda_t), \qquad t \in [0, t_1].$$

But the right-hand side of the Hamiltonian system is obtained from differential of the Hamiltonian by a standard linear transformation, thus equality (16) follows.

Conversely, let $\lambda_t = (p(t), q(t))$, $p(t) \neq 0$, be a trajectory of the Hamiltonian system (14). One can show that it is possible to choose an admissible control $\tilde{u}(t)$ that realizes maximum along λ_t:

$$H(\lambda_t) = h_{\tilde{u}(t)}(\lambda_t) = \max_{u \in U} h_u(\lambda_t).$$

As we have shown above, then there holds equality (16). So $(\tilde{u}(t), \lambda_t)$ satisfies PMP. □

2.2 Geometric statement of PMP for free time

In the previous section we discussed Pontryagin Maximum Principle for the case of fixed terminal time t_1. Now we consider the case of free t_1.

Theorem 2. *Let $\tilde{u}(\cdot)$ be an admissible control for control system* (9) *and $\tilde{q}(t) = q_{\tilde{u}}(t)$ the corresponding solution of* (9), (10). *If*

$$\tilde{q}(t_1) \in \partial \left(\cup_{|t-t_1|<\varepsilon} \mathcal{A}_{q_0}(t) \right)$$

for some $t_1 > 0$ and $\varepsilon \in (0, t_1)$, then there exists a Lipschitz vector-function

$$\lambda_t = (p(t), \tilde{q}(t)) \in (\mathbb{R}^n \setminus 0) \times M, \quad \lambda_t \neq 0, \qquad 0 \leq t \leq t_1,$$

such that

$$\begin{aligned} \dot{p}(t) &= -\frac{\partial h_{\tilde{u}(t)}}{\partial p}(\lambda_t), \\ h_{\tilde{u}(t)}(\lambda_t) &= \max_{u \in U} h_u(\lambda_t), \\ h_{\tilde{u}(t)}(\lambda_t) &= 0 \end{aligned} \tag{17}$$

for almost all $t \in [0, t_1]$.

Remark. In problems with free time, there appears one more variable, the terminal time t_1. In order to eliminate it, we have one additional condition — equality (17). This condition is indeed scalar since the previous two equalities imply that $h_{\tilde{u}(t)}(\lambda_t) = \text{const}$, see remark after formulation of Theorem 1.

Proof. We reduce the case of free time to the case of fixed time by extension of the control system via substitution of time. Admissible trajectories of the extended system are reparametrized admissible trajectories of the initial system (the positive direction of time on trajectories is preserved).

Let a new time be a smooth function

$$\varphi : \mathbb{R} \to \mathbb{R}, \quad \dot{\varphi} > 0.$$

We find an ODE for a reparametrized trajectory:

$$\frac{d}{dt} q_u(\varphi(t)) = \dot{\varphi}(t) f_{u(\varphi(t))}(q_u(\varphi(t))),$$

so the required equation is

$$\dot{q} = \dot{\varphi}(t) f_{u(\varphi(t))}(q).$$

Now consider along with the initial control system

$$\dot{q} = f_u(q), \qquad u \in U,$$

an extended system of the form

$$\dot{q} = v f_u(q), \qquad u \in U, \ |v - 1| < \delta, \tag{18}$$

where $\delta = \varepsilon / t_1 \in (0, 1)$. Admissible controls of the new system are

$$w(t) = (v(t), u(t)),$$

and the reference control corresponding to the control $\tilde{u}(\cdot)$ of the initial system is

$$\tilde{w}(t) = (1, \tilde{u}(t)).$$

It is easy to see that since $\tilde{q}(t_1) \in \partial \left(\cup_{|t - t_1| < \varepsilon} \mathcal{A}_{q_0}(t) \right)$, then the trajectory of the new system through the point q_0 corresponding to the control $\tilde{w}(\cdot)$ comes at the moment t_1 to the boundary of the attainable set of the new system for time t_1. Thus $\tilde{w}(t)$ satisfies PMP with fixed time. We apply Theorem 1 to the new system (18). The Hamiltonian for the new system is $v h_u(\lambda)$. Then the maximality condition (13) reads

$$1 \cdot h_{\tilde{u}(t)}(\lambda_t) = \max_{u \in U, \ |v - 1| < \delta} v h_u(\lambda_t).$$

We take $u = \tilde{u}(t)$ under the maximum and obtain

$$h_{\tilde{u}(t)}(\lambda_t) = 0,$$

then we restrict the maximum to the set $v = 1$ and come to

$$h_{\tilde{u}(t)}(\lambda_t) = \max_{u \in U} h_u(\lambda_t).$$

The Hamiltonian systems along $\tilde{w}(\cdot)$ and $\tilde{u}(\cdot)$ coincide one with another, thus the proposition follows. □

2.3 PMP for optimal control problems

Now we apply PMP in geometric form to optimal control problems, starting from problems with fixed time.

For a control system

$$\dot{q} = f_u(q), \qquad q \in M, \quad u \in U, \tag{19}$$

with the boundary conditions

$$q(0) = q_0, \quad q(t_1) = q_1, \quad q_0, q_1 \in M \text{ fixed}, \tag{20}$$

$$t_1 > 0 \text{ fixed}, \tag{21}$$

and the cost functional

$$J(u) = \int_0^{t_1} \varphi(q_u(t), u(t))\, dt \tag{22}$$

we consider the optimal control problem

$$J(u) \to \min. \tag{23}$$

We transform the problem as in Sec. 1.2. We extend the state space:

$$\widehat{q} = \begin{pmatrix} y \\ q \end{pmatrix} \in \mathbb{R} \times M,$$

define the extended vector field

$$\widehat{f}_u(q) = \begin{pmatrix} \varphi(q, u) \\ f_u(q) \end{pmatrix},$$

and come to the new control system:

$$\frac{d\widehat{q}}{dt} = \widehat{f}_u(q) \quad \Leftrightarrow \quad \begin{cases} \dot{y} = \varphi(q,u), \\ \dot{q} = f_u(q) \end{cases} \tag{24}$$

with the boundary conditions

$$\widehat{q}(0) = \widehat{q}_0 = \begin{pmatrix} 0 \\ q_0 \end{pmatrix}, \qquad \widehat{q}(t_1) = \begin{pmatrix} J(u) \\ q_1 \end{pmatrix}.$$

If a control $\tilde{u}(\cdot)$ is optimal for problem (19)–(23), then the trajectory $\widehat{q}_{\tilde{u}}(t)$ of the extended system (24) starting from $\widehat{q}_0$ satisfies the condition

$$\widehat{q}_{\tilde{u}}(t_1) \in \partial \widehat{\mathcal{A}}_{\widehat{q}_0}(t_1),$$

where $\widehat{\mathcal{A}}_{\widehat{q}_0}(t_1)$ is the attainable set of system (24) from the point $\widehat{q}_0$ for time t_1. So we can apply Theorem 1.

But the geometric form of PMP applied to the extended system (24) does not distinguish minimum and maximum of the cost $J(u)$. In order to have conditions valid only for minimum, we introduce a new control parameter v and consider a new system of the form

$$\begin{cases} \dot{y} = \varphi(q,u) + v, \\ \dot{q} = f_u(q), \end{cases} \qquad v \geq 0, \quad u \in U. \tag{25}$$

Now the trajectory of system (25) corresponding to the controls $\tilde{v}(t) \equiv 0$, $\tilde{u}(t)$, comes to the boundary of the attainable set of this system at time t_1. We apply Theorem 1 to system (25). The Hamiltonian function for system (25) has the form

$$\widehat{h}_{(v,u)}(\nu, p, q) = \langle p, f_u(q) \rangle + \nu(\varphi + v),$$

and the Hamiltonian system of PMP is

$$\begin{cases} \dot{\nu} = \frac{\partial \widehat{h}}{\partial y} = 0, \\ \dot{y} = \varphi(q,u) + v, \\ \dot{p} = -\frac{\partial h_{\tilde{u}(t)}}{\partial p}(\nu, \lambda_t), \\ \dot{q} = f_{\tilde{u}(t)}(q(t)), \end{cases} \tag{26}$$

where

$$h_u(\nu, p, q) = \langle p, f_u(q) \rangle + \nu \varphi(q,u).$$

The first of equations (26) means that

$$\nu = \text{const}$$

along the reference trajectory.

The maximality condition has the form

$$\langle p(t), f_{\tilde{u}(t)}(\tilde{q}(t))\rangle + \nu\varphi(\tilde{q}(t), \tilde{u}(t)) = \max_{u\in U,\ v\geq 0} \left(\langle p(t), f_u(\tilde{q}(t))\rangle + \nu\varphi(\tilde{q}(t), u) + \nu v\right).$$

Since the previous maximum is attained, we have

$$\nu \leq 0,$$

thus $v = 0$ and

$$\langle p(t), f_{\tilde{u}(t)}(\tilde{q}(t))\rangle + \nu\varphi(\tilde{q}(t), \tilde{u}(t)) = \max_{u\in U} \left(\langle p(t), f_u(\tilde{q}(t))\rangle + \nu\varphi(\tilde{q}(t), u)\right).$$

So we obtain the following result.

Theorem 3. *Let $\tilde{u}(t)$, $\tilde{q}(t)$, $t \in [0, t_1]$, be an optimal control and the corresponding trajectory for problem* (19)–(23)*:*

$$J(\tilde{u}) = \min\{J(u) \mid q_u(t_1) = q_1\}.$$

Define a Hamiltonian function

$$h_u^\nu(p, q) = \langle p, f_u\rangle + \nu\varphi(q, u), \qquad (p, q) \in \mathbb{R}^n \times M, \quad u \in U, \quad \nu \in \mathbb{R}.$$

Then there exists a nontrivial pair:

$$(\nu, p(t)) \neq 0, \qquad \nu \in \mathbb{R}, \quad p(t) \in \mathbb{R}^n,$$

such that the following conditions hold:

$$\begin{aligned}
&\dot{p}(t) = -\frac{\partial h^\nu_{\tilde{u}(t)}}{\partial q}(p(t), \tilde{q}(t)),\\
&h^\nu_{\tilde{u}(t)}(p(t), \tilde{q}(t)) = \max_{u\in U} h^\nu_u(p(t), \tilde{q}(t)) \quad \forall \text{ a.e. } t \in [0, t_1],\\
&\nu \leq 0.
\end{aligned}$$

Remarks. (1) If we have a maximization problem instead of minimization problem (23), then the preceding inequality for ν should be reversed:

$$\nu \geq 0.$$

(2) For the problem with free time t_1: (19), (20), (22), (23), necessary optimality conditions of PMP are the same as in Theorem 3 plus one additional scalar equality $h^\nu_{\tilde{u}(t)}(p(t), \tilde{q}(t)) \equiv 0$.

There are two distinct possibilities for the constant parameter ν in Theorem 3:

(a) if $\nu \neq 0$, then $\lambda_t = (p(t), \tilde{q}(t))$ is called a *normal extremal*. Since the pair (ν, λ_t) can be multiplied by any positive number, we can normalize $\nu < 0$ and assume that $\nu = -1$ in the normal case;

(b) if $\nu = 0$, then λ_t is an *abnormal extremal*.

So we can always assume that $\nu = -1$ or 0.

Now consider the time-optimal problem:

$$\begin{aligned} &\dot{q} = f_u(q), \qquad q \in M, \quad u \in U, \\ &q(0) = q_0, \quad q(t_1) = q_1, \quad q_0, q_1 \text{ fixed}, \\ &t_1 = \int_0^{t_1} 1 \, dt \to \min. \end{aligned}$$

For the time-optimal problem, Pontryagin Maximum Principle takes the following form.

Corollary 1. *Let an admissible control $\tilde{u}(t)$, $t \in [0, t_1]$, be time-optimal. Define a Hamiltonian function*

$$h_u(p, q) = \langle p, f_u(q) \rangle, \qquad p \in \mathbb{R}^n, \quad u \in U.$$

Then there exists a Lipschitz vector-function

$$p(t) \in \mathbb{R}^n, \quad p(t) \neq 0, \qquad t \in [0, t_1],$$

such that the following conditions hold for almost all $t \in [0, t_1]$:

$$\begin{aligned} &\dot{p}(t) = -\frac{\partial h_{\tilde{u}(t)}}{\partial q}(p(t), \tilde{q}(t)), \\ &h_{\tilde{u}(t)}(p(t), \tilde{q}(t)) = \max_{u \in U} h_u(p(t), \tilde{q}(t)), \\ &h_{\tilde{u}(t)}(p(t), \tilde{q}(t)) \geq 0. \end{aligned} \tag{27}$$

Proof. Apply Theorem 3 and the second remark after it, taking $\varphi \equiv 1$. Then the Hamiltonian system and the maximality condition follow. Inequality (27) is equivalent to conditions $h_{\tilde{u}(t)}(p(t), \tilde{q}(t)) + \nu = 0$ and $\nu \leq 0$.

The inequality $p(t) \neq 0$ is obtained as follows: if $p(t) = 0$, then $h_{\tilde{u}(t)}(p(t), \tilde{q}(t)) = 0$, thus $\nu = 0$. But the pair $(\nu, p(t))$ must be nontrivial, consequently, $p(t) \neq 0$. □

In all previous problems, boundary conditions for a trajectory $q(t)$ were of the form $q(0) = q_0$, $q(t_1) = q_1$. Consider more general boundary conditions:

$$q(0) \in N_0, \qquad q_u(t_1) \in N_1,$$

where $N_0, N_1 \subset M$ are smooth submanifolds. It is easy to see that optimal solutions in the new problem are optimal for the problem with fixed $q(0)$, $q(t_1)$ as well. So all conditions of Pontryagin Maximum Principle should be satisfied. In addition to them, we need $(\dim N_1 + \dim N_2)$ extra conditions for the initial and terminal points. They are called *transversality conditions*: the adjoint covector $p(t)$ must be orthogonal to the submanifold N_0 at $q(0)$ and to the submanifold N_1 at $q(t_1)$ at the moments of time 0 and t_1 respectively:

$$p(0) \perp T_{q(0)} N_0 \quad \Leftrightarrow \quad \langle p(0), T_{q(0)} N_0 \rangle = 0,$$
$$p(t_1) \perp T_{q(t_1)} N_1 \quad \Leftrightarrow \quad \langle p(t_1), T_{q_1} N_1 \rangle = 0,$$

where $T_q N$ is the tangent space to the submanifold N at the point $q \in N$.

3 Existence of Optimal controls

In this section, $\mathcal{U}$ is the set all of measurable bounded vector-functions $t \mapsto u(t)$ with values in U.

3.1 Compactness of attainable sets

Due to the reduction of optimal control problems to the study of attainable sets, existence of optimal solutions to these problems is reduced to compactness of attainable sets.

For control system (1), sufficient conditions for compactness of the attainable sets $\mathcal{A}_{q_0}(t)$ for time t and $\mathcal{A}^t_{q_0}$ for time not greater than t are given in the following proposition.

Theorem 4 (Filippov). *Let the space of control parameters $U \Subset \mathbb{R}^m$ be compact. Let there exist a compact $K \Subset M$ such that $f_u(q) = 0$ for $q \notin K$, $u \in U$. Moreover, let the velocity sets*

$$f_U(q) = \{f_u(q) \mid u \in U\} \subset T_q M, \qquad q \in M,$$

be convex. Then the attainable sets $\mathcal{A}_{q_0}(t)$ and $\mathcal{A}^t_{q_0}$ are compact for all $q_0 \in M$, $t > 0$.

Remark. The condition of convexity of the velocity sets $f_U(q)$ is natural since trajectories of the ODE

$$\dot{q} = \alpha(t) f_{u_1}(q) + (1 - \alpha(t)) f_{u_2}(q), \qquad 0 \leq \alpha(t) \leq 1,$$

can be uniformly approximated by the "fast switching" trajectories of the systems of the form

$$\dot{q} = f_v(q), \quad \text{where} \quad v(t) \in \{u_1(t), u_2(t)\}.$$

Now we give a sketch of the proof of Theorem 4.

Proof. Notice first of all that all nonautonomous vector fields $f_u(q)$ with admissible controls u have a common compact support, thus are complete. Further, under hypotheses of the theorem, velocities $f_u(q)$, $q \in M$, $u \in U$, are uniformly bounded, thus all trajectories $q(t)$ of control system (1) starting at q_0 are Lipschitzian with the same Lipschitz constant. Thus the set of admissible trajectories is precompact in the topology of uniform convergence. For any sequence $q_n(t)$ of admissible trajectories:

$$\dot{q}_n(t) = f_{u_n}(q_n(t)), \qquad 0 \leq t \leq t_1, \quad q_n(0) = q_0,$$

there exists a uniformly converging subsequence, we denote it again by $q_n(t)$:

$$q_n(\cdot) \to q(\cdot) \text{ in } C[0, t_1] \text{ as } n \to \infty.$$

Now we show that $q(t)$ is an admissible trajectory of control system (1).

Fix a sufficiently small $\varepsilon > 0$. Then

$$\frac{1}{\varepsilon}(q_n(t+\varepsilon) - q_n(t)) = \frac{1}{\varepsilon} \int_t^{t+\varepsilon} f_{u_n}(q_n(\tau))\, d\tau$$
$$\in \operatorname{conv} \bigcup_{\tau \in [t, t+\varepsilon]} f_U(q_n(\tau)) \subset \operatorname{conv} \bigcup_{q \in O_{q(t)}(c\varepsilon)} f_U(q)$$

where c is the doubled Lipschitz constant of admissible trajectories. Then we pass to the limit $n \to \infty$ and obtain

$$\frac{1}{\varepsilon}(q(t+\varepsilon) - q(t)) \in \operatorname{conv} \bigcup_{q \in O_{q(t)}(c\varepsilon)} f_U(q).$$

Now let $\varepsilon \to 0$. If t is a point of differentiability of $q(t)$, then

$$\dot{q}(t) \in f_U(q)$$

since $f_U(q)$ is convex.

In order to show that $q(t)$ is an admissible trajectory of control system (1), we should find a measurable selection $u(t) \in U$ that generates $q(t)$. We do this via the lexicographic order on the set $U = \{(u_1, \dots, u_m)\} \subset \mathbb{R}^m$.

The set

$$V_t = \{v \in U \mid \dot{q}(t) = f_v(q(t))\}$$

is a compact subset of U, thus of $\mathbb{R}^m$. There exists a vector $v_{\min}(t) \in V_t$ minimal in the sense of lexicographic order: to find $v_{\min}(t)$, we minimize the first coordinate v_1 among all $v = (v_1, \dots, v_m) \in V_t$, then minimize the second coordinate v_2 on the compact set found at the first step, etc. The control $u(t) = v_{\min}(t)$ is measurable, thus $q(t)$ is an admissible solution of control system (1).

The proof of compactness of the attainable set $\mathcal{A}_{q_0}(t)$ is complete. Compactness of $\mathcal{A}_{q_0}^t$ is proved by a slightly modified argument. □

Remark. In Filippov's theorem, the hypothesis of common compact support of the vector fields in the right-hand side is essential to ensure the uniform boundedness of velocities and comple teness of vector fields. In the domain M, sufficient conditions for completeness of a vector field cannot be given in terms of boundedness of the vector field and its derivatives: a constant vector field is not complete in a bounded domain in $\mathbb{R}^n$. Nevertheless, one can prove compactness of attainable sets for many systems without the assumption of common compact support. If for such a system we have a priori bounds on solutions, then we can multiply its right-hand side by a cut-off function, and obtain a system with vector fields having compact support. We can apply Filippov's theorem to the new system. Since trajectories of the initial and new systems coincide in a domain of interest for us, we obtain a conclusion on compactness of attainable sets for the initial system.

For control systems on $M = \mathbb{R}^n$, there exist well-known sufficient conditions for completeness of vector fields: if the right-hand side grows at infinity not faster than a linear field, i.e.,

$$|f_u(x)| \le C(1 + |x|), \qquad x \in \mathbb{R}^n, \quad u \in U, \tag{28}$$

for some constant C, then the nonautonomous vector fields $f_u(x)$ are complete (here $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ is the norm of a point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$).

These conditions provide an a priori bound for solutions: any solution $x(t)$ of the control system

$$\dot{x} = f_u(x), \qquad x \in \mathbb{R}^n, \quad u \in U, \tag{29}$$

with the right-hand side satisfying (28) admits the bound

$$|x(t)| \leq e^{2Ct} \left(|x(0)| + 1\right), \qquad t \geq 0.$$

So Filippov's theorem plus the previous remark imply the following sufficient condition for compactness of attainable sets for systems in $\mathbb{R}^n$.

Corollary 2. *Let system* (29) *have a compact space of control parameters* $U \Subset \mathbb{R}^m$ *and convex velocity sets* $f_U(x)$, $x \in \mathbb{R}^n$. *Suppose moreover that the right-hand side of the system satisfies a bound of the form* (28). *Then the attainable sets* $\mathcal{A}_{x_0}(t)$ *and* $\mathbb{C}A^t_{x_0}$ *are compact for all* $x_0 \in \mathbb{R}^n$, $t > 0$.

3.2 Time-optimal problem

Given a pair of points $q_0 \in M$, $q_1 \in \mathcal{A}_{q_0}$, the *time-optimal problem* consists in minimizing the time of motion from q_0 to q_1 via admissible controls of control system (1):

$$\min_u \{t_1 \mid q_u(t_1) = q_1\}. \tag{30}$$

That is, we consider the optimal control problem described in Sec. 1.1 with the integrand $\varphi(q,u) \equiv 1$ and free terminal time t_1.

Reduction of optimal control problems to the study of attainable sets and Filippov's Theorem yield the following existence result.

Corollary 3. *Under hypotheses of Theorem* 4, *time-optimal problem* (1), (30) *has a solution for any points* $q_0 \in M$, $q_1 \in \mathcal{A}_{q_0}$.

3.3 Relaxations

Consider a control system of the form (1) with a compact set of control parameters U. There is a standard procedure called *relaxation* of control system (1), which extends the velocity set $f_U(q)$ of this system to its convex hull $\operatorname{conv} f_U(q)$.

Recall that the *convex hull* $\operatorname{conv} S$ of a subset S of a linear space is the minimal convex set that contains S. A constructive description of convex hull is given by the following classical proposition: any point in the convex hull of a set S in the n-dimensional linear space is contained in the convex hull of some $n+1$ points in S.

Lemma 1 (Carathéodory). *For any subset $S \subset \mathbb{R}^n$, its convex hull has the form*

$$\operatorname{conv} S = \left\{ \sum_{i=0}^{n} \alpha_i x_i \mid x_i \in S,\ \alpha_i \geq 0,\ \sum_{i=0}^{n} \alpha_i = 1 \right\}.$$

Relaxation of control system (1) is constructed as follows. Let $n = \dim M$ be dimension of the state space. The set of control parameters of the relaxed system is

$$V = \Delta^n \times \underbrace{U \times \cdots \times U}_{n+1 \text{ times}},$$

where

$$\Delta^n = \left\{ (\alpha_0, \ldots, \alpha_n) \mid \alpha_i \geq 0,\ \sum_{i=0}^{n} \alpha_i = 1 \right\} \subset \mathbb{R}^{n+1}$$

is the standard n-dimensional simplex. So the control parameter of the new system has the form

$$v = (\alpha, u_0, \ldots, u_n) \in V, \qquad \alpha = (\alpha_0, \ldots, \alpha_n) \in \Delta^n,\ u_i \in U.$$

If U is compact, then V is compact as well.

The *relaxed system* is

$$\dot{q} = g_v(q) = \sum_{i=0}^{n} \alpha_i f_{u_i}(q), \qquad v = (\alpha, u_0, \ldots, u_n) \in V, \quad q \in M. \tag{31}$$

By Carathéodory's lemma, the velocity set $g_V(q)$ of system (31) is convex, moreover,

$$g_V(q) = \operatorname{conv} f_U(q).$$

If all vector fields in the right-hand side of (31) have a common compact support, we obtain by Filippov's theorem that attainable sets for the relaxed system are compact. Any trajectory of relaxed system (31) can be uniformly approximated by families of trajectories of initial system (1). Thus attainable sets of the relaxed system coincide with closure of attainable sets of the initial system.

4 Examples of optimal control problems

In this chapter we apply Pontryagin Maximum Principle to solve concrete optimal control problems.

4.1 The fastest stop of a train at a station

Consider a train moving on a railway. The problem is to drive the train to a station and stop it there in a minimal time.

Describe position of the train by a coordinate x_1 on the real line; the origin $0 \in \mathbb{R}$ corresponds to the station. Assume that the train moves without friction, and we can control acceleration of the train by applying a force bounded by absolute value. Using rescaling if necessary, we can assume that absolute value of acceleration is bounded by 1.

We obtain the control system

$$\ddot{x}_1 = u, \qquad x_1 \in \mathbb{R}, \quad |u| \le 1,$$

or, in the standard form,

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u, \end{cases} \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2, \quad |u| \le 1.$$

The time-optimal control problem is

$$\begin{aligned} & x(0) = x^0, \quad x(t_1) = 0, \\ & t_1 \to \min . \end{aligned}$$

First we verify existence of optimal controls by Filippov's theorem. The set of control parameters $U = [-1, 1]$ is compact, the vector fields in the right-hand side

$$f(x, u) = \begin{pmatrix} x_2 \\ u \end{pmatrix}, \qquad |u| \le 1,$$

are linear, and the set of admissible velocities at a point

$$f(x, U) = \{ f(x, u) \mid |u| \le 1 \}$$

is convex. By Corollary 3, the time-optimal control problem has a solution if the origin $0 \in \mathbb{R}^2$ is attainable from the initial point x^0. We will show that any point $x \in \mathbb{R}^2$ can be connected with the origin by an extremal curve.

Now we apply Pontryagin Maximum Principle. Introduce canonical coordinates:

$$M = \mathbb{R}^2 = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\}.$$

So x is a specialization of the state variable q of previous sections. An adjoint vector (a specialization of the vector p of Sec. 2) is denoted by ξ and presented as a row: $\xi = (\xi_1, \xi_2)$. The control-dependent Hamiltonian function of PMP is

$$h_u(\xi, x) = (\xi_1, \xi_2) \begin{pmatrix} x_2 \\ u \end{pmatrix} = \xi_1 x_2 + \xi_2 u,$$

and the corresponding Hamiltonian system has the form

$$\begin{cases} \dot{x} = \dfrac{\partial h_u}{\partial \xi}, \\ \dot{\xi} = -\dfrac{\partial h_u}{\partial x}. \end{cases}$$

In coordinates this system splits into two independent subsystems:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u, \end{cases} \qquad \begin{cases} \dot{\xi}_1 = 0, \\ \dot{\xi}_2 = -\xi_1. \end{cases} \tag{32}$$

By PMP, if a control $\tilde{u}(\cdot)$ is time-optimal, then the Hamiltonian system has a nontrivial solution $(\xi(t), x(t))$, $\xi(t) \not\equiv 0$, such that

$$h_{\tilde{u}(t)}(\xi(t), x(t)) = \max_{|u| \le 1} h_u(\xi(t), x(t)) \ge 0.$$

From this maximality condition, if $\xi_2(t) \ne 0$, then $\tilde{u}(t) = \operatorname{sgn} \xi_2(t)$. Notice that the maximized Hamiltonian

$$\max_{|u| \le 1} h_u(\xi, x) = \xi_1 x_2 + |\xi_2|$$

is not smooth. So we cannot apply Proposition 1, but we can obtain description of optimal controls directly from Pontryagin Maximum Principle, without preliminary maximization of Hamiltonian.

Since

$$\ddot{\xi}_2 = 0,$$

then ξ_2 is linear:

$$\xi_2(t) = \alpha + \beta t, \qquad \alpha,\ \beta = \text{const},$$

hence the optimal control has the form

$$\tilde{u}(t) = \operatorname{sgn}(\alpha + \beta t).$$

So $\tilde{u}(t)$ is piecewise constant, takes only the extremal values ± 1, and has not more than one switching (discontinuity point).

New we find all trajectories $x(t)$ that correspond to such controls and come to the origin. For controls $u = \pm 1$, the first of subsystems (32) reads

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \pm 1. \end{cases}$$

Trajectories of this system satisfy the equation

$$\frac{d\,x_1}{d\,x_2} = \pm x_2,$$

thus are parabolas of the form

$$x_1 = \pm \frac{x_2^2}{2} + C, \qquad C = \text{const}\,.$$

First we find trajectories from this family that come to the origin without switchings: these are two semiparabolas

$$x_1 = \frac{x_2^2}{2}, \qquad x_2 < 0, \quad \dot{x}_2 > 0, \tag{33}$$

and

$$x_1 = -\frac{x_2^2}{2}, \qquad x_2 > 0, \quad \dot{x}_2 < 0, \tag{34}$$

for $u = +1$ and -1 respectively.

Now we find all extremal trajectories with one switching. Let $(x_{1s}, x_{2s}) \in \mathbb{R}^2$ be a switching point for anyone of curves (33), (34). Then extremal trajectories with one switching coming to the origin have the form

$$x_1 = \begin{cases} -x_2^2/2 + x_{2s}^2/2 + x_{1s}, & x_2 > x_{2s}, \quad \dot{x}_2 < 0, \\ \\ x_2^2/2 & 0 > x_2 > x_{2s}, \quad \dot{x}_2 > 0, \end{cases} \tag{35}$$

and

$$x_1 = \begin{cases} x_2^2/2 - x_{2s}^2/2 + x_{1s}, & x_2 < x_{2s}, \quad \dot{x}_2 > 0, \\ \\ -x_2^2/2 & 0 < x_2 < x_{2s}, \quad \dot{x}_2 < 0. \end{cases} \tag{36}$$

It is easy to see that through any point (x_1, x_2) of the plane passes exactly one curve of the forms (33)–(36). So for any point of the plane there exists exactly one extremal trajectory steering this point to the origin. Since optimal trajectories exist, then the solutions found are optimal.

4.2 Control of a linear oscillator

Consider a linear oscillator whose motion can be controlled by force bounded in absolute value. The corresponding control system (after appropriate rescaling) is

$$\ddot{x}_1 + x_1 = u, \qquad |u| \leq 1, \quad x_1 \in \mathbb{R},$$

or, in the canonical form:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + u, \end{cases} \qquad |u| \leq 1, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2.$$

We consider the time-optimal problem for this system.

By Filippov's theorem, optimal control exists. Similarly to the previous problem, we apply Pontryagin Maximum Principle: the Hamiltonian function is

$$h_u(\xi, x) = \xi_1 x_2 - \xi_2 x_1 + \xi_2 u, \qquad (\xi, x) \in T^*\mathbb{R}^2 = \mathbb{R}^{2*} \times \mathbb{R}^2,$$

and the Hamiltonian system reads

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + u, \end{cases} \qquad \begin{cases} \dot{\xi}_1 = \xi_2, \\ \dot{\xi}_2 = -\xi_1. \end{cases}$$

The maximality condition of PMP yields

$$\xi_2(t)\tilde{u}(t) = \max_{|u| \leq 1} \xi_2(t) u,$$

thus optimal controls satisfy the condition

$$\tilde{u}(t) = \operatorname{sgn} \xi_2(t) \qquad \text{if } \xi_2(t) \neq 0.$$

For the variable ξ_2 we have the ODE

$$\ddot{\xi}_2 = -\xi_2,$$

hence

$$\xi_2 = \alpha \sin(t+\beta), \qquad \alpha,\ \beta = \text{const}\,.$$

Notice that $\alpha \neq 0$: indeed, if $\xi_2 \equiv 0$, then $\xi_1 = -\dot{\xi}_2(t) \equiv 0$, thus $\xi(t) = (\xi_1(t), \xi_2(t)) \equiv 0$, which is impossible by PMP. Consequently,

$$\tilde{u}(t) = \operatorname{sgn}(\alpha \sin(t+\beta)).$$

This equality yields a complete description of possible structure of optimal control. The interval between successive switching points of $\tilde{u}(t)$ has the length π. Let $\tau \in [0,\pi)$ be the first switching point of $\tilde{u}(t)$. Then

$$\tilde{u}(t) = \begin{cases} \operatorname{sgn} \tilde{u}(0), & t \in [0,\tau) \cup [\tau+\pi, \tau+2\pi) \cup [\tau+3\pi, \tau+4\pi) \cup \dots \\ -\operatorname{sgn} \tilde{u}(0), & t \in [\tau, \tau+\pi) \cup [\tau+2\pi, \tau+3\pi) \cup \dots \end{cases}$$

That is, $\tilde{u}(t)$ is parametrized by two numbers: the first switching time $\tau \in [0,\pi)$ and the initial sign $\operatorname{sgn} \tilde{u}(0) \in \{\pm 1\}$.

Optimal control $\tilde{u}(t)$ takes only the extremal values ± 1. Thus optimal trajectories $(x_1(t), x_2(t))$ consist of pieces that satisfy the system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 \pm 1, \end{cases} \tag{37}$$

i.e., arcs of the circles

$$(x_1 \pm 1)^2 + x_2^2 = C, \qquad C = \text{const},$$

passed clockwise.

Now we describe all optimal trajectories coming to the origin. Let γ be any such trajectory. If γ has no switchings, then it is an arc belonging to one of the semicircles

$$(x_1 - 1)^2 + x_2^2 = 1, \qquad x_2 \leq 0, \tag{38}$$

$$(x_1 + 1)^2 + x_2^2 = 1, \qquad x_2 \geq 0 \tag{39}$$

and containing the origin. If γ has switchings, then the last switching can occur at any point of these semicircles except the origin. Assume that γ has the last switching on semicircle (38). Then the part of γ before the last switching and after the next to last switching is a semicircle of the circle $(x_1+1)^2 + x_2^2 = C$ passing through the last switching point. The next to last switching of γ occurs on the curve obtained by rotation of semicircle (38) around the point $(-1,0)$ in the plane (x_1,x_2) by the angle π, i.e., on the semicircle

$$(x_1+3)^2+x_2^2=1, \qquad x_2\geq 0. \tag{40}$$

To obtain the geometric locus of the previous switching of γ, we have to rotate semicircle (40) around the point $(1,0)$ by the angle π; we come to the semicircle

$$(x_1-5)^2+x_2^2=1, \qquad x_2\leq 0.$$

The previous switching of γ takes place on the semicircle

$$(x_1+7)^2+x_2^2=1, \qquad x_2\geq 0,$$

and so on.

The case when the last switching of γ occurs on semicircle (39) is obtained from the case just considered by the central symmetry of the plane (x_1,x_2) w.r.t. the origin: $(x_1,x_2)\mapsto(-x_1,-x_2)$. Then the successive switchings of γ (in the reverse order starting from the end) occur on the semicircles

$$\begin{aligned}
(x_1+1)^2+x_2^2=1, &\qquad x_2\geq 0,\\
(x_1-3)^2+x_2^2=1, &\qquad x_2\leq 0,\\
(x_1+5)^2+x_2^2=1, &\qquad x_2\geq 0,\\
(x_1-7)^2+x_2^2=1, &\qquad x_2\leq 0,
\end{aligned}$$

etc. We obtained the switching curve in the plane (x_1,x_2):

$$\begin{aligned}
(x_1-(2k-1))^2+x_2^2=1, &\qquad x_2\leq 0, \quad k\in\mathbb{N},\\
(x_1+(2k-1))^2+x_2^2=1, &\qquad x_2\geq 0, \quad k\in\mathbb{N}.
\end{aligned} \tag{41}$$

This switching curve divides the plane (x_1,x_2) into two parts. Any extremal trajectory $(x_1(t),x_2(t))$ in the upper part of the plane is a solution of ODE (37) with -1 in the second equation, and in the lower part it is a solution of (37) with $+1$. For any point of the plane (x_1,x_2) there exists exactly one curve of this family of extremal trajectories that comes to the origin (it has the form of a “spiral” with a finite number of switchings). Since optimal trajectories exist, the constructed extremal trajectories are optimal.

The time-optimal control problem is solved: in the part of the plane (x_1,x_2) over the switching curve (41) the optimal control is $\tilde{u}=-1$, and below this curve $\tilde{u} = +1$. Through any point of the plane passes one optimal trajectory which corresponds to this optimal control rule. After finite number of switchings, any optimal trajectory comes to the origin.

Now we consider optimal control problems with the same dynamics as in the previous two sections, but with another cost functional.

4.3 The cheapest stop of a train

As in Section 4.1, we control motion of a train. Now the goal is to stop the train at a fixed instant of time with a minimum expenditure of energy, which is assumed proportional to the integral of squared acceleration.

So the optimal control problem is as follows:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u, \end{cases} \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2, \quad u \in \mathbb{R},$$

$$x(0) = x^0, \quad x(t_1) = 0, \quad t_1 \text{ fixed},$$

$$\frac{1}{2}\int_0^{t_1} u^2\, dt \to \min .$$

Filippov's theorem cannot be applied directly since the right-hand side of the control system is not compact. Although, one can choose a new time $t \mapsto \frac{1}{2}\int_0^t u^2(\tau)\, d\tau + C$ and obtain a bounded right-hand side, then compactify it and apply Filippov's theorem. In such a way existence of optimal control can be proved. See also the general theory of linear quadratic problems below in Chapter 6.

To find optimal control, we apply PMP. The Hamiltonian function is

$$h_u^\nu(\xi, x) = \xi_1 x_2 + \xi_2 u + \frac{\nu}{2}u^2, \qquad (\xi, x) \in \mathbb{R}^{2*} \times \mathbb{R}^2 .$$

Along optimal trajectories

$$\nu \leq 0, \quad \nu = \text{const} .$$

From the Hamiltonian system of PMP, we have

$$\begin{cases} \dot{\xi}_1 = 0, \\ \dot{\xi}_2 = -\xi_1 . \end{cases} \tag{42}$$

Consider first the case of abnormal extremals:

$$\nu = 0.$$

The triple (ξ_1, ξ_2, ν) must be nonzero, thus

$$\xi_2(t) \not\equiv 0.$$

But the maximality condition of PMP yields

$$\tilde{u}(t)\xi_2(t) = \max_{u \in \mathbb{R}} u\, \xi_2(t). \tag{43}$$

Since $\xi_2(t) \neq 0$, the maximum above does not exist. Consequently, there are no abnormal extremals.

Consider the normal case: $\nu \neq 0$, we can take $\nu = -1$. The normal Hamiltonian function is

$$h_u(\xi, x) = h_u^{-1}(\xi, x) = \xi_1 x_2 + \xi_2 u - \frac{1}{2}u^2.$$

Maximality condition of PMP is equivalent to $\frac{\partial h_u}{\partial u} = 0$, thus

$$\tilde{u}(t) = \xi_2(t)$$

along optimal trajectories. Taking into account system (42), we conclude that optimal control is linear:

$$\tilde{u}(t) = \alpha t + \beta, \qquad \alpha,\ \beta = \text{const}.$$

The maximized Hamiltonian function

$$H(\xi, x) = \max_u h_u(\xi, x) = \xi_1 x_2 + \frac{1}{2}\xi_2^2$$

is smooth. That is why optimal trajectories satisfy the Hamiltonian system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \xi_2, \\ \dot{\xi}_1 = 0, \\ \dot{\xi}_2 = -\xi_1. \end{cases}$$

For the variable x_1 we obtain the boundary value problem

$$\begin{aligned} & x_1^{(4)} = 0, \\ & x_1(0) = x_1^0, \quad \dot{x}_1(0) = x_2^0, \quad x_1(t_1) = 0, \quad \dot{x}_1(t_1) = 0. \end{aligned} \tag{44}$$

For any (x_1^0, x_2^0), there exists exactly one solution $x_1(t)$ of this problem — a cubic spline. The function $x_2(t)$ is found from the equation $x_2 = \dot{x}_1$.

So through any initial point $x^0 \in \mathbb{R}^2$ passes a unique extremal trajectory arriving at the origin. It is a curve $(x_1(t), x_2(t))$, $t \in [0, t_1]$, where $x_1(t)$ is a cubic polynomial that satisfies the boundary conditions (44), and $x_2(t) = \dot{x}_1(t)$. In view of existence, this is an optimal trajectory.

4.4 Control of a linear oscillator with cost

We control a linear oscillator, say a pendulum with a small amplitude, by an unbounded force u, but take into account expenditure of energy measured by the integral $\frac{1}{2}\int_0^{t_1} u^2(t)\,dt$. The optimal control problem reads

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + u, \end{cases} \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2, \quad u \in \mathbb{R},$$

$$x(0) = x^0, \quad x(t_1) = 0, \quad t_1 \text{ fixed},$$

$$\frac{1}{2}\int_0^{t_1} u^2\,dt \to \min.$$

Existence of optimal control can be proved by the same argument as in the previous section.

The Hamiltonian function of PMP is

$$h_u^\nu(\xi, x) = \xi_1 x_2 - \xi_2 x_1 + \xi_2 u + \frac{\nu}{2}u^2.$$

The corresponding Hamiltonian system yields

$$\begin{cases} \dot{\xi}_1 = \xi_2, \\ \dot{\xi}_2 = -\xi_1. \end{cases}$$

In the same way as in the previous problem, we show that there are no abnormal extremals, thus we can assume $\nu = -1$. Then the maximality condition yields

$$\tilde{u}(t) = \xi_2(t).$$

In particular, optimal control is a harmonic:

$$\tilde{u}(t) = \alpha \sin(t + \beta), \qquad \alpha,\ \beta = \text{const}.$$

The system of ODEs for extremal trajectories

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + \alpha \sin(t + \beta) \end{cases}$$

is solved explicitly:

$$\begin{aligned} x_1(t) &= -\frac{\alpha}{2} t \cos(t + \beta) + a \sin(t + b), \\ x_2(t) &= \frac{\alpha}{2} t \sin(t + \beta) - \frac{\alpha}{2} \cos(t + \beta) + a \cos(t + b), \quad a,\ b \in \mathbb{R}. \end{aligned} \tag{45}$$

Exercise 1. Show that exactly one extremal trajectory of the form (45) satisfies the boundary conditions.

In view of existence, these extremal trajectories are optimal.

4.5 Dubins car

Consider a car moving in the plane. The car can move forward with a fixed linear velocity and simultaneously rotate with a bounded angular velocity. Given initial and terminal position and orientation of the car in the plane, the problem is to drive the car from the initial configuration to the terminal one for a minimal time.

Admissible paths of the car are curves with bounded curvature. Suppose that curves are parametrized by length, then our problem can be stated geometrically. Given two points in the plane and two unit velocity vectors attached respectively at these points, one has to find a curve in the plane that starts at the first point with the first velocity vector and comes to the second point with the second velocity vector, has curvature bounded by a given constant, and has the minimal length among all such curves.

Remark. If curvature is unbounded, then the problem, in general, has no solutions. Indeed, the infimum of lengths of all curves that satisfy the boundary conditions without bound on curvature is the distance between the initial and terminal points: the segment of the straight line through these points can be approximated by smooth curves with the required boundary conditions. But this infimum is not attained when the boundary velocity vectors do not lie on the line through the boundary points and are not collinear one to another.

After rescaling, we obtain a time-optimal problem for a nonlinear system:

$$\begin{cases} \dot{x}_1 = \cos\theta, \\ \dot{x}_2 = \sin\theta, \\ \dot{\theta} = u, \end{cases} \tag{46}$$

$$x = (x_1, x_2) \in \mathbb{R}^2, \quad \theta \in S^1, \quad |u| \le 1,$$
$$x(0),\ \theta(0),\ x(t_1),\ \theta(t_1) \text{ fixed},$$
$$t_1 \to \min.$$

Existence of solutions is guaranteed by Filippov's Theorem. We apply Pontryagin Maximum Principle.

We have $(x_1, x_2, \theta) \in M = \mathbb{R}^2_x \times S^1_\theta$, let (ξ_1, ξ_2, μ) be the corresponding coordinates of the adjoint vector. Then

$$\lambda = (x, \theta, \xi, \mu) \in T^*M,$$

and the control-dependent Hamiltonian is

$$h_u(\lambda) = \xi_1 \cos\theta + \xi_2 \sin\theta + \mu u.$$

The Hamiltonian system of PMP yields

$$\dot{\xi} = 0, \tag{47}$$
$$\dot{\mu} = \xi_1 \sin\theta - \xi_2 \cos\theta, \tag{48}$$

and the maximality condition reads

$$\mu(t)u(t) = \max_{|u|\le 1} \mu(t)u. \tag{49}$$

Equation (47) means that ξ is constant along optimal trajectories, thus the right-hand side of (48) can be rewritten as

$$\xi_1 \sin\theta - \xi_2 \cos\theta = \alpha \sin(\theta + \beta), \qquad \alpha,\ \beta = \text{const}, \quad \alpha = \sqrt{\xi_1^2 + \xi_2^2} \ge 0. \tag{50}$$

So the Hamiltonian system of PMP (46)–(48) yields the following system:

$$\begin{cases} \dot{\mu} = \alpha \sin(\theta + \beta), \\ \dot{\theta} = u. \end{cases}$$

Maximality condition (49) implies that

$$u(t) = \operatorname{sgn} \mu(t) \quad \text{if } \mu(t) \neq 0. \tag{51}$$

If $\alpha = 0$, then $(\xi_1, \xi_2) \equiv 0$ and $\mu = \text{const} \neq 0$, thus $u = \text{const} = \pm 1$. So the curve $x(t)$ is an arc of a circle of radius 1.

Let $\alpha \neq 0$, then in view of (50), we have $\alpha > 0$. Conditions (47), (48), (49) are preserved if the adjoint vector (ξ, μ) is multiplied by any positive constant. Thus we can choose (ξ, μ) such that $\alpha = \sqrt{\xi_1^2 + \xi_2^2} = 1$. That is why we suppose in the sequel that

$$\alpha = 1.$$

Condition (51) means that behavior of sign of the function $\mu(t)$ is crucial for the structure of optimal control. We consider several possibilities for $\mu(t)$.

(0) If the function $\mu(t)$ does not vanish on the segment $[0, t_1]$, then the optimal control is constant:

$$u(t) = \text{const} = \pm 1, \qquad t \in [0, t_1], \tag{52}$$

and the optimal trajectory $x(t)$, $t \in [0, t_1]$, is an arc of a circle. Notice that an optimal trajectory cannot contain a full circle: a circle can be eliminated so that the resulting trajectory satisfy the same boundary conditions and is shorter. Thus controls (52) can be optimal only if $t_1 < 2\pi$.

In the sequel we can assume that the set

$$N = \{\tau \in [0, t_1] \mid \mu(\tau) = 0\}$$

is nonempty. Since N is open, it is a union of open intervals in $[0, t_1]$, plus, may be, semiopen intervals of the form $[0, \tau_1)$, $(\tau_2, t_1]$.

(1) Suppose that the set N contains an interval of the form

$$(\tau_1, \tau_2) \subset [0, t_1], \qquad \tau_1 < \tau_2. \tag{53}$$

We can assume that the interval (τ_1, τ_2) is maximal w.r.t. inclusion:

$$\mu(\tau_1) = \mu(\tau_2) = 0, \qquad \mu|_{(\tau_1, \tau_2)} \neq 0.$$

From PMP we have the inequality

$$h_{u(t)}(\lambda(t)) = \cos(\theta(t) + \beta) + \mu(t) u(t) \geq 0.$$

Thus

$$\cos(\theta(\tau_1) + \beta) \geq 0.$$

This inequality means that the angle

$$\widehat{\theta} = \theta(\tau_1) + \beta$$

satisfies the inclusion

$$\widehat{\theta} \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right).$$

Consider first the case

$$\widehat{\theta} \in \left(0, \frac{\pi}{2}\right].$$

Then $\dot{\mu}(\tau_1) = \sin\widehat{\theta} > 0$, thus at τ_1 control switches from -1 to $+1$, so

$$\dot{\theta}(t) = u(t) \equiv 1, \qquad t \in (\tau_1, \tau_2).$$

We evaluate the distance $\tau_2 - \tau_1$. Since

$$\mu(\tau_2) = \int_{\tau_1}^{\tau_2} \sin(\widehat{\theta} + \tau - \tau_1)\, d\tau = 0,$$

then $\tau_2 - \tau_1 = 2(\pi - \widehat{\theta})$, thus

$$\tau_2 - \tau_1 \in [\pi, 2\pi). \tag{54}$$

In the case

$$\widehat{\theta} \in \left[\frac{3\pi}{2}, 2\pi\right)$$

inclusion (54) is proved similarly, and in the case $\widehat{\theta} = 0$ we obtain no optimal controls (the curve $x(t)$ contains a full circle, which can be eliminated).

Inclusion (54) means that successive roots τ_1, τ_2 of the function $\mu(t)$ cannot be arbitrarily close one to another. Moreover, the previous argument shows that at such instants τ_i optimal control switches from one extremal value to another, and along any optimal trajectory the distance between any successive switchings τ_i, τ_{i+1} is the same.

So in case (1) an optimal control can only have the form

$$u(t) = \begin{cases} \varepsilon, & t \in (\tau_{2k-1}, \tau_{2k}), \\ -\varepsilon, & t \in (\tau_{2k}, \tau_{2k+1}), \end{cases} \tag{55}$$

$$\varepsilon = \pm 1,$$

$$\tau_{i+1} - \tau_i = \text{const} \in [\pi, 2\pi), \qquad i = 1, \ldots, N-1, \tag{56}$$

$$\tau_1 \in (0, 2\pi),$$

here we do not indicate values of u in the intervals before the first switching, $t \in (0, \tau_1)$, and after the last switching, $t \in (\tau_N, t_1)$. For such trajectories, control takes only extremal values ± 1 and the number of switchings is finite on any compact time segment. Such a control is called *bang-bang*.

Controls $u(t)$ given by (55), (56) satisfy PMP for arbitrarily large t, but they are not optimal if the number of switchings is $N > 3$. Indeed, suppose that such a control has at least 4 switchings. Then the piece of trajectory $x(t)$, $t \in [\tau_1, \tau_4]$, is a concatenation of three arcs of circles corresponding to the segments of time $[\tau_1, \tau_2]$, $[\tau_2, \tau_3]$, $[\tau_3, \tau_4]$ with

$$\tau_4 - \tau_3 = \tau_3 - \tau_2 = \tau_2 - \tau_1 \in [\pi, 2\pi).$$

Draw the segment of line

$$\tilde{x}(t), \qquad t \in [(\tau_1 + \tau_2)/2, (\tau_3 + \tau_4)/2], \quad \left| \frac{d\,\tilde{x}}{d\,t} \right| \equiv 1,$$

the common tangent to the first and third circles through the points $x\left((\tau_1 + \tau_2)/2\right)$ and $x\left((\tau_3 + \tau_4)/2\right)$. Then the curve

$$y(t) = \begin{cases} x(t), & t \notin [(\tau_1 + \tau_2)/2, (\tau_3 + \tau_4)/2], \\ \tilde{x}(t), & t \in [(\tau_1 + \tau_2)/2, (\tau_3 + \tau_4)/2], \end{cases}$$

is an admissible trajectory and shorter than $x(t)$. We proved that optimal bang-bang control can have not more than 3 switchings.

(2) It remains to consider the case where the set N does not contain intervals of the form (53). Then N consists of at most two semiopen intervals:

$$N = [0, \tau_1) \cup (\tau_2, t_1], \qquad \tau_1 \leq \tau_2,$$

where one or both intervals may be absent. If $\tau_1 = \tau_2$, then the function $\mu(t)$ has a unique root on the segment $[0, t_1]$, and the corresponding optimal control is determined by condition (51). Otherwise

$$\tau_1 < \tau_2,$$

and

$$\mu|_{[0,\tau_1)} \neq 0, \quad \mu|_{[\tau_1,\tau_2]} \equiv 0, \quad \mu|_{(\tau_2,t_1]} \neq 0. \tag{57}$$

In this case the maximality condition of PMP (51) does not determine optimal control $u(t)$ uniquely since the maximum is attained for more than one value of control parameter u. Such a control is called *singular*. Nevertheless, singular controls in this problem can be determined from PMP. Indeed, the following identities hold on the interval (τ_1, τ_2):

$$\dot{\mu} = \sin(\theta + \beta) = 0 \quad \Rightarrow \quad \theta + \beta = \pi k \quad \Rightarrow \quad \theta = \text{const} \quad \Rightarrow \quad u = 0.$$

Consequently, if an optimal trajectory $x(t)$ has a singular piece, which is a line, then τ_1 and τ_2 are the only switching times of the optimal control. Then

$$u|_{(0,\tau_1)} = \text{const} = \pm 1, \quad u|_{(\tau_2,t_1)} = \text{const} = \pm 1,$$

and the whole trajectory $x(t)$, $t \in [0, t_1]$, is a concatenation of an arc of a circle of radius 1

$$x(t), \quad u(t) = \pm 1, \qquad t \in [0, \tau_1],$$

a line

$$x(t), \quad u(t) = 0, \qquad t \in [\tau_1, \tau_2],$$

and one more arc of a circle of radius 1

$$x(t), \quad u(t) = \pm 1, \qquad t \in [\tau_2, t_1].$$

So optimal trajectories in the problem have one of the following two types:

(1) concatenation of a bang-bang piece (arc of a circle, $u = \pm 1$), a singular piece (segment of a line, $u = 0$), and a bang-bang piece, or

(2) concatenation of bang-bang pieces with not more than 3 switchings, the arcs of circles between switchings having the same central angle $\in [\pi, 2\pi)$.

If boundary points $x(0)$, $x(t_1)$ are sufficiently far one from another, then they can be connected only by trajectories containing singular piece. For such boundary points, we obtain a simple algorithm for construction of an optimal trajectory. Through each of the points $x(0)$ and $x(t_1)$, construct a pair of circles of radius 1 tangent respectively to the velocity vectors $\dot{x}(0) = (\cos\theta(0), \sin\theta(0))$ and $\dot{x}(t_1) = (\cos\theta(t_1), \sin\theta(t_1))$. Then draw common tangents to the circles at $x(0)$ and $x(t_1)$ respectively, so that direction of motion along these tangents was compatible with direction of rotation along the circles determined by the boundary velocity vectors $\dot{x}(0)$ and $\dot{x}(t_1)$. Finally, choose the shortest curve among the candidates obtained. This curve is the optimal trajectory.

5 Linear time-optimal problem

5.1 Problem statement

In this chapter we study the following optimal control problem:

$$\begin{aligned} &\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n, \quad u \in U \subset \mathbb{R}^m, \\ &x(0) = x_0, \quad x(t_1) = x_1, \quad x_0,\ x_1 \in \mathbb{R}^n \text{ fixed}, \\ &t_1 \to \min, \end{aligned} \tag{58}$$

where U is a compact convex polytope in $\mathbb{R}^m$, and A and B are constant matrices of order $n \times n$ and $n \times m$ respectively. Such problem is called *linear time-optimal problem.*

The polytope U is the convex hull of a finite number of points $a_1, \dots, a_k$ in $\mathbb{R}^m$:

$$U = \operatorname{conv}\{a_1, \dots, a_k\}.$$

We assume that the points a_i do not belong to the convex hull of all the rest points a_j, $j \neq i$, so that each a_i is a vertex of the polytope U.

In the sequel we assume the following *General Position Condition*:

For any edge $[a_i, a_j]$ of U, the vector $e_{ij} = a_j - a_i$ satisfies the equality

$$\operatorname{span}(Be_{ij}, ABe_{ij}, \dots, A^{n-1}Be_{ij}) = \mathbb{R}^n. \tag{59}$$

This condition means that no vector Be_{ij} belongs to a proper invariant subspace of the matrix A. This is equivalent to controllability of the linear system $\dot{x} = Ax + Bu$ with the set of control parameters $u \in \mathbb{R}e_{ij}$. Condition (59) can be achieved by a small perturbation of matrices A, B.

We already considered examples of linear time-optimal problems in Sections 4.1, 4.2. Here we study the structure of optimal control, prove its uniqueness, evaluate the number of switchings.

Existence of optimal control for any points x_0, x_1 such that $x_1 \in \mathcal{A}(x_0)$ is guaranteed by Filippov's theorem. Notice that for the analogous problem with an unbounded set of control parameters, optimal control may not exist: it is easy to show this using linearity of the system.

Before proceeding with the study of linear time-optimal problems, we recall some basic facts on polytopes.

5.2 Geometry of polytopes

The convex hull of a finite number of points $a_1, \dots, a_k \in \mathbb{R}^m$ is the set

$$U = \operatorname{conv}\{a_1, \dots, a_k\} \stackrel{\text{def}}{=} \left\{ \sum_{i=1}^{k} \alpha_i a_i \mid \alpha_i \geq 0, \ \sum_{i=1}^{k} \alpha_i = 1 \right\}.$$

An affine hyperplane in $\mathbb{R}^m$ is a set of the form

$$\Pi = \{u \in \mathbb{R}^m \mid \langle \xi, u \rangle = c\}, \qquad \xi \in \mathbb{R}^{m*} \setminus \{0\}, \quad c \in \mathbb{R}.$$

A supporting hyperplane to a polytope U is a hyperplane Π such that

$$\langle \xi, u \rangle \leq c \quad \forall u \in U$$

for the covector ξ and number c that define Π, and this inequality turns into equality at some point $u \in \partial U$, i.e., $\Pi \cap U \neq \emptyset$.

A polytope $U = \operatorname{conv}\{a_1, \dots, a_k\}$ intersects with any its supporting hyperplane $\Pi = \{u \mid \langle \xi, u \rangle = c\}$ by another polytope:

$$\begin{aligned} &U \cap \Pi = \operatorname{conv}\{a_{i1}, \dots, a_{il}\}, \\ &\langle \xi, a_{i1} \rangle = \cdots = \langle \xi, a_{il} \rangle = c, \\ &\langle \xi, a_j \rangle < c, \quad j \notin \{i_1, \dots, i_l\}. \end{aligned}$$

Such polytopes $U \cap \Pi$ are called faces of the polytope U. Zero-dimensional and one-dimensional faces are called respectively vertices and edges. A polytope has a finite number of faces, each of which is the convex hull of a finite number of vertices. A face of a face is a face of the initial polytope. Boundary of a polytope is a union of all its faces. This is a straightforward corollary of the separation theorem for convex sets (or the Hahn-Banach Theorem).

5.3 Bang-bang theorem

Optimal control in the linear time-optimal problem is bang-bang, i.e., it is piecewise constant and takes values in vertices of the polytope U.

Theorem 5. *Let $u(t)$, $0 \leq t \leq t_1$, be an optimal control in the linear time-optimal control problem* (58)*. Then there exists a finite subset*

$$\mathcal{T} \subset [0, t_1], \qquad \#\mathcal{T} < \infty,$$

such that

$$u(t) \in \{a_1, \dots, a_k\}, \qquad t \in [0, t_1] \setminus \mathcal{T}, \tag{60}$$

and restriction $u(t)|_{t \in [0,t_1] \setminus \mathcal{T}}$ is locally constant.

Proof. Apply Pontryagin Maximum Principle to the linear time-optimal problem (58). State vector and adjoint vectors are

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \qquad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^{n*}.$$

The control-dependent Hamiltonian is

$$h_u(\xi, x) = \xi A x + \xi B u$$

(we multiply raws by columns). The Hamiltonian system and maximality condition of PMP take the form:

$$\begin{cases} \dot{x} = Ax + Bu, \\ \dot{\xi} = -\xi A, \end{cases}$$
$$\xi(t) \neq 0,$$
$$\xi(t) B u(t) = \max_{u \in U} \xi(t) B u. \tag{61}$$

The Hamiltonian system implies that adjoint vector

$$\xi(t) = \xi(0) e^{-tA}, \qquad \xi(0) \neq 0, \tag{62}$$

is analytic along the optimal trajectory.

Consider the set of indices corresponding to vertices where maximum (61) is attained:

$$J(t) = \left\{ 1 \leq j \leq k \mid \xi(t) B a_j = \max_{u \in U} \xi(t) B u = \max\{\xi(t) B a_i \mid i = 1, \dots, k\} \right\}.$$

At each instant t the linear function $\xi(t)B$ attains maximum at vertices of the polytope U. We show that this maximum is attained at one vertex always except a finite number of moments.

Define the set

$$\mathcal{T} = \{ t \in [0, t_1] \mid \# J(t) > 1 \}.$$

By contradiction, suppose that $\mathcal{T}$ is infinite: there exists a sequence of distinct moments

$$\{\tau_1, \dots, \tau_n, \dots\} \subset \mathcal{T}.$$

Since there is a finite number of choices for the subset $J(\tau_n) \subset \{1, \dots, k\}$, we can assume, without loss of generality, that

$$J(\tau_1) = J(\tau_2) = \cdots = J(\tau_n) = \cdots .$$

Denote $J = J(\tau_i)$.

Further, since the convex hull

$$\operatorname{conv}\{a_j \mid j \in J\}$$

is a face of U, then there exist indices $j_1, j_2 \in J$ such that the segment $[a_{j_1}, a_{j_2}]$ is an edge of U. We have

$$\xi(\tau_i)Ba_{j_1} = \xi(\tau_i)Ba_{j_2}, \qquad i = 1, 2, \dots .$$

For the vector $e = a_{j_2} - a_{j_1}$ we obtain

$$\xi(\tau_i)Be = 0, \qquad i = 1, 2, \dots .$$

But $\xi(\tau_i) = \xi(0)e^{-\tau_i A}$ by (62), so the analytic function

$$t \mapsto \xi(0)e^{-tA}Be$$

has an infinite number of zeros on the segment $[0, t_1]$, thus it is identically zero:

$$\xi(0)e^{-tA}Be \equiv 0.$$

We differentiate this identity successively at $t = 0$ and obtain

$$\xi(0)Be = 0, \quad \xi(0)ABe = 0, \quad \dots, \quad \xi(0)A^{n-1}Be = 0.$$

By General Position Condition (59), we have $\xi(0) = 0$, a contradiction to (62). So the set $\mathcal{T}$ is finite.

Out of the set $\mathcal{T}$, the function $\xi(t)B$ attains maximum on U at one vertex $a_{j(t)}$, $\{j(t)\} = J(t)$, thus the optimal control $u(t)$ takes value in the vertex $a_{j(t)}$. Condition (60) follows. Further,

$$\xi(t)Ba_{j(t)} > \xi(t)Ba_i, \qquad i \neq j(t).$$

But all functions $t \mapsto \xi(t)Ba_i$ are continuous, so the preceding inequality preserves for instants close to t. The function $t \mapsto j(t)$ is locally constant on $[0, t_1] \setminus \mathcal{T}$, thus the optimal control $u(t)$ is also locally constant on $[0, t_1] \setminus \mathcal{T}$. □

In the sequel we will need the following statement proved in the preceding argument.

Corollary 4. *Let $\xi(t)$, $t \in [0, t_1]$, be a nonzero solution of the adjoint equation $\dot{\xi} = -\xi A$. Then everywhere in the segment $[0, t_1]$, except a finite number of points, there exists a unique control $u(t) \in U$ such that $\xi(t)Bu(t) = \max_{u \in U} \xi(t)Bu$.*

5.4 Uniqueness of optimal controls and extremals

Theorem 6. *Let the terminal point x_1 be reachable from the initial point x_0:*

$$x_1 \in \mathcal{A}(x_0).$$

Then linear time-optimal control problem (58) *has a unique solution.*

Proof. As we already noticed, existence of an optimal control follows from Filippov's Theorem.

Suppose that there exist two optimal controls: $u_1(t)$, $u_2(t)$, $t \in [0, t_1]$. By Cauchy's formula:

$$x(t_1) = e^{t_1 A}\left(x_0 + \int_0^{t_1} e^{-tA} B u(t)\, dt\right),$$

we obtain

$$e^{t_1 A}\left(x_0 + \int_0^{t_1} e^{-tA} B u_1(t)\, dt\right) = e^{t_1 A}\left(x_0 + \int_0^{t_1} e^{-tA} B u_2(t)\, dt\right),$$

thus

$$\int_0^{t_1} e^{-tA} B u_1(t)\, dt = \int_0^{t_1} e^{-tA} B u_2(t)\, dt. \tag{63}$$

Let $\xi_1(t) = \xi_1(0) e^{-tA}$ be the adjoint vector corresponding by PMP to the control $u_1(t)$. Then equality (63) can be written in the form

$$\int_0^{t_1} \xi_1(t) B u_1(t)\, dt = \int_0^{t_1} \xi_1(t) B u_2(t)\, dt. \tag{64}$$

By the maximality condition of PMP

$$\xi_1(t) B u_1(t) = \max_{u \in U} \xi_1(t) B u,$$

thus

$$\xi_1(t) B u_1(t) \geq \xi_1(t) B u_2(t).$$

But this inequality together with equality (64) implies that almost everywhere on $[0, t_1]$

$$\xi_1(t) B u_1(t) = \xi_1(t) B u_2(t).$$

By Corollary 4,

$$u_1(t) \equiv u_2(t)$$

almost everywhere on $[0, t_1]$. □

So for linear time-optimal problem, optimal control is unique. The standard procedure to find the optimal control for a given pair of boundary points x_0, x_1 is to find all extremals $(\xi(t), x(t))$ steering x_0 to x_1 and then to seek for the best among them. In the examples considered in Sections 4.1, 4.2, there was one extremal for each pair x_0, x_1 with $x_1 = 0$. We prove now that this is a general property of linear time-optimal problems.

Theorem 7. *Let $x_1 = 0 \in \mathcal{A}(x_0)$ and $0 \in U \setminus \{a_1, \dots, a_k\}$. Then there exists a unique control $u(t)$ that steers x_0 to 0 and satisfies Pontryagin Maximum Principle.*

Proof. Assume that there exist two controls

$$u_1(t), \quad t \in [0, t_1], \qquad \text{and} \qquad u_2(t), \quad t \in [0, t_2],$$

that steer x_0 to 0 and satisfy PMP.

If $t_1 = t_2$, then the argument of the proof of preceding theorem shows that $u_1(t) \equiv u_2(t)$ a.e., so we can assume that

$$t_1 > t_2.$$

Cauchy's formula gives

$$e^{t_1 A}\left(x_0 + \int_0^{t_1} e^{-tA} B u_1(t)\, dt\right) = 0,$$
$$e^{t_2 A}\left(x_0 + \int_0^{t_2} e^{-tA} B u_2(t)\, dt\right) = 0,$$

thus

$$\int_0^{t_1} e^{-tA} B u_1(t)\, dt = \int_0^{t_2} e^{-tA} B u_2(t)\, dt. \tag{65}$$

According to PMP, there exists an adjoint vector $\xi_1(t)$, $t \in [0, t_1]$, such that

$$\xi_1(t) = \xi_1(0) e^{-tA}, \qquad \xi_1(0) \neq 0, \tag{66}$$
$$\xi_1(t) B u_1(t) = \max_{u \in U} \xi_1(t) B u. \tag{67}$$

Since $0 \in U$, then

$$\xi_1(t) B u_1(t) \geq 0, \qquad t \in [0, t_1]. \tag{68}$$

Equality (65) can be rewritten as

$$\int_0^{t_1} \xi_1(t)Bu_1(t)\,dt = \int_0^{t_2} \xi_1(t)Bu_2(t)\,dt. \tag{69}$$

Taking into account inequality (68), we obtain

$$\int_0^{t_2} \xi_1(t)Bu_1(t)\,dt \le \int_0^{t_2} \xi_1(t)Bu_2(t)\,dt. \tag{70}$$

But maximality condition (67) implies that

$$\xi_1(t)Bu_1(t) \ge \xi_1(t)Bu_2(t), \qquad t \in [0, t_2]. \tag{71}$$

Now inequalities (70) and (71) are compatible only if

$$\xi_1(t)Bu_1(t) = \xi_1(t)Bu_2(t), \qquad t \in [0, t_2],$$

thus inequality (70) should turn into equality. In view of (69), we have

$$\int_{t_1}^{t_2} \xi_1(t)Bu_1(t)\,dt = 0.$$

Since the integrand is nonnegative, see (68), then it vanishes identically:

$$\xi_1(t)Bu_1(t) \equiv 0, \qquad t \in [t_1, t_2].$$

By the argument of Theorem 5, the control $u_1(t)$ is bang-bang, so there exists an interval $I \subset [t_1, t_2]$ such that

$$u_1(t)|_I \equiv a_j \ne 0.$$

Thus

$$\xi_1(t)Ba_j \equiv 0, \qquad t \in I.$$

But $\xi_1(t)0 \equiv 0$, this is a contradiction with uniqueness of the control for which maximum in PMP is obtained, see Corollary 4. □

5.5 Switchings of optimal control

Now we evaluate the number of switchings of optimal control in linear time-optimal problems. In the examples of Sections 4.1, 4.2 we had respectively one switching and an arbitrarily large number of switchings, although finite on any segment. It turns out that in general there are two cases: non-oscillating and oscillating, depending on whether the matrix A of the control system has real spectrum or not. Recall that in the example with one switching, Section 4.1, we had

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \operatorname{Sp}(A) = \{0\} \subset \mathbb{R},$$

and in the example with arbitrarily large number of switchings, Section 4.2,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \operatorname{Sp}(A) = \{\pm i\} \not\subset \mathbb{R}.$$

We consider systems with scalar control:

$$\dot{x} = Ax + ub, \qquad u \in U = [\alpha, \beta] \subset \mathbb{R}, \quad x \in \mathbb{R}^n,$$

under the General Position Condition

$$\operatorname{span}(b, Ab, \dots, A^{n-1}b) = \mathbb{R}^n.$$

Then attainable set of the system is full-dimensional for arbitrarily small times. We can evaluate the minimal number of switchings necessary to fill a full-dimensional domain. Optimal control is piecewise constant with values in $\{\alpha, \beta\}$. Assume that we start from the initial point x_0 with the control α. Without switchings we fill a piece of a 1-dimensional curve $e^{(Ax+\alpha b)t}x_0$, with 1 switching we fill a piece of a 2-dimensional surface $e^{(Ax+\beta b)t_2} \circ e^{(Ax+\alpha b)t_1} x_0$, with 2 switchings we can attain points in a 3-dimensional surface, etc. So the minimal number of switchings required to reach an n-dimensional domain is $n-1$.

We prove now that in the non-oscillating case we never need more than $n-1$ switchings of optimal control.

Theorem 8. *Assume that the matrix A has only real eigenvalues:*

$$\operatorname{Sp}(A) \subset \mathbb{R}.$$

Then any optimal control in linear time-optimal problem (58) *has no more than $n-1$ switchings.*

Proof. Let $u(t)$ be an optimal control and $\xi(t) = \xi(0)e^{-tA}$ the corresponding solution of the adjoint equation $\dot{\xi} = -\xi A$. The maximality condition of PMP reads

$$\xi(t)bu(t) = \max_{u\in[\alpha,\beta]} \xi(t)bu,$$

thus

$$u(t) = \begin{cases} \beta & \text{if } \xi(t)b > 0, \\ \alpha & \text{if } \xi(t)b < 0. \end{cases}$$

So the number of switchings of the control $u(t)$, $t \in [0, t_1]$, is equal to the number of changes of sign of the function

$$y(t) = \xi(t)b, \qquad t \in [0, t_1].$$

We show that $y(t)$ has not more than $n-1$ real roots.

Derivatives of the adjoint vector have the form

$$\xi^{(k)}(t) = \xi(0)e^{-tA}(-A)^k.$$

By Cayley Theorem, the matrix A satisfies its characteristic equation:

$$A^n + c_1 A^{n-1} + \cdots + c_n \operatorname{Id} = 0,$$

where

$$\det(t \operatorname{Id} - A) = t^n + c_1 t^{n-1} + \cdots + c_n,$$

thus

$$(-A)^n - c_1(-A)^{n-1} + \cdots + (-1)^n c_n \operatorname{Id} = 0.$$

Then the function $y(t)$ satisfies an n-th order ODE:

$$y^{(n)}(t) - c_1 y^{(n-1)}(t) + \cdots + (-1)^n c_n y(t) = 0. \tag{72}$$

It is well known that any solution of this equation is a quasipolynomial:

$$y(t) = \sum_{i=1}^{k} e^{-\lambda_i t} P_i(t),$$

$P_i(t)$ a polynomial,

$\lambda_i \neq \lambda_j$ for $i \neq j$, where λ_i are eigenvalues of the matrix A and degree of each polynomial P_i is less than multiplicity of the corresponding eigenvalue λ_i, thus

$$\sum_{i=1}^{k} \deg P_i \leq n - k.$$

Now the statement of this theorem follows from the next general lemma. □

Lemma 2. *A quasipolynomial*

$$y(t) = \sum_{i=1}^{k} e^{\lambda_i t} P_i(t), \qquad \sum_{i=1}^{k} \deg P_i \leq n - k, \tag{73}$$
$$\lambda_i \neq \lambda_j \textit{ for } i \neq j,$$

has no more than $n - 1$ real roots.

Proof. Apply induction on k.

If $k = 1$, then a quasipolynomial

$$y(t) = e^{\lambda t} P(t), \qquad \deg P \leq n - 1,$$

has no more than $n - 1$ roots.

We prove the induction step for $k > 1$. Denote

$$n_i = \deg P_i, \qquad i = 1, \dots, k.$$

Suppose that the quasipolynomial $y(t)$ has n real roots. Rewrite the equation

$$y(t) = \sum_{i=1}^{k-1} e^{\lambda_i t} P_i(t) + e^{\lambda_k t} P_k(t) = 0$$

as follows:

$$\sum_{i=1}^{k-1} e^{(\lambda_i - \lambda_k)t} P_i(t) + P_k(t) = 0. \tag{74}$$

The quasipolynomial in the left-hand side has n roots. We differentiate this quasipolynomial successively $(n_k + 1)$ times so that the polynomial $P_k(t)$ disappear. After $(n_k + 1)$ differentiations we obtain a quasipolynomial

$$\sum_{i=1}^{k-1} e^{(\lambda_i - \lambda_k)t} Q_i(t), \qquad \deg Q_i \leq \deg P_i,$$

which has $(n - n_k - 1)$ real roots by Rolle's Theorem. But by induction assumption the maximal possible number of real roots of this quasipolynomial is

$$\sum_{i=1}^{k-1} n_i + k - 2 < n - n_k - 1.$$

The contradiction finishes the proof of the lemma. □

So we completed the proof of Theorem 8: in the non-oscillating case an optimal control has no more than $n-1$ switchings on the whole domain (recall that $n-1$ switchings are always necessary even on short time segments since the attainable sets $\mathcal{A}_{q_0}(t)$ are full-dimensional for all $t > 0$).

For an arbitrary matrix A, one can obtain the upper bound of $(n-1)$ switchings for sufficiently short intervals of time.

Theorem 9. *Consider the characteristic polynomial of the matrix A:*

$$\det(t \operatorname{Id} - A) = t^n + c_1 t^{n-1} + \cdots + c_n,$$

and let

$$c = \max_{1 \le i \le n} |c_i|.$$

Then for any time-optimal control $u(t)$ and any $\bar{t} \in \mathbb{R}$, the real segment

$$\left[\bar{t}, \bar{t} + \ln\left(1 + \frac{1}{c}\right)\right]$$

contains not more than $(n-1)$ switchings of an optimal control $u(t)$.

In the proof of this theorem we will require the following general proposition, which I learned from S. Yakovenko.

Lemma 3. *Consider an ODE*

$$y^{(n)} + c_1(t) y^{(n-1)} + \cdots + c_n(t) y = 0$$

with measurable and bounded coefficients:

$$c_i = \max_{t \in [\bar{t}, \bar{t}+\delta]} |c_i(t)|.$$

If

$$\sum_{k=1}^{n} c_k \frac{\delta^k}{k!} < 1, \tag{75}$$

then any nonzero solution $y(t)$ of the ODE has not more than $n-1$ roots on the segment $t \in [\bar{t}, \bar{t} + \delta]$.

Proof. By contradiction, suppose that the function $y(t)$ has at least n roots on the segment $t \in [\bar{t}, \bar{t} + \delta]$. By Rolle's Theorem, derivative $\dot{y}(t)$ has not less than $n - 1$ roots, etc. Then $y^{(n-1)}$ has a root $t_{n-1} \in [\bar{t}, \bar{t} + \delta]$. Thus

$$y^{(n-1)}(t) = \int_{t_{n-1}}^{t} y^{(n)}(\tau)\, d\tau.$$

Let $t_{n-2} \in [\bar{t}, \bar{t} + \delta]$ be a root of $y^{(n-2)}(t)$, then

$$y^{(n-2)}(t) = \int_{t_{n-2}}^{t} d\tau_1 \int_{t_{n-1}}^{\tau_1} y^{(n)}(\tau_2)\, d\tau_2.$$

We continue this procedure by integrating $y^{(n-i+1)}(t)$ from a root $t_{n-i} \in [\bar{t}, \bar{t} + \delta]$ of $y^{(n-i)}(t)$ and obtain

$$y^{(n-i)}(t) = \int_{t_{n-i}}^{t} d\tau_1 \int_{t_{n-i+1}}^{\tau_1} d\tau_2 \cdots \int_{t_{n-1}}^{\tau_{i-1}} y^{(n)}(\tau_i)\, d\tau_i, \qquad i = 1, \dots, n.$$

There holds a bound:

$$\begin{aligned}
\left|y^{(n-i)}(t)\right| &\leq \int_{t_{n-i}}^{t} d\tau_1 \int_{t_{n-i+1}}^{\tau_1} d\tau_2 \cdots \int_{t_n-1}^{\tau_{i-1}} \left|y^{(n)}(\tau_i)\right| d\tau_i \\
&\leq \int_{\bar{t}}^{\bar{t}+\delta} d\tau_1 \int_{\bar{t}}^{\tau_1} d\tau_2 \cdots \int_{\bar{t}}^{\tau_{i-1}} \left|y^{(n)}(\tau_i)\right| d\tau_i \leq \frac{\delta^k}{k!} \sup_{t \in [\bar{t}, \bar{t}+\delta]} \left|y^{(n)}(t)\right|.
\end{aligned}$$

Then

$$\left|\sum_{i=1}^{n} c_i(t) y^{(n-i)}(t)\right| \leq \sum_{i=1}^{n} |c_i(t)| \left|y^{(n-i)}(t)\right| \leq \sum_{i=1}^{n} c_i \frac{\delta^k}{k!} \sup_{t \in [\bar{t}, \bar{t}+\delta]} \left|y^{(n)}(t)\right|,$$

i.e.,

$$\left|y^{(n)}(t)\right| \leq \sum_{i=1}^{n} c_i \frac{\delta^k}{k!} \sup_{t \in [\bar{t}, \bar{t}+\delta]} \left|y^{(n)}(t)\right|,$$

a contradiction with (75). The lemma is proved. □

Now we prove Theorem 9.

Proof. As we showed in the proof of Theorem 8, the number of switchings of $u(t)$ is not more than the number of roots of the function $y(t) = \xi(t) b$, which satisfies ODE (72).

We have

$$\sum_{k=1}^{n} |c_k| \frac{\delta^k}{k!} < c(e^\delta - 1) \quad \forall \delta > 0.$$

By Lemma 3, if

$$c(e^\delta - 1) \leq 1, \tag{76}$$

then the function $y(t)$ has not more than $n-1$ real roots on any interval of length δ. But inequality (76) is equivalent to the following one:

$$\delta \leq \ln\left(1 + \frac{1}{c}\right),$$

so $y(t)$ has not more than $n-1$ roots on any interval of the length $\ln\left(1+\frac{1}{c}\right)$. □

6 Linear-quadratic problem

6.1 Problem statement and assumptions

In this chapter we study a class of optimal control problems very popular in applications, *linear-quadratic problems*. That is, we consider linear systems with quadratic cost functional:

$$\begin{aligned} &\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \qquad (77)\\ &x(0) = x_0, \quad x(t_1) = x_1, \qquad x_0, x_1, t_1 \text{ fixed},\\ &J(u) = \frac{1}{2}\int_0^{t_1} \langle Ru(t), u(t)\rangle + \langle Px(t), u(t)\rangle + \langle Qx(t), x(t)\rangle \, dt \to \min. \end{aligned}$$

Here A, B, R, P, Q are constant matrices of appropriate dimensions, R, Q are symmetric:

$$R^* = R, \qquad Q^* = Q,$$

and angle brackets $\langle \cdot, \cdot \rangle$ denote the standard inner product in $\mathbb{R}^m$ and $\mathbb{R}^n$.

One can show that the condition $R \geq 0$ is necessary for existence of optimal control. We do not touch here the case of degenerate R and assume that $R > 0$. The substitution of variables $u \mapsto v = R^{1/2}u$ transforms the functional $J(u)$ to a similar functional with the identity matrix instead of R. That is why we assume in the sequel that $R = \mathrm{Id}$. Another change of variables kills the matrix P (exercise: find this change of variables). So we can write the cost functional as follows:

$$J(u) = \frac{1}{2} \int_0^{t_1} |u(t)|^2 + \langle Qx(t), x(t) \rangle \, dt.$$

For dynamics of the problem, we assume that the linear system is controllable:

$$\operatorname{rank}(B, AB, \dots, A^{n-1}B) = n. \tag{78}$$

6.2 Existence of optimal control

Since the set of control parameters $U = \mathbb{R}^m$ is noncompact, Filippov's Theorem does not apply, and existence of optimal controls in linear-quadratic problems is a nontrivial problem.

In this chapter we assume that admissible controls are square-integrable:

$$u \in L_2^m[0, t_1]$$

and use the L_2^m norm for controls:

$$\|u\| = \left(\int_0^{t_1} |u(t)|^2 \, dt \right)^{1/2} = \left(\int_0^{t_1} u_1^2(t) + \dots + u_m^2(t) \, dt \right)^{1/2}.$$

Consider the set of all admissible controls that steer the initial point to the terminal one:

$$U(x_0, x_1) = \left\{ u \in L_2^m[0, t_1] \mid x(t_1, u, x_0) = x_1 \right\}.$$

We denote by $x(t, u, x_0)$ the trajectory of system (77) corresponding to an admissible control $u \in L_2^m$ starting at a point $x_0 \in \mathbb{R}^n$. By Cauchy's formula, the endpoint mapping

$$u \mapsto x(t_1, u, x_0) = e^{t_1 A} x_0 + \int_0^{t_1} e^{(t_1 - \tau)A} B u(\tau) \, d\tau$$

is an affine mapping from $L_2^m[0, t_1]$ to $\mathbb{R}^n$. Controllability of the linear system (77) means that for any $x_0 \in \mathbb{R}^n$, $t_1 > 0$, the image of the endpoint mapping is the whole $\mathbb{R}^n$. Thus

$$U(x_0, x_1) \subset L_2^m[0, t_1]$$

is an affine subspace,

$$U(0,0) \subset L_2^m[0,t_1]$$

is a linear subspace, and

$$U(x_0,x_1) = u + U(0,0) \quad \text{for any } u \in U(x_0,x_1).$$

Thus it is natural that existence of optimal controls is closely related to behavior of the cost functional $J(u)$ on the linear subspace $U(0,0)$.

Proposition 2. (1) *If there exist points $x_0, x_1 \in \mathbb{R}^n$ such that*

$$\inf_{u \in U(x_0,x_1)} J(u) > -\infty, \tag{79}$$

then

$$J(u) \geq 0 \quad \forall u \in U(0,0).$$

(2) *Conversely, if*

$$J(u) > 0 \quad \forall u \in U(0,0) \setminus 0,$$

then the minimum is attained:

$$\exists \min_{u \in U(x_0,x_1)} J(u) \quad \forall x_0,\ x_1 \in \mathbb{R}^n.$$

Remark. That is, the inequality

$$J|_{U(0,0)} \geq 0$$

is necessary for existence of optimal controls, at least for one pair (x_0, x_1), and the strict inequality

$$J|_{U(0,0)\setminus 0} > 0$$

is sufficient for existence of optimal controls for all pairs (x_0, x_1).

In the proof of Proposition 2, we will need the following auxiliary proposition.

Lemma 4. *If $J(v) > 0$ for all $v \in U(0,0) \setminus 0$, then*

$$J(v) \geq \alpha \|v\|^2 \qquad \textit{for some } \alpha > 0 \textit{ and all } v \in U(0,0),$$

or, which is equivalent,

$$\inf\{J(v) \mid \|v\| = 1,\ v \in U(0,0)\} > 0.$$

Proof. Let v_n be a minimizing sequence of the functional $J(v)$ on the sphere $\{\|v\| = 1\} \cap U(0,0)$. Closed balls in Hilbert spaces are weakly compact, thus we can find a subsequence weakly converging in the unit ball and preserve the notation v_n for its terms, so that

$$\begin{aligned} &v_n \to \widehat{v} \text{ weakly as } n \to \infty, \qquad \|\widehat{v}\| \le 1, \quad \widehat{v} \in U(0,0), \\ &J(v_n) \to \inf\{J(v) \mid \|v\| = 1,\ v \in U(0,0)\}, \qquad n \to \infty. \end{aligned} \tag{80}$$

We have

$$J(v_n) = \frac{1}{2} + \frac{1}{2}\int_0^{t_1} \langle Qx_n(\tau), x_n(\tau)\rangle \, d\tau.$$

Since the controls converge weakly, then the corresponding trajectories converge strongly:

$$x_n(\cdot) \to x_{\widehat{v}}(\cdot), \qquad n \to \infty,$$

thus

$$J(v_n) \to \frac{1}{2} + \frac{1}{2}\int_0^{t_1} \langle Qx_{\widehat{v}}(\tau), x_{\widehat{v}}(\tau)\rangle \, d\tau, \qquad n \to \infty.$$

In view of (80), the infimum in question is equal to

$$\frac{1}{2} + \frac{1}{2}\int_0^{t_1} \langle Qx_{\widehat{v}}(\tau), x_{\widehat{v}}(\tau)\rangle \, d\tau = \frac{1}{2}\left(1 - \|\widehat{v}\|^2\right) + J(\widehat{v}) > 0.$$

□

Now we prove Proposition 2.

Proof. (1) By contradiction, suppose that there exists $v \in U(0,0)$ such that $J(v) < 0$. Take any $u \in U(x_0, x_1)$, then $u + sv \in U(x_0, x_1)$ for any $s \in \mathbb{R}$.

Let $y(t)$, $t \in [0, t_1]$, be the solution to the Cauchy problem

$$\dot{y} = Ay + Bv, \qquad y(0) = 0,$$

and

$$J(u, v) = \frac{1}{2}\int_0^{t_1} \langle u(\tau), v(\tau)\rangle + \langle Qx(\tau), y(\tau)\rangle \, d\tau.$$

Then the quadratic functional J on the family of controls $u + sv$, $s \in \mathbb{R}$, is computed as follows:

$$J(u + sv) = J(u) + 2sJ(u,v) + s^2 J(v).$$

Since $J(v) < 0$, then $J(u + sv) \to -\infty$ as $s \to \infty$. The contradiction with hypothesis (79) finishes the proof of item (1) of this proposition.

(2) We have

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_0^{t_1} \langle Qx(\tau), x(\tau)\rangle \, d\tau.$$

The norm $\|u\|$ is lower semicontinuous in the weak topology on L_2^m, and the functional $\int_0^{t_1} \langle Qx(\tau), x(\tau)\rangle \, d\tau$ is weakly continuous on L_2^m. Thus $J(u)$ is weakly lower semicontinuous on L_2^m. Since balls are weakly compact in L_2^m and the affine subspace $U(x_0, x_1)$ is weakly compact, it is enough to prove that $J(u) \to \infty$ when $u \to \infty$, $u \in U(x_0, x_1)$.

Take any control $u \in U(x_0, x_1)$. Then for any $v \in U(0,0) \setminus 0$, the control $u + v$ belongs to $U(x_0, x_1)$ and

$$J(u + v) = J(u) + 2\|v\| J\left(u, \frac{v}{\|v\|}\right) + J(v).$$

Denote $J(u) = C_0$. Further, $\left| J\left(u, \frac{v}{\|v\|}\right)\right| \leq C_1 = \text{const}$ for all $v \in U(0,0) \setminus 0$. Finally, by Lemma 4, $J(v) \geq \alpha \|v\|^2$, $\alpha > 0$, for all $v \in U(0,0) \setminus 0$. Consequently,

$$J(u + v) \geq C_0 - 2\|v\| C_1 + \alpha \|v\|^2 \to \infty, \qquad v \to \infty, \ v \in U(0,0).$$

Item (2) of this proposition follows. □

So we reduced the question of existence of optimal controls in linear-quadratic problems to the study of the restriction $J|_{U(0,0)}$. We will consider this restriction in detail later.

6.3 Extremals

Now we write PMP for linear-quadratic problems. The control-dependent Hamiltonian is

$$h_u(\xi, x) = \xi A x + \xi B u - \frac{\nu}{2}(\|u\|^2 + \langle Qx, x\rangle), \qquad x \in \mathbb{R}^n, \ \xi \in \mathbb{R}^{n*}.$$

Consider first the abnormal case:

$$\nu = 0.$$

By PMP, adjoint vector along an extremal satisfies the ODE $\dot{\xi} = -\xi A$, thus $\xi(t) = \xi(0)e^{-tA}$. The maximality condition

$$\xi(t)Bu(t) = \max_{u \in \mathbb{R}^n} \xi(t)Bu \tag{81}$$

implies that

$$0 \equiv \xi(t)B = \xi(0)e^{-tA}B.$$

We differentiate this identity $n-1$ times, take into account the controllability condition (78) and obtain $\xi(0) = 0$. This contradicts PMP, thus there are no abnormal extremals.

In the sequel we consider the normal case: $\nu \neq 0$, thus we can assume

$$\nu = 1.$$

Then the control-dependent Hamiltonian takes the form

$$h_u(\xi, x) = \xi A x + \xi B u - \frac{1}{2}(\|u\|^2 + \langle Qx, x\rangle), \qquad x \in \mathbb{R}^n, \ \xi \in \mathbb{R}^{n*}.$$

The term $\xi Bu - \frac{1}{2}\|u\|^2$ depending on u has a unique maximum in $u \in \mathbb{R}^m$ at the point where

$$\frac{\partial h_u}{\partial u} = \xi B - u^* = 0,$$

thus

$$u = B^* \xi^*.$$

So the maximized Hamiltonian is

$$\begin{aligned} H(\xi, x) &= \max_{u \in \mathbb{R}^m} h_u(\xi, x) = \xi A x - \frac{1}{2}\langle Qx, x\rangle + \frac{1}{2}|B^*\xi^*|^2 \\ &= \xi A x - \frac{1}{2}\langle Qx, x\rangle + \frac{1}{2}|B\xi|^2. \end{aligned}$$

The Hamiltonian function $H(\xi, x)$ is smooth, thus extremals are solutions of the corresponding Hamiltonian system

$$\begin{cases} \dot{x} = Ax + BB^*\xi^*, \\ \dot{\xi} = x^*Q - \xi A. \end{cases}$$

6.4 Conjugate points

Now we study conditions of existence and uniqueness of optimal controls depending upon the terminal time. So we write the cost functional to be minimized as follows:

$$J_t(u) = \frac{1}{2}\int_0^t |u(\tau)|^2 + \langle Qx(\tau), x(\tau)\rangle \, d\tau.$$

Denote

$$\begin{aligned} U_t(0,0) &= \{u \in L_2^m[0,t] \mid x(t,u,x_0) = x_1\}, \\ \mu(t) &\stackrel{\text{def}}{=} \inf\{J_t(u) \mid u \in U_t(0,0),\ \|u\| = 1\}. \end{aligned} \tag{82}$$

We showed in Proposition 2 that if $\mu(t) > 0$ then the problem has solution for any boundary conditions, and if $\mu(t) < 0$ then there are no solutions for any boundary conditions. The case $\mu(t) = 0$ is doubtful. Now we study properties of the function $\mu(t)$ in detail.

Proposition 3. (1) *The function $t \mapsto \mu(t)$ is monotone nonincreasing and continuous.*

(2)

$$1 \geq 2\mu(t) \geq 1 - \frac{t^2}{2} e^{2t\|A\|}\|B\|^2\|Q\|. \tag{83}$$

(3) *If $1 > 2\mu(t)$, then the infimum in* (82) *is attained, i.e., it is minimum.*

Proof. (3) Denote

$$I_t(u) = \frac{1}{2}\int_0^t \langle Qx(\tau), x(\tau)\rangle \, d\tau,$$

the functional $I_t(u)$ is weakly continuous on L_2^m. Notice that

$$J_t(u) = \frac{1}{2} + I_t(u) \quad \text{on the sphere } \|u\| = 1.$$

Take a minimizing sequence of the functional $I_t(u)$ on the sphere $\{\|u\| = 1\} \cap U_t(0,0)$. Since the ball $\{\|u\| \leq 1\}$ is weakly compact, we can find a weakly converging subsequence:

$$\begin{aligned} &u_n \to \widehat{u} \text{ weakly as } n \to \infty, \qquad \|\widehat{u}\| \leq 1, \quad \widehat{u} \in U_t(0,0), \\ &I_t(u_n) \to I_t(\widehat{u}) = \inf\{I_t(u) \mid \|u\| = 1,\ u \in U_t(0,0)\}, \qquad n \to \infty. \end{aligned}$$

If $\widehat{u} = 0$, then $I_t(\widehat{u}) = 0$, thus $\mu(t) = \frac{1}{2}$, which contradicts hypothesis of item (3).

So $\widehat{u} \neq 0$, $I_t(\widehat{u}) < 0$, and $I_t\left(\frac{\widehat{u}}{\|\widehat{u}\|}\right) \leq I_t(\widehat{u})$. Thus $\|\widehat{u}\| = 1$, and $J_t(u)$ attains minimum on the sphere $\{\|u\| = 1\} \cap U_t(0,0)$ at the point $\widehat{u}$.

(2) Let $\|u\| = 1$ and $x_0 = 0$. By Cauchy's formula,

$$x(t) = \int_0^t e^{(t-\tau)A} Bu(\tau)\, d\tau,$$

thus

$$|x(t)| \leq \int_0^t e^{(t-\tau)\|A\|} \|B\| \cdot |u(\tau)|\, d\tau$$

by Cauchy-Schwartz inequality

$$\begin{aligned} &\leq \|u\| \left(\int_0^t e^{(t-\tau)2\|A\|} \|B\|^2\, d\tau \right)^{1/2} \\ &= \left(\int_0^t e^{(t-\tau)2\|A\|} \|B\|^2\, d\tau \right)^{1/2}. \end{aligned}$$

We substitute this estimate of $x(t)$ into J_t and obtain the second inequality in (83).

The first inequality in (83) is obtained by considering a weakly converging sequence $u_n \to 0$, $n \to \infty$, in the sphere $\|u_n\| = 1$, $u_n \in U_t(0,0)$.

(1) Monotonicity of $\mu(t)$. Take any $\hat{t} > t$. Then the space $U_t(0,0)$ is isometrically embedded into $U_{\hat{t}}(0,0)$ by extending controls $u \in U_t(0,0)$ by zero:

$$\begin{aligned} &u \in U_t(0,0) \quad \Rightarrow \quad \widehat{u} \in U_{\hat{t}}(0,0), \\ &\widehat{u}(\tau) = \begin{cases} u(\tau), & \tau \leq t, \\ 0, & \tau > t. \end{cases} \end{aligned}$$

Moreover,

$$J_{\hat{t}}(\widehat{u}) = J_t(u).$$

Thus

$$\begin{aligned} \mu(t) &= \inf\{J_t(u) \mid u \in U_t(0,0), \|u\| = 1\} \\ &\geq \inf\{J_{\hat{t}}(u) \mid u \in U_{\hat{t}}(0,0), \|u\| = 1\} = \mu(\hat{t}). \end{aligned}$$

Continuity of $\mu(t)$: we show separately continuity from the right and from the left.

Continuity from the right. Let $t_n \searrow t$. We can assume that $\mu(t_n) < \frac{1}{2}$ (otherwise $\mu(t_n) = \mu(t) = \frac{1}{2}$), thus minimum in (82) is attained:

$$\mu(t_n) = \frac{1}{2} + I_{t_n}(u_n), \qquad u_n \in U_{t_n}(0,0), \quad \|u_n\| = 1.$$

Extend the functions $u_n \in L_2^m[0, t_n]$ to the segment $[0, t]$ by zero. Choosing a weakly converging subsequence in the unit ball, we can assume that

$$u_n \to u \text{ weakly as } n \to \infty, \qquad u \in U_t(0,0), \quad \|u_n\| \leq 1,$$

thus

$$I_{t_n}(u_n) \to I_t(u) \geq \inf\{I_t(v) \mid v \in U_t(0,0), \quad \|v\| = 1\}, \quad t_n \searrow t.$$

Then

$$\mu(t) \leq \frac{1}{2} + \lim_{t_n \searrow t} I_{t_n}(u_n) = \lim_{t_n \searrow t} \mu(t_n).$$

By monotonicity of μ,

$$\mu(t) = \lim_{t_n \searrow t} \mu(t_n),$$

i.e., continuity from the right is proved.

Continuity from the left. We can assume that $\mu(t) < \frac{1}{2}$ (otherwise $\mu(\tau) = \mu(t) = \frac{1}{2}$ for $\tau < t$). Thus minimum in (82) is attained:

$$\mu(t) = \frac{1}{2} + I_t(\widehat{u}), \qquad \widehat{u} \in U_t(0,0), \quad \|\widehat{u}\| = 1.$$

For the trajectory

$$\widehat{x}(\tau) = x(\tau, \widehat{u}, 0),$$

we have

$$\widehat{x}(\tau) = \int_0^\tau e^{(\tau-\theta)A} B\widehat{u}(\theta)\, d\theta.$$

Denote

$$\alpha(\varepsilon) = \| \, \widehat{u}|_{[0,\varepsilon]} \, \|$$

and notice that

$$\alpha(\varepsilon) \to 0, \qquad \varepsilon \to 0.$$

Denote the ball

$$B_\delta = \{u \in L_2^m \mid \|u\| \leq \delta, \; u \in U(0,0)\}.$$

Obviously,

$$x(\varepsilon, B_{\alpha(\varepsilon)}, 0) \ni \widehat{x}(\varepsilon).$$

The mapping $u \mapsto x(\varepsilon, u(\cdot), 0)$ from L_2^m to $\mathbb{R}^n$ is linear, and the system $\dot{x} = Ax + Bu$ is controllable, thus $x(\varepsilon, B_{\alpha(\varepsilon)}, 0)$ is a convex full-dimensional set in $\mathbb{R}^n$ such that the positive cone generated by this set is the whole $\mathbb{R}^n$. That is why

$$x(\varepsilon, 2B_{\alpha(\varepsilon)}, 0) = 2x(\varepsilon, B_{\alpha(\varepsilon)}, 0) \supset O_{x(\varepsilon, B_{\alpha(\varepsilon)}, 0)}$$

for some neighborhood $O_{x(\varepsilon, B_{\alpha(\varepsilon)}, 0)}$ of the set $x(\varepsilon, B_{\alpha(\varepsilon)}, 0)$. Further, there exists an instant $t_\varepsilon > \varepsilon$ such that

$$\widehat{x}(t_\varepsilon) \in x(\varepsilon, 2B_{\alpha(\varepsilon)}, 0),$$

consequently,

$$\widehat{x}(t_\varepsilon) = x(\varepsilon, v_\varepsilon, 0), \qquad \|v_\varepsilon\| \leq 2\alpha(\varepsilon).$$

Consider the following family of controls that approximate $\widehat{u}$:

$$u_\varepsilon(\tau) = \begin{cases} v_\varepsilon(\tau), & 0 \leq \tau \leq t_\varepsilon, \\ \widehat{u}(\tau + t_\varepsilon - \varepsilon), & t_\varepsilon < \tau \leq t + \varepsilon - t_\varepsilon. \end{cases}$$

We have

$$\begin{aligned} & u_\varepsilon \in U_{t+\varepsilon-t_\varepsilon}(0,0), \\ & \|\widehat{u} - u_\varepsilon\| \to 0, \qquad \varepsilon \to 0. \end{aligned}$$

But $t + \varepsilon - t_\varepsilon < t$ and μ is nonincreasing, thus it is continuous from the left.

Continuity from the right was already proved, hence μ is continuous. □

Now we prove that the function μ can have not more than one root.

Proposition 4. *If $\mu(t) = 0$ for some $t > 0$, then $\mu(\tau) < 0$ for all $\tau > t$.*

Proof. Let $\mu(t) = 0$, $t > 0$. By Proposition 3, infimum in (82) is attained at some control $\widehat{u} \in U_t(0,0)$, $\|\widehat{u}\| = 1$:

$$\begin{aligned} \mu(t) &= \min\{J_t(u) \mid u \in U_t(0,0),\ \|u\| = 1\} \\ &= J_t(\widehat{u}) = 0. \end{aligned}$$

Then

$$J_t(u) \geq J_t(\widehat{u}) = 0 \quad \forall u \in U_t(0,0),$$

i.e., the control $\widehat{u}$ is optimal, thus it satisfies PMP. There exists a solution $(\xi(\tau), x(\tau))$, $\tau \in [0, t]$, of the Hamiltonian system

$$\begin{cases} \dot{\xi} = x^* Q - \xi A, \\ \dot{x} = Ax + BB^* \xi, \end{cases}$$

with the boundary conditions

$$x(0) = x(t) = 0,$$

and

$$u(\tau) = B^* \xi^*(\tau), \qquad \tau \in [0, t].$$

We proved that for any root t of the function μ, any control $u \in U_t(0,0)$, $\|u\| = 1$, with $J_t(u) = 0$ satisfies PMP.

Now we prove that $\mu(\tau) < 0$ for all $\tau > t$. By contradiction, suppose that the function μ vanishes at some instant $t' > t$. Since μ is monotone, then

$$\mu|_{[t,t']} \equiv 0.$$

Consequently, the control

$$u'(\tau) = \begin{cases} \widehat{u}(\tau), & \tau \le t, \\ 0, & \tau \in [t, t'], \end{cases}$$

satisfies the conditions:

$$u' \in U_{t'}(0,0), \quad \|u'\| = 1,$$
$$J_{t'}(u') = 0.$$

Thus u' satisfies PMP, i.e.,

$$u'(\tau) B^* \xi^{*'}(\tau), \qquad \tau \in [0, t'],$$

is an analytic function. But $u'|_{[t,t']} \equiv 0$, thus $u' \equiv 0$, a contradiction with $\|u'\| = 1$. □

It would be nice to have a way to solve the equation $\mu(t) = 0$ without performing the minimization procedure in (82). This can be done in terms of the following notion.

Definition 1. A point $t > 0$ is *conjugate* to 0 for the linear-quadratic problem in question if there exists a nontrivial solution $(\xi(\tau), x(\tau))$ of the Hamiltonian system

$$\begin{cases} \dot{\xi} = x^* Q - \xi A, \\ \dot{x} = Ax + BB^* \xi \end{cases}$$

such that $x(0) = x(t) = 0$.

Proposition 5. *The function μ vanishes at a point $t > 0$ if and only if t is the closest to 0 conjugate point.*

Proof. Let $\mu(t) = 0$, $t > 0$. First of all, t is conjugate to 0, we showed this in the proof of Proposition 4.

Suppose that $t' > 0$ is conjugate to 0. Compute the functional $J_{t'}$ on the corresponding control $u(\tau) = B^* \xi^*(\tau)$, $\tau \in [0, t']$:

$$\begin{aligned} J_{t'}(u) &= \frac{1}{2} \int_0^{t'} \langle B^* \xi^*(\tau), B^* \xi^*(\tau) \rangle + \langle Qx(\tau), x(\tau) \rangle \, d\tau \\ &= \frac{1}{2} \int_0^{t'} \langle BB^* \xi^*(\tau), \xi^*(\tau) \rangle + \langle Qx(\tau), x(\tau) \rangle \, d\tau \\ &= \frac{1}{2} \int_0^{t'} \xi(\tau)(\dot{x}(\tau) - Ax(\tau)) + x^*(\tau) Q x(\tau) \, d\tau \\ &= \frac{1}{2} \int_0^{t'} (\xi \dot{x} + \dot{\xi} x) \, d\tau \\ &= \frac{1}{2} (\xi(t') x(t') - \xi(0) x(0)) = 0. \end{aligned}$$

Thus $\mu(t') \le J_{t'}\left(\frac{u}{\|u\|}\right) = 0$. Now the result follows since μ is nonincreasing. □

The first (closest to zero) conjugate point determines existence and uniqueness properties of optimal control in linear-quadratic problems.

Before the first conjugate point, optimal control exists and is unique for any boundary conditions (if there are two optimal controls, then their difference gives rise to a conjugate point).

At the first conjugate point, there is existence and nonuniqueness for some boundary conditions, and nonexistence for other boundary conditions.

And after the first conjugate point, the problem has no optimal solutions for any boundary conditions.

Exercises

1. Optimal U-turn of the Dubins car.
Consider the system

$$\begin{cases} \dot{x}^1 &= \cos\theta \\ \dot{x}^2 &= \sin\theta \\ \dot{\theta} &= u \end{cases} \qquad |u| \leq 1.$$

Find a time-optimal control and trajectory for the boundary conditions: $z(0) = (0,0,0)$, $z(t_1) = (0,0,\pi)$, where $z = (x^1, x^2, \theta)$.

2. Time-optimal stabilization of the oscillator with friction.
Consider the system

$$\begin{cases} \dot{x}^1 &= x^2 \\ \dot{x}^2 &= -x^1 - kx^2 + u \end{cases} \qquad |u| \leq 1.$$

Design a time-optimal synthesis with the target $(x^1, x^2) = (0,0)$ for any friction coefficient $k > 0$.

3. Conjugate points.
Consider the following linear-quadratic problem:

$$\min \int_0^T (u^2(t) - x^2(t))\, dt, \quad \ddot{x} = u.$$

Find an approximate value (up to 0.01) of the nearest to zero conjugate point.

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