which contradicts the exponential character of the increasing function H(s). The theorem is proved.

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CURVATURE OF A DISTRIBUTION

A. F. Solov'ev

We introduce the concept of curvature of a distribution on a Riemannian manifold.† Some special properties of the curvature of the horizontal distribution of a Riemannian submersion, a horizontal distribution on the tangent bundle, provided with a Sasaki metric, and left-invariant distributions on a Lie group with invariant Riemannian metric are obtained.

1. Curvature of a Distribution

We consider a Riemannian manifold M with metric tensor $\langle,\rangle$ and Levi-Civita connection ∇. For any differentiable distribution Δ without singularities on M, we denote by $\Delta^{\perp}$ its orthogonal complement, and by H and $H^{\perp}$ the orthogonal projectors onto Δ and $\Delta^{\perp}$, respectively. Let $\mathfrak{X}(M)$ be the set of differentiable vector fields on M. All differentiable objects considered are assumed to be of class C^{∞}.

By the induced connection of the distribution Δ is meant the linear connection $\bar{\nabla}_X Y = H\nabla_X HY + H^{\perp}\nabla_X H^{\perp}Y$, and its second fundamental form [1, 2] is the tensor field $h = \nabla - \bar{\nabla}$. Let h^+ and h^- be the symmetric and skew-symmetric parts of the field h, respectively.

Proposition 1.1 [2]. The distribution Δ on the Riemannian manifold M is completely geodesic (respectively involutive), if and only if $h^+(HX, HY) = 0$ (respectively, $h^-(HX, HY) = 0$) for any $X, Y \in \mathfrak{X}(M)$.

A diffeomorphism $f: M \to M^*$ of Riemannian manifolds is called a Δ-isometry [1, 2], where Δ is some distribution on M, if $\langle f_*X, f_*Y\rangle^* \circ f = \langle X, Y\rangle$ for any vector fields $X, Y \in \Delta$ and $f_*(\Delta^{\perp}) = (f_*\Delta)^{\perp}$. We set $W_* = f_*W$ for any $W \in \mathfrak{X}(M)$.

LEMMA 1.2 [1]. If $f: M \to M^*$ is a Δ-isometry, T and T* are the torsion tensors of the induced connections $\bar{\nabla}$ and $\bar{\nabla}^*$ of the distributions Δ and $f_*\Delta$ respectively, then

$$\{\langle\bar{\nabla}^*_{X_*}Y_*, Z_*\rangle^* - (1/2)\langle X_*, T^*(Y_*, Z_*)\rangle^*\} \circ f = \langle\bar{\nabla}_X Y, Z\rangle - (1/2)\langle X, T(Y, Z)\rangle$$

for any vector fields $Y, Z \in \Delta$ and X on M.

Hence on a Riemannian manifold M with a distribution Δ given on it, we define a new linear connection D, by setting

$$\langle D_X HY, Z\rangle = \langle\bar{\nabla}_X HY, HZ\rangle - (1/2)\langle X, T(HY, HZ)\rangle$$

and $D_X H^{\perp}Y$ arbitrarily for any $X, Y, Z \in \mathfrak{X}(M)$. If $\hat{K}$ is a curvature tensor of this connection, then $K = \hat{K}(,) \circ H$ will be called the curvature tensor of the distribution Δ. By Lemma 1.2, for any given Δ-isometry f, the curvature tensors of the distributions Δ and $f_*\Delta$ correspond. Let T and $\bar{R}$ be the torsion and curvature tensors of the connection $\bar{\nabla}$. Since

†Its connection with the familiar concept of curvature of nonholonomic manifolds is indicated at the end of the present paper.

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$$T(X, Y) = -H^{\perp}[X, Y] = -2h^{-}(X, Y), \quad X, Y \in \Delta, \tag{1.1}$$

by definition one has

$$\langle K(X, Y) Z, W\rangle = \langle \bar{R}(X, Y) Z, W\rangle - (1/2)\langle T(X, Y), T(Z, W)\rangle \tag{1.2}$$

for any $X, Y, Z, W \in \Delta$ and hence for such vector fields we have

$$\langle K(X, Y) Z, W\rangle = \langle R(X, Y) Z, W\rangle - 2\langle h^{-}(X, Y), h^{-}(Z, W)\rangle + \langle h(X, W), h(Y, Z)\rangle - \langle h(Y, W), h(X, Z)\rangle, \tag{1.3}$$

where R is the curvature tensor of the Riemannian manifold. (1.3) completely determines the value of $K(HX, HY)HZ$, since Δ is parallel with respect to D, and consequently, $K(X, Y)HZ \in \Delta$ for any $X, Y, Z \in \mathfrak{X}(M)$. It can be considered as an analog of the Gauss equation of a submanifold.

We consider at a point $p \in M$ the two-dimensional plane $x \wedge y \subset \Delta_p$. The quantity $K_{xy} = \langle K(x, y) y, x\rangle \| x \wedge y \|^{-2}$, where $\| x \wedge y \|^2 = \| x \|^2 \| y \|^2 - \langle x, y\rangle^2$, is independent of the choice of basis {x, y} of this area element. We call it the sectional curvature of the distribution Δ at the point p. The sectional curvature of a two-dimensional distribution will be called its Gaussian curvature. The sum $k_x = \sum_a K_{xe_a}$ is independent of the choice of tangent vectors e_α, which together with x form an orthogonal basis of the subspace $\Delta_p \subset M_p$. We call k_x the Ricci curvature of the distribution Δ at the point p. Analogously, the number S = $\sum_i k_{e_i}$ is independent of the choice of orthogonal basis $\{e_i\}$ of the subspace Δ_p; we call S the scalar curvature of the distribution Δ at the point p. By definition the sectional, Ricci, and scalar curvatures of the distribution Δ are invariant with respect to any Δ-isometry.

For the geometric interpretation of the curvature of a distribution, we consider its sectional torsion, defined in [1] by $t_{xy} = \| T(x, y)\|^2 \| x \wedge y \|^{-2}$, $x \wedge y \subset \Delta_p$, where T is the torsion tensor of the connection $\bar{\nabla}$. Let U be the domain of definition of the exponential map $\overline{\exp}\,|_p$ of this connection. The submanifold $\delta(p) = \overline{\exp}(U \cap \Delta_p)$ is called the osculating geodesic surface [1] of the distribution Δ at the point p (in [1] the formulation of the definition of this surface contains a misprint). One should note that $D_X Y = \bar{\nabla}_X Y$ for any $X, Y \in \Delta$. Hence in the definitions of sectional torsion and osculating geodesic surface of the distribution Δ one can replace the connection $\bar{\nabla}$ by D.

THEOREM 1.3. Let K and t be the sectional curvature and torsion of the distribution Δ on a Riemannian manifold, and $K^{(1)}$ be the sectional curvature of the osculating geodesic surface $\delta(p)$ of this distribution. Then for any element of area $x \wedge y \subset \Delta_p$

$$K_{xy} = K^{(1)}_{xy} + (3/4)\, t_{xy}.$$

Considering the torsion in a one-dimensional direction and the scalar torsion of the distribution, one can give the analogous interpretation as the Ricci curvature and scalar curvature.

Proof. Let $\tilde{h}$ and h be the second fundamental forms of the surface δ(p) and the distribution Δ. It is shown in [1] that $\tilde{h}(x, y) = h^{+}(x, y)$ for any $x, y \in \Delta_p$. Hence, as follows from the Gauss equation of the submanifold δ(p) (see, e.g., [3]),

$$K^{(1)}_{xy} = R_{xy} + \{\langle h^{+}(x, x), h^{+}(y, y)\rangle - \| h^{+}(x, y)\|^2\} \| x \wedge y \|^{-2},$$

where $x \wedge y \subset \Delta_p$ and R_{xy} is the sectional curvature of the Riemannian manifold. On the other hand, by (1.3),

$$K_{xy} = R_{xy} + \{2\| h^{-}(x, y)\|^2 + \langle h(x, x), h(y, y)\rangle - \langle h(x, y), h(y, x)\rangle\} \| x \wedge y \|^{-2}, \tag{1.4}$$

$x \wedge y \subset \Delta_p$. Consequently, $K_{xy} = K^{(1)}_{xy} + 3\| h^{-}(x, y)\|^2 \cdot \| x \wedge y \|^{-2} = K^{(1)}_{xy} + (3/4)\, t_{xy}$. The theorem is proved.

If Δ is completely geodesic, then $K^{(1)}_{xy} = R_{xy}$ and hence

$$K_{xy} = R_{xy} + (3/4)\, t_{xy}, \quad x \wedge y \subset \Delta_p. \tag{1.5}$$

Let $K_{xy}^{(2)}$ be the sectional curvature of the distribution Δ, defined in the usual way in terms of the curvature tensor $\bar{R}$ of the induced connection $\bar{\nabla}$. Then $K_{xy}^{(2)} = K_{xy}^{(1)} + (1/4)\, t_{xy}$, $x \wedge y \subset \Delta$ (see [1]), and consequently, the sectional curvature K of the distribution Δ is connected with its sectional curvatures of the first and second kinds $K^{(1)}$ and $K^{(2)}$ [1] by the relation

$$K_{xy} = 3K_{xy}^{(2)} - 2K_{xy}^{(1)}, \quad x \wedge y \subset \Delta_p. \tag{1.6}$$

2. Horizontal Distribution of a Riemannian Submersion

Let M and B be Riemannian manifolds. A differentiable surjective map $\pi: M \to B$ is called a Riemannian submersion [4, 5], if π has maximal rank and $\pi_* |_{(\ker \pi_*)^\perp}$ is a linear isometry. The involutive distribution on M, whose maximal integral manifolds are the fibers $\pi^{-1}(b)$, $b \in B$, is called the vertical, and its orthogonal complement the horizontal distributions [4] of the Riemannian submersion. We denote these distributions by V(M) and H(M), and the orthogonal projectors onto V(M) and H(M) by V and H, respectively. Let $\langle , \rangle$ and $\langle , \rangle^*$ be the Riemannian metrics on M and B, and ∇ and ∇* be their Levi–Civita connections. A vector field X on M is called basic [4], if it is horizontal and π-connected with some vector field X_* on B.

LEMMA 2.1 [4]. If X and Y are basic vector fields, then

1) $\langle X, Y\rangle = \langle X_*, Y_*\rangle^* \circ \pi$;

2) H[X, Y] is basic and $\pi_* \{H[X, Y]\} = [X_*, Y_*]$;

3) $H\nabla_X Y$ is basic and $\pi_* \{H\nabla_X Y\} = \nabla^*_{X_*} Y_*$.

Assertion 3) can be formulated as follows:

3') $\bar{\nabla}_X Y$ is basic and $\pi_* \bar{\nabla}_X Y = \nabla^*_{X_*} Y_*$, where $\bar{\nabla}$ is the induced connection of the horizontal distribution.

It is shown in [4] that for any vector fields $X, Y \in H(M)$

$$V\nabla_X Y = (1/2)\, V[X, Y]. \tag{2.1}$$

Hence, in particular, the horizontal distribution H(M) is completely geodesic (see [2]).

THEOREM 2.2. Let K be the curvature tensor of the horizontal distribution of the Riemannian submersion $\pi: M \to B$ and R* be the curvature tensor of the Riemannian manifold B. Then for any basic vector fields X, Y, Z, the vector field K(X, Y)Z is basic and

$$\pi_* K(X, Y) Z = R^*(X_*, Y_*) Z_*.$$

Proof. We consider the value $\bar{R}(X, Y) Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X, Y]} Z$ of the curvature tensor of the induced connection $\bar{\nabla}$ of the distribution H(M) for basic X, Y, Z. According to Lemma 2.1, the vector fields $\bar{\nabla}_X \bar{\nabla}_Y Z$, $\bar{\nabla}_{H[X, Y]} Z$, and consequently, $\bar{R}(X, Y) Z + \bar{\nabla}_{V[X, Y]} Z$ are basic, while $\pi_* \{\bar{R}(X, Y) Z + \bar{\nabla}_{V[X, Y]} Z\} = R^*(X_*, Y_*) Z_*$. Hence due to (1.2) and (1.1), it remains to prove $\langle \nabla_{V[X, Y]} Z, W\rangle = -(1/2) \langle V[X, Y], V[Z, W]\rangle$ for any basic X, Y, Z, W. If Z is basic, then $\pi_* [V[X, Y], Z] = 0$, $H\nabla_{V[X, Y]} Z = H\nabla_Z V[X, Y]$ and consequently, $\langle \nabla_{V[X, Y]} Z, W\rangle = \langle \nabla_Z V[X, Y], W\rangle = -\langle \nabla_Z W, V[X, Y]\rangle = -(1/2) \langle V[Z, W], V[X, Y]\rangle$ for such X, Y, Z, W by (2.1). The theorem is proved.

COROLLARY 2.3. Let T be the torsion tensor of the induced connection of the horizontal distribution of the Riemannian submersion $\pi: M \to B$, R and R* be the curvature tensors of the Riemannian manifolds M and B respectively. Then for any basic X, Y, Z, and W,

$$\langle R^*(X_*, Y_*) Z_*, W_*\rangle^* \circ \pi = \langle R(X, Y) Z, W\rangle - (1/2) \langle T(X, Y), T(Z, W)\rangle + (1/4) \langle T(X, W), T(Y, Z)\rangle + (1/4) \langle T(Y, W), T(Z, X)\rangle. \tag{2.2}$$

Proof. Since $h(X, Y) = h^-(X, Y)$ for any $X, Y \in H(M)$, from (1.3) and (1.1) we have

$$\langle K(X, Y) Z, W\rangle = \langle R(X, Y) Z, W\rangle - (1/2) \langle T(X, Y), T(Z, W)\rangle + (1/4) \langle T(X, W), T(Y, Z)\rangle + (1/4) \times \langle T(Y, W), T(Z, X)\rangle \quad \text{for any} \quad X, Y, Z, W \in H(M).$$

Now one should apply Theorem 2.2 and Lemma 2.1. The corollary is proved.

THEOREM 2.4. Let K be the sectional curvature of the horizontal distribution of the Riemannian submersion $\pi: M \to B$ and R* be the sectional curvature of the Riemannian manifold B. Then for any basic X and Y, $\| X \wedge Y \| \neq 0$, we have

$$R^*_{X_* Y_*} \circ \pi = K_{XY}.$$

The proof follows from Theorem 2.2 and Lemma 2.1.

By virtue of Theorem 2.4, under the map $\pi: M \to B$ the Ricci curvatures (and scalar curvatures) of the horizontal distribution on M and of the Riemannian manifold B also correspond. According to the same theorem and (1.5), one has

COROLLARY 2.5. If t is the sectional torsion of the horizontal distribution of the Riemannian submersion $\pi: M \to B$, R and R* are the sectional curvatures of the Riemannian manifolds M and B, respectively, then for any basic X and Y, $\| X \wedge Y \| \neq 0$,

$$R^*_{X_* Y_*} \circ \pi = R_{XY} + (3/4)\, t_{XY}. \tag{2.3}$$

Equations of the form (2.2) and (2.3) were previously obtained in [4] and [5].

3. Horizontal Distribution on the Tangent Bundle

Let M be a Riemannian manifold, TM be its Tangent bundle, and $\pi: TM \to M$ be the canonical projection. We consider on M the Levi–Civita connection ∇^* and the corresponding connection map $\mathscr{K}: TTM \to TM$ in the sense of [6]. It is linear on each fiber $(TM)_\xi$ of the bundle TTM. The kernel of the map $\mathscr{K}|_{(TM)_\xi}$ is called the horizontal subspace† [6] of the space $(TM)_\xi$. We denote the horizontal† distribution on TM by H(TM). Let X^V and X^H respectively denote the vertical and horizontal lifts of the field $X \in \mathfrak{X}(M)$ on TM (see [6]). For a given Riemannian metric $\langle , \rangle^*$ on M we consider on TM the Sasaki metric defined by

$$\langle A, B \rangle = \langle \pi_* A, \pi_* B \rangle^* \circ \pi + \langle \mathscr{K} A, \mathscr{K} B \rangle^* \circ \pi,$$

$A, B \in (TM)_\xi$, $\xi \in TM$. Let ∇ be its Levi–Civita connection. Then

$$(\nabla_{X^V} Y^H)_\xi = (1/2)\,(R^*(\xi, X_p)\, Y_p)^H_\xi,$$
$$(\nabla_{X^H} Y^H)_\xi = (\nabla^*_X Y)^H_\xi - (1/2)\,(R^*(X_p, Y_p)\,\xi)^V_\xi, \tag{3.1}$$

where $\xi \in TM$, $p = \pi(\xi)$ and R* is the curvature tensor for ∇^* (see [7]).

THEOREM 3.1. Let M be a Riemannian manifold and H(TM) be the horizontal distribution on the tangent bundle TM with the Sasaki metric. Then the following assertions are equivalent: 1) H(TM) is parallel with respect to the Levi–Civita connection of this metric; 2) the induced connection of the distribution H(TM) is symmetric; 3) H(TM) is involutive; 4) M is locally Euclidean; 5) the distribution H(TM) has curvature tensor zero; and 6) H(TM) has sectional curvature zero.

Proof. The equivalence of assertions 1) and 2) for an arbitrary distribution on a Riemannian manifold is proved in [2]. The equivalence of assertions 3) and 4) is well known (see, e.g., [7]). Since any distribution which is parallel with respect to a symmetric connection is involutive, 3) follows from 1). The implication 4) → 1) is obvious due to (3.1). Finally, assertions 4)-6) are equivalent due to Theorems 3.2 and 3.3 below. Theorem 3.1 is proved.

In relation to the Riemannian metric on M and the corresponding Sasaki metric on TM, the canonical projection $\pi: TM \to M$ is a Riemannian submersion. The horizontal distribution on TM with respect to the connection ∇^* is at the same time the horizontal distribution of this submersion. Hence the following theorem is a specialization of Theorem 2.2.

THEOREM 3.2. Let R* be the curvature tensor of the Riemannian manifold M and K be the curvature tensor of the horizontal distribution on the tangent bundle TM, provided with the Sasaki metric. Then $\{R^*(X, Y)\, Z\}^H = K(X^H, Y^H)\, Z^H$ for any $X, Y, Z \in \mathfrak{X}(M)$.

An obvious consequence of Theorem 3.2 is

THEOREM 3.3. The sectional curvature K of the horizontal distribution on the tangent bundle TM with Sasaki metric is the vertical lift of the sectional curvature R* of the

†Relative to the Levi–Civita connection ∇^* on M.

Riemannian manifold M, i.e., $R^*_{XY}\circ\pi = K_{X^HY^H}$ for any $X, Y \in \mathfrak{X}(M)$, $\| X \wedge Y \|^* \neq 0$.

Let T be the torsion tensor of the induced connection $\overline{\nabla}$ of the distribution H(TM). Then from (3.1) we get

$$T(X^H, Y^H)_\xi = (R^*(X_p, Y_p)\,\xi)^V_\xi,$$

$\xi \in TM$, $p = \pi(\xi)$. Since $\mathcal{K}(W^V_\xi) = W_{\pi(\xi)}$, $\pi_*(W^V) = 0$ (see [6]), and consequently, $\langle W^V, W^V\rangle_\xi = \langle W, W\rangle^*_{\pi(\xi)}$ for any $W \in \mathfrak{X}(M)$, the sectional torsion of this distribution is equal to

$$t_{X^HY^H}\circ Z = \| R^*(X, Y)\,Z \|^{*2} \, \| X \wedge Y \|^{*-2}$$

for any $X, Y, Z \in \mathfrak{X}(M)$, $\| X \wedge Y \|^* \neq 0$, where on the left side of the equation Z is considered as a section of the bundle TM. Hence from (2.3) we have

$$R^*_{XY} = R_{X^HY^H}\circ Z + (3/4)\, \| R^*(X, Y)\,Z \|^{*2} \, \| X \wedge Y \|^{*-2}$$

for any $X, Y, Z \in \mathfrak{X}(M)$, $\| X \wedge Y \|^* \neq 0$, where R is the sectional curvature of the Sasaki metric. This equation was obtained previously in [5].

4. Left-Invariant Distributions

Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. To each subspace $\Delta \subset \mathfrak{g}$ there corresponds uniquely a left-invariant distribution on G, whose value at the identity $e \in G$ coincides with Δ. In what follows we shall identify the subspace $\Delta \subset \mathfrak{g}$ with the left-invariant distribution on G corresponding to it. As usual, by ad (ξ) we mean the linear transformation $X \to [\xi, X]$ of the Lie algebra

LEMMA 4.1. Let G be a Lie group with left-invariant Riemannian metric $\langle\, ,\rangle$, Δ be a subspace of its Lie algebra $\mathfrak{g}$ and $\{e_i\}$ be some orthonormal basis of this algebra, adapted to Δ. The sectional curvature of the left-invariant distribution Δ can be calculated in terms of the structural constants $C_{ijk} = \langle [e_i, e_j], e_k\rangle$ of the Lie algebra $\mathfrak{g}$ as follows:

$$K_{e_ae_b} = (1/2)\sum_{i=1}^{n} C_{abi}(C_{bia} + C_{iab}) + \sum_{c=1}^{m}\{(1/4)(C_{cab} + C_{cba})^2 - (3/4)(C_{abc})^2 - C_{caa}C_{cbb}\},$$

where $n = \dim \mathfrak{g}$ and $m = \dim \Delta$.

Proof. Since the Levi–Civita connection of the left-invariant metric can be calculated from the formula

$$\langle \nabla_X Y, Z\rangle = (1/2)\,\{\langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle\},$$

$X, Y, Z \in \mathfrak{g}$, one has

$$h(e_a, e_b) = (1/2)\sum_{\alpha=m+1}^{n} (C_{ab\alpha} - C_{b\alpha a} + C_{\alpha ab})\, e_\alpha \tag{4.1}$$

for any $e_a, e_b \in \Delta$ and by (1.4),

$$K_{e_ae_b} = R_{e_ae_b} + \sum_{\alpha=m+1}^{n}\{C_{\alpha aa}C_{\alpha bb} + (3/4)(C_{ab\alpha})^2 - (1/4)(C_{\alpha ab} + C_{\alpha ba})^2\}.$$

Now it remains to use the following expression from [8] for the sectional curvature of a left-invariant metric:

$$R_{e_ae_b} = \sum_{i=1}^{n}\{(1/2)\,C_{abi}(-C_{abi} + C_{bia} + C_{iab}) - (1/4)(C_{abi} - C_{bia} + C_{iab})(C_{abi} + C_{bia} - C_{iab}) - C_{iaa}C_{ibb}\}.$$

The lemma is proved.

Proposition 4.2. If X belongs to the center of the Lie algebra $\mathfrak{g}$ of the Lie group G, then for any two-dimensional left-invariant distribution $\Delta \ni X$ on G, in relation to a left-inariant metric on G its Gaussian curvature is equal to zero.

Proof. Let $C_{aij} = 0$ for some value a and any values i, j = 1, . . ., n. Then the formula of Lemma 4.1 reduces to $K_{e_ae_b} = 0$. The proposition is proved.

It also follows from Lemma 4.1 that any two-dimensional left-invariant involutive distribution $(e_1 \wedge e_2)$ has constant nonpositive Gaussian curvature (equal to $(C_{121})^2 - (C_{122})^2$). Hence any connected two-dimensional Lie subgroup of a Lie group with left-invariant metric has zero or constant negative Gaussian curvature, depending on whether this subgroup is commutative or not.

LEMMA 4.3. Let the linear transformations ad(X) and ad(Y) of the Lie algebra $\mathfrak{g}$ of the Lie group G be skew adjoint with respect to some left-invariant metric on G. Then the sectional curvature of any left-invariant distribution $\Delta \supset X \wedge Y$ has the property

$$K_{XY} = \| H^{\perp} [X, Y] \|^2 + (1/4) \| H [X, Y] \|^2, \quad \| X \wedge Y \| = 1. \tag{4.2}$$

Proof. If the quantities C_{aij} and C_{bij} for some values a and b are skew-symmetric in i and j, then the formula of Lemma 4.1 assumes the form

$$K_{e_a e_b} = \sum_{\alpha=m+1}^{n} (C_{ab\alpha})^2 + (1/4) \sum_{C=1}^{m} (C_{abc})^2,$$

which proves the lemma.

A left-invariant metric on a connected Lie group G is also right-invariant if and only if ad (ξ) is skew adjoint for any $\xi \in \mathfrak{g}$ (see [8]). Hence any left-invariant distribution on a connected Lie group G is completely geodesic in relation to any bi-invariant metric on G, in fact $h^{+}(e_a, e_b) = (1/2) \sum_{\alpha=m+1}^{n} (C_{\alpha ab} + C_{\alpha ba}) e_{\alpha}$ according to (4.1) and Proposition 1.1 holds. The sectional curvature of any left-invariant distribution Δ on a connected Lie group with bi-invariant metric can be calculated from (4.2). Since here $K_{XY} \geqslant 0$ for any linearly independent $X, Y \in \Delta$, one has

THEOREM 4.4. A left-invariant distribution on a connected Lie group with bi-invariant metric has curvature zero (sectional, Ricci, or scalar) if and only if it is a commutative Lie subalgebra of this Lie group.

THEOREM 4.5. Let G and G* be connected Lie groups with bi-invariant metrics $\langle , \rangle$ and $\langle , \rangle^*$, respectively, and $f: G \to G^*$ be an isomorphism of Lie groups. If there exists on G a left-invariant distribution Δ such that $\langle f_* X, f_* Y \rangle^* \circ f = \langle X, Y \rangle$ for any $X, Y \in \Delta$, $[\Delta, \Delta] = \Delta^{\perp}$ and $[f_* \Delta, f_* \Delta] = (f_* \Delta)^{\perp}$, then f is an isometry.

Proof. Let $\mathfrak{g}$ and $\mathfrak{g}^*$ be the Lie algebras of the groups G and G*. Since f is an isomorphism, $f_* Z \in \mathfrak{g}^*$ for any $Z \in \mathfrak{g}$. By hypothesis f is a Δ-isometry, since $f_*(\Delta^{\perp}) = f_*[\Delta, \Delta] = [f_* \Delta, f_* \Delta] = (f_* \Delta)^{\perp}$. Hence the sectional curvatures K and K* of the distributions Δ and $f_* \Delta$ correspond, i.e., $K_{XY} = K^*_{f_* X f_* Y} \circ f$ for any $X, Y \in \Delta$, $\| X \wedge Y \| \neq 0$. Then from Lemma 4.3 we have $\| [X, Y] \| = \| [f_* X, f_* Y] \|^* \circ f$ for any $X, Y \in \Delta$. Since $[\Delta, \Delta] = \Delta^{\perp}$, for any $Z \in \Delta^{\perp}$ one can find $X, Y \in \Delta$ such that $[X, Y] = Z$. Consequently, the Δ-isometry f is also a $\Delta^{\perp}$-isometry, and the theorem is proved.

COROLLARY 4.6. Let Δ be a left-invariant distribution on the connected Lie group G, such that $[\Delta, \Delta] = \Delta^{\perp}$ in relation to two certain biinvariant metrics on G. Then these metrics coincide if their restrictions to Δ coincide.

To prove this it suffices to set G* = G in Theorem 4.5 and to take as f the identity transformation of the Lie group.

Proposition 4.7. The curvature tensor of a left-invariant distribution Δ on a connected Lie group with biinvariant metric is equal to

$$K(X, Y) Z = (1/4) H [X, H [Y, Z]] + (1/4) H [Y, H [Z, X]] + (1/2) H [Z, H [X, Y]] + H [Z, H^{\perp} [X, Y]], \quad X, Y, Z \in \Delta.$$

Proof. First of all we note that the curvature tensor of a bi-invariant metric is equal to $R(X, Y) Z = -(1/4) [[X, Y], Z]$ (see [8]). Further, since Δ is completely geodesic, by (1.3) and (1.1), $\langle K(X, Y) Z, W \rangle = -(1/4) \langle [[X, Y], Z], W \rangle + (1/4) \langle H^{\perp} [Y, Z], H^{\perp} [X, W] \rangle + (1/4) \langle H^{\perp} [Z, X], H^{\perp} [Y, W] \rangle - (1/2) \langle H^{\perp} [X, Y], H^{\perp} [Z, W] \rangle$ for any $X, Y, Z, W \in \Delta$. Since the 3-form $\langle [X, Y], Z \rangle$ is antisymmetric in Y and Z, one has $\langle H^{\perp} [X, Y], H^{\perp} [Z, W] \rangle = \langle [H^{\perp} [X, Y], Z], W \rangle$. Moreover, $H^{\perp} K(X, Y) HZ = 0$. Consequently, for any $X, Y, Z \in \Delta$

$$K(X, Y) Z = -(1/4) H [[X, Y], Z] + (1/4) H [H^{\perp} [Y, Z], X] + (1/4) H [H^{\perp} [Z, X], Y] - (1/2) H [H^{\perp} [X, Y], Z].$$

To complete the proof one should apply the Jacobi identity.

5. Remarks

(1) Suppose that on the manifold M with Riemannian metric $\langle , \rangle$ there is given a distribution Δ. With respect to the trivial Δ-isometry id: $(M, \langle , \rangle) \to (M, \langle , \rangle^*)$ the curvature of

Δ is invariant. Correspondingly, the curvature of the horizontal distribution of the Riemannian submersion $\pi: M \to B$ is unchanged if the given projected metric on M is deformed on the fibers of this submersion. In particular, the properties of the curvature noted in Sec. 3 holds not only for the Sasaki metric but also for any Riemannian metric on the tangent bundle obtained by altering the first on its fibers.

(2) Let Δ° be an involutive distribution of least dimension containing the given distribution Δ on M. It is natural to call Δ a distribution of constant curvature, if its sectional curvature K_{xy} at each point $p \in M$ is independent of the area element $x \wedge y \subset \Delta_p$ and is constant along each maximal integral manifold of the distribution Δ°. By Theorem 2.4, the horizontal distribution of a Riemannian submersion $\pi: M \to B$ has constant curvature if and only if B is a manifold of (the same) constant curvature. In particular, as Theorem 3.3 shows, there is connected with any manifold M of constant curvature a (horizontal) distribution H(TM) of the same constant curvature on its tangent bundle TM, provided with a Sasaki metric or another suitable (see Remark (1)) Riemmanian metric.

(3) Let G be a connected Lie group with biinvariant metric and N be a closed Lie subgroup of it. The normal Riemannian metric on the homogeneous space G/N is defined by the requirement that the canonical projection $\pi: G \to G/N$ be a Riemannian submersion (see [4, 5]). Let Δ be the horizontal distribution of this submersion. Using the formula of Proposition 4.7 for calculating the curvature tensor of the left-invariant distribution Δ on G, with the help of Theorem 2.2 we get an expression (familiar from [3, 5]) for the curvature tensor of the Riemannian homogeneous space G/N. According to Theorem 4.4 this space is flat (Ricci flat or of scalar curvature zero) if and only if, Δ is a commutative Lie subalgebra of the Lie group G. Now if $[\Delta, \Delta]$ coincides with the Lie algebra of the Lie subgroup N, then by Corollary 4.6, there exists on G a unique biinvariant metric, which induces on G/N the given normal metric and in relation to which $\Delta^\perp = [\Delta, \Delta]$. In this case the curvature tensor of the distribution Δ (and of the space G/N) is equal to $K(X, Y)Z = -[[X, Y], Z]$, $X, Y, Z \in \Delta$.

(4) The concept of curvature tensor of a framed nonholonomic manifold was introduced by Schouten and van Kampen [9]. This tensor (the Schouten tensor in the terminology of Vagner [10]) depends only on the metric in the given nonholonomic manifold and on its framing, but is independent of the metric of the ambient space. If the distribution Δ is completely geodesic, then its curvature tensor in the sense of the present paper (with all three arguments belonging to Δ) is the Schouten tensor. V. V. Vagner, generalizing the concept of intrinsic geometry of a nonholonomic manifold introduced in [9], constructed in [10] a new curvature tensor, with the help of the Schouten tensor, and the new tensor is essentially different from the one considered here. The total and Gaussian curvatures of a two-dimensional nonholonomic surface in Euclidean space E^n, considered by Sintsov [11] for n = 3 and by Glova [12] for n = 4, are the sectional curvatures of the first and second kinds $K^{(1)}$ and $K^{(2)}$, respectively (see Sec. 1). They are connected with the Gaussian curvature K in our sense by (1.6). In the integrable case these three curvatures coincide. Nonholonomic manifolds were also studied from the point of view of their curvatures by Vranceanu [13], Mihailescu [14], and Slukhaev [15].

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PROBABILITIES OF LARGE DEVIATIONS IN THE CASE OF STABLE LIMIT DISTRIBUTIONS

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Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with common distribution function $F(x)$. Let the distribution law $F(x)$ belong to the domain of attraction of a stable law with exponent $\alpha, 0<\alpha<2$. As is known, one then has

$$1-F(x)=\frac{c_1+o(1)}{x^\alpha}h(x), \qquad x\to+\infty,$$

$$F(-x)=\frac{c_2+o(1)}{x^\alpha}h(x), \qquad x\to+\infty,$$

where $h(x)$ is a slowly varying function, and c_1 and c_2 are certain constants, $c_1\geqslant 0$, $c_2\geqslant 0$ and $c_1+c_2>0$. Let us assume in addition that $\mathbf{E}X_1=0$ if $\alpha>1$, and we set $S_n=\sum_{i=1}^n X_i$.

We consider the case when $c_1=0$ (so that $c_2>0$ automatically). Let $F(-x)=x^{-\alpha}h(x)$, and

$$1-F(x)=\frac{\varepsilon(x)h(x)}{x^\alpha} \qquad (x\geqslant x_0), \tag{1}$$

where the function $\varepsilon(x)$ as $x\to+\infty$ satisfies the following conditions:

a) $\varepsilon(x)\to 0$, $\frac{\varepsilon(\gamma x)}{\varepsilon(x)}=O(1)$ for any $\gamma>0$,

b) for any function $\delta(x)$ such that $\delta(x)\to 0$,

$$\frac{\varepsilon(x(1\pm\delta(x)))}{\varepsilon(x)}=1+o(1).$$

One should note that it follows in particular from a) that for any function $\varepsilon(x)$ one can find a constant $A_\varepsilon>0$ such that

$$\varepsilon(x)\geqslant\exp(-A_\varepsilon\ln x),\ x>x_1.$$

Let $\int_{x_0}^x\frac{h(y)}{y}dy=\hat{h}(x)$, $x>x_0$. The following theorem holds.

THEOREM. If (1) holds with some function $\varepsilon(x)$, satisfying a) and b), and $h(x)$ is a slowly varying function, then one has*

*Here and below, the limits are indicated as $n\to\infty$, if nothing is said to the contrary.

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