Graduate Texts in Mathematics 235

# Graduate Texts in Mathematics

1 TAKEUTI/ZARING. Introduction to Axiomatic Set Theory. 2nd ed.
2 OXTOBY. Measure and Category. 2nd ed.
3 SCHAEFER. Topological Vector Spaces. 2nd ed.
4 HILTON/STAMMBACH. A Course in Homological Algebra. 2nd ed.
5 MAC LANE. Categories for the Working Mathematician. 2nd ed.
6 HUGHES/PIPER. Projective Planes.
7 J.-P. SERRE. A Course in Arithmetic.
8 TAKEUTI/ZARING. Axiomatic Set Theory.
9 HUMPHREYS. Introduction to Lie Algebras and Representation Theory.
10 COHEN. A Course in Simple Homotopy Theory.
11 CONWAY. Functions of One Complex Variable I. 2nd ed.
12 BEALS. Advanced Mathematical Analysis.
13 ANDERSON/FULLER. Rings and Categories of Modules. 2nd ed.
14 GOLUBITSKY/GUILLEMIN. Stable Mappings and Their Singularities.
15 BERBERIAN. Lectures in Functional Analysis and Operator Theory.
16 WINTER. The Structure of Fields.
17 ROSENBLATT. Random Processes. 2nd ed.
18 HALMOS. Measure Theory.
19 HALMOS. A Hilbert Space Problem Book. 2nd ed.
20 HUSEMOLLER. Fibre Bundles. 3rd ed.
21 HUMPHREYS. Linear Algebraic Groups.
22 BARNES/MACK. An Algebraic Introduction to Mathematical Logic.
23 GREUB. Linear Algebra. 4th ed.
24 HOLMES. Geometric Functional Analysis and Its Applications.
25 HEWITT/STROMBERG. Real and Abstract Analysis.
26 MANES. Algebraic Theories.
27 KELLEY. General Topology.
28 ZARISKI/SAMUEL. Commutative Algebra. Vol.I.
29 ZARISKI/SAMUEL. Commutative Algebra. Vol.II.
30 JACOBSON. Lectures in Abstract Algebra I. Basic Concepts.
31 JACOBSON. Lectures in Abstract Algebra II. Linear Algebra.
32 JACOBSON. Lectures in Abstract Algebra III. Theory of Fields and Galois Theory.
33 HIRSCH. Differential Topology.
34 SPITZER. Principles of Random Walk. 2nd ed.
35 ALEXANDER/WERMER. Several Complex Variables and Banach Algebras. 3rd ed.
36 KELLEY/NAMIOKA et al. Linear Topological Spaces.
37 MONK. Mathematical Logic.
38 GRAUERT/FRITZSCHE. Several Complex Variables.
39 ARVESON. An Invitation to $C^*$-Algebras.
40 KEMENY/SNELL/KNAPP. Denumerable Markov Chains. 2nd ed.
41 APOSTOL. Modular Functions and Dirichlet Series in Number Theory. 2nd ed.
42 J.-P. SERRE. Linear Representations of Finite Groups.
43 GILLMAN/JERISON. Rings of Continuous Functions.
44 KENDIG. Elementary Algebraic Geometry.
45 LOÈVE. Probability Theory I. 4th ed.
46 LOÈVE. Probability Theory II. 4th ed.
47 MOISE. Geometric Topology in Dimensions 2 and 3.
48 SACHS/WU. General Relativity for Mathematicians.
49 GRUENBERG/WEIR. Linear Geometry. 2nd ed.
50 EDWARDS. Fermat's Last Theorem.
51 KLINGENBERG. A Course in Differential Geometry.
52 HARTSHORNE. Algebraic Geometry.
53 MANIN. A Course in Mathematical Logic.
54 GRAVER/WATKINS. Combinatories with Emphasis on the Theory of Graphs.
55 BROWN/PEARCY. Introduction to Operator Theory I: Elements of Functional Analysis.
56 MASSEY. Algebraic Topology: An Introduction.
57 CROWELL/FOX. Introduction to Knot Theory.
58 KOBLITZ. $p$-adic Numbers, $p$-adic Analysis, and Zeta-Functions. 2nd ed.
59 LANG. Cyclotomic Fields.
60 ARNOLD. Mathematical Methods in Classical Mechanics. 2nd ed.
61 WHITEHEAD. Elements of Homotopy Theory.
62 KARGAPOLOV/MERLZIAKOV. Fundamentals of the Theory of Groups.
63 BOLLOBAS. Graph Theory.

*(continued after index)*

Mark R. Sepanski

# Compact Lie Groups

Springer

Mark R. Sepanski
Department of Mathematics
Baylor University
Waco, TX 76798-7328
USA
mark_sepanski@baylor.edu



Mathematics Subject Classification (2000): 22E15, 17B20

Library of Congress Control Number: 2006937104

ISBN-10: 0-387-30263-8 e-ISBN-10: 0-387-49158-9
ISBN-13: 978-0-387-30263-8 e-ISBN-13: 978-0-387-49158-5

Printed on acid-free paper.



9 8 7 6 5 4 3 2 1

springer.com (TXQ/SB)

*To Laura, Sarah, Ben, and Shannon*

# Contents

# Preface

As an undergraduate, I was offered a reading course on the representation theory of finite groups. When I learned this basically meant studying homomorphisms from groups into matrices, I was not impressed. In its place I opted for a reading course on the much more glamorous sounding topic of multilinear algebra. Ironically, when I finally took a course on representation theory from B. Kostant in graduate school, I was immediately captivated.

In broad terms, representation theory is simply the study of symmetry. In practice, the theory often begins by classifying all the ways in which a group acts on vector spaces and then moves into questions of decomposition, unitarity, geometric realizations, and special structures. In general, each of these problems is extremely difficult. However in the case of compact Lie groups, answers to most of these questions are well understood. As a result, the theory of compact Lie groups is used extensively as a stepping stone in the study of noncompact Lie groups.

Regarding prerequisites for this text, the reader must first be familiar with the definition of a group and basic topology. Secondly, elementary knowledge of differential geometry is assumed. Students lacking a formal course in manifold theory will be able to follow most of this book if they are willing to take a few facts on faith. This mostly consists of accepting the existence of an invariant integral in §1.4.1. In a bit more detail, the notion of a submanifold is used in §1.1.3, the theory of covering spaces is used in §1.2, §1.3, §4.2.3, and §7.3.6, integral curves are used in §4.1.2, and Frobenius' theorem on integral submanifolds is used in the proof of Theorem 4.14. A third prerequisite is elementary functional analysis. Again, students lacking formal course work in this area can follow most of the text if they are willing to assume a few facts. In particular, the Spectral Theorem for normal bounded operators is used in the proof of Theorem 3.12, vector-valued integration is introduced in §3.2.2, and the Spectral Theorem for compact self-adjoint operators is used in the proof of Lemma 3.13.

The text assumes no prior knowledge of Lie groups or Lie algebras and so all the necessary theory is developed here. Students already familiar with Lie groups can quickly skim most of Chapters 1 and 4. Similarly, students familiar with Lie algebras can quickly skim most of Chapter 6.

The book is organized as follows. Chapter 1 lays out the basic definitions, examples, and theory of compact Lie groups. Though the construction of the spin groups in §1.3 is very important to later representation theory and mathematical physics, this material can be easily omitted on a first reading. Doing so allows for a more rapid transition to the harmonic analysis in Chapter 3. A similar remark holds for the construction of the spin representations in §2.1.2.4. Chapter 2 introduces the concept of a finite-dimensional representation. Examples, Schur's Lemma, unitarity, and the canonical decomposition are developed here. Chapter 3 begins with matrix coefficients and character theory. It culminates in the celebrated Peter–Weyl Theorem and its corresponding Fourier theory.

Up through Chapter 3, the notion of a Lie algebra is unnecessary. In order to progress further, Chapter 4 takes up their study. Since this book works with compact Lie groups, it suffices to consider linear Lie groups which allows for a fair amount of differential geometry to be bypassed. Chapter 5 examines maximal tori and Cartan subalgebras. The Maximal Torus Theorem, Dynkin's Formula, the Commutator Theorem, and basic structural results are given. Chapter 6 introduces weights, roots, the Cartan involution, the Killing form, the standard $\mathfrak{sl}(2, \mathbb{C})$, various lattices, and the Weyl group. Chapter 7 uses all this technology to prove the Weyl Integration Formula, the Weyl Character Formula, the Highest Weight Theorem, and the Borel–Weil Theorem.

Since this work is intended as a textbook, most references are given only in the bibliography. The interested reader may consult [61] or [34] for brief historical outlines of the theory. With that said, there are a number of resources that had a powerful impact on this work and to which I am greatly indebted. First, the excellent lectures of B. Kostant and D. Vogan shaped my view of the subject. Notes from those lectures were used extensively in certain sections of this text. Second, any book written by A. Knapp on Lie theory is a tremendous asset to all students in the field. In particular, [61] was an extremely valuable resource. Third, although many other works deserve recommendation, there are four outstanding texts that were especially influential: [34] by Duistermaat and Kolk, [72] by Rossmann, [70] by Onishchik and Vinberg, and [52] by Hoffmann and Morris. Many thanks also go to C. Conley who took up the onerous burden of reading certain parts of the text and making helpful suggestions. Finally, the author is grateful to the Baylor Sabbatical Committee for its support during parts of the preparation of this text.

Mark Sepanski
March 2006

# Compact Lie Groups