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Semisimple Lie Groups and Algebras

In this chapter, unless expressly stated otherwise, by Lie algebra we mean a *complex Lie algebra.* Since every real Lie algebra can be complexified, most of our results also have immediate consequences for real Lie algebras.

1 Semisimple Lie Algebras

1.1 $sl(2, \mathbb{C})$

The first and most basic example, which in fact needs to be developed first, is $sl(2, \mathbb{C})$. For this one takes the usual basis

$$e := \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}, \quad f := \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, \quad h := \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}.$$

These elements satisfy the commutation relations

$$(1.1.1) \qquad [e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

From the theory developed in Chapter 9, we know that the symmetric powers $S^k(V)$ of the 2-dimensional vector space V are the list of rational irreducible representations for $SL(V)$. Hence they are irreducible representations of $sl(2, \mathbb{C})$. To prove that they are also all the irreducible representations of $sl(2, \mathbb{C})$ we start with

Lemma. *Let M be a representation of $sl(2, \mathbb{C})$, $v \in M$ a vector such that $hv = kv, k \in \mathbb{C}$.*

(i) For all i we have that $he^i v = (k + 2i)e^i v$, $hf^i v = (k - 2i) f^i v$.
(ii) If furthermore $ev = 0$, then $ef^i v = i(k - i + 1) f^{i-1} v$.
(iii) Finally if $ev = 0$, $f^m v \neq 0$, $f^{m+1} v = 0$, we have $k = m$ and the elements $f^i v, i = 0, \dots, m$, are a basis of an irreducible representation of $sl(2, \mathbb{C})$.

Proof. (i) We have $2ev = (he - eh)v = hev - kev \implies hev = (k+2)ev$. Hence ev is an eigenvector for h of weight $k+2$. Similarly $hfv = (k-2)fv$. Then the first statement follows by induction.

(ii) From $[e, f] = h$ we see that $ef^i v = (k - 2i + 2)f^{i-1}v + fef^{i-1}v$ and thus, recursively, we check that $ef^i v = i(k - i + 1)f^{i-1}v$.

(iii) If we assume $f^{m+1}v = 0$ for some minimal m, then by the previous identity we have $0 = ef^{m+1}v = (m+1)(k-m)f^m v$. This implies $m = k$ and the vectors $v_i := \frac{1}{i!}f^i v$, $i = 0, \dots, k$, span a submodule N with the explicit action

(1.1.2) $$hv_i = (k - 2i)v_i, \ fv_i = (i+1)v_{i+1}, \ ev_i = (k - i + 1)v_{i-1}.$$

The fact that N is irreducible is clear from these formulas. □

Theorem. *The representations $S^k(V)$ form the list of irreducible finite-dimensional representations of $sl(2, \mathbb{C})$.*

Proof. Let N be a finite-dimensional irreducible representation. Since h has at least one eigenvector, by the previous lemma, if we choose one v_0 with maximal eigenvalue, we have $ev_0 = 0$. Since N is finite dimensional, $f^{m+1}v = 0$ for some minimal m, and we have the module given by formula 1.1.1. Call this representation V_k. Notice that $V = V_1$. We identify V_k with $S^k(V)$ since, if V has basis x, y, the elements $\binom{k}{i}x^{k-i}y^i$ behave as the elements v_i under the action of the elements e, f, h. □

Remark. It is useful to distinguish among the *even* and *odd* representations,[83] according to the parity of k. In an even representation all weights for h are even, and there is a unique weight vector for h of weight 0. In the odd case, all weights are odd and there is a unique weight vector of weight 1.

It is natural to call a vector v with $ev = 0$ a *highest weight vector*. This idea carries over to all semisimple Lie algebras with the appropriate modifications (§5).

There is one expository difficulty in the theory. We have proved that rational representations of $SL(2, \mathbb{C})$ are completely reducible and we have seen that its irreducible representations correspond exactly to the irreducible representations of $sl(2, \mathbb{C})$. It is not clear though why representations of the Lie algebra $sl(2, \mathbb{C})$ are completely reducible, nor why they correspond to rational representations of $SL(2, \mathbb{C})$. There are in fact several ways to prove this which then extend to all semisimple Lie algebras and their corresponding groups.

1. One proves by algebraic means that all finite-dimensional representations of the Lie algebra $sl(2, \mathbb{C})$ are completely reducible.
2. One integrates a finite-dimensional representation of $su(2, \mathbb{C})$ to $SU(2, \mathbb{C})$. Since $SU(2, \mathbb{C})$ is compact, the representation under the group is completely reducible.
3. One integrates a finite-dimensional representation of $sl(2, \mathbb{C})$ to $SL(2, \mathbb{C})$ and then proves that it is rational.

[83] In physics it is usual to divide the highest weight by 2 and talk of integral or half-integral spin.

1.2 Complete Reducibility

Let us discuss the algebraic approach to complete reducibility.[84] First, we remark that a representation of a 1-dimensional Lie algebra is just a linear operator. Since not all linear operators are semisimple it follows that if a Lie algebra $L \supsetneq [L, L]$, then it has representations which are not completely reducible.

If $L = [L, L]$ we have that a 1-dimensional representation is necessarily trivial, and we denote it by $\mathbb{C}$.

Theorem 1. *For a Lie algebra $L = [L, L]$, the following properties are equivalent.*

(1) Every finite-dimensional representation is completely reducible.
(2) If M is a finite-dimensional representation, $N \subset M$ a submodule with M/N 1-dimensional, then $M = N \oplus \mathbb{C}$ as modules.
(3) If M is a finite-dimensional representation, $N \subset M$ an irreducible submodule, with M/N 1-dimensional, then $M = N \oplus \mathbb{C}$ as modules.

Proof. Clearly (1) $\implies$ (2) $\implies$ (3). Let us show the converse.

(3) $\implies$ (2) by a simple induction on $\dim N$. Suppose we are in the hypotheses of (2) assuming (3). If N is irreducible we can just apply (3). Otherwise N contains a nonzero irreducible submodule P and we have the new setting $M' := M/P$, $N' := N/P$ with M'/N' 1-dimensional. Thus, by induction, there is a complement $\mathbb{C}$ to N' in M'. Consider the submodule Q of M with $Q/P = \mathbb{C}$. By part (3) P has a 1-dimensional complement in Q and this is also a 1-dimensional complement of N in M.

(2) $\implies$ (1) is delicate. We check complete reducibility as in Chapter 6 and show that, given a module M and a submodule N, N has a complementary submodule P, i.e., $M = N \oplus P$.

Consider the space of linear maps $\hom(M, N)$. The formula $(l\phi)(m) := l(\phi(m)) - \phi(lm)$ makes this space a module under L. It is immediately verified that a linear map $\phi : M \to N$ is an L homomorphism if and only if $L\phi = 0$.

Since N is a submodule, the restriction $\pi : \hom(M, N) \to \hom(N, N)$ is a homomorphism of L-modules. In $\hom(N, N)$ we have the trivial 1-dimensional submodule $\mathbb{C}1_N$ formed by the multiples of the identity map. Thus take $A := \pi^{-1}(\mathbb{C}1_N)$ and let $B := \pi^{-1}(0)$. Both A, B are L modules and A/B is the trivial module. Assuming (2) we can thus find an element $\phi \in A$, $\phi \notin B$ with $L\phi = 0$. In other words, $\phi : M \to N$ is an L-homomorphism, which restricted to N, is a nonzero multiple of the identity. Its kernel is thus a complement to N which is a submodule. □

The previous theorem gives us a criterion for complete reducibility which can be used for semisimple algebras once we develop enough of the theory, in particular after we introduce the *Casimir element*. Let us use it immediately to prove that all finite-dimensional representations of the Lie algebra $sl(2, \mathbb{C})$ are completely reducible.

[84] We work over the complex numbers just for simplicity.

Take a finite-dimensional representation M of $sl(2,\mathbb{C})$ and identify the elements e, f, h with the operators they induce on M. We claim that the operator $C := ef + h(h-2)/4 = fe + h(h+2)/4$ commutes with e, f, h. For instance,

$$[C, e] = e[f, e] + [h, e](h-2)/4 + h[h, e]/4$$
$$= -eh + e(h-2)/2 + he/2 = [h, e]/2 - e = 0.$$

Let us show that $sl(2,\mathbb{C})$ satisfies (3) of the previous theorem. Consider $N \subset M$, $M/N = \mathbb{C}$ with N irreducible of highest weight k. On $\mathbb{C}$ the operator C acts as 0 and on N as a scalar by Schur's lemma. To compute which scalar, we find its value on the highest weight vector, getting $k(k+2)/4$. So if $k > 0$, we have a nonzero scalar. On the other hand, on the trivial module, it acts as 0. If $\dim N > 1$, we have that C has the two eigenvalues $k(k+2)/4$ on N and 0 necessarily on a complement of N. It remains to understand the case $\dim N = 1$. In this case the matrices that represent L are a priori 2×2 matrices of type $\begin{vmatrix} 0 & a \\ 0 & 0 \end{vmatrix}$. The commutators of these matrices are all 0. Since $sl(2,\mathbb{C})$ is spanned by commutators, the representation is trivial. From Theorem 1 we have proved:

Theorem 2. *All finite-dimensional representations of $sl(2,\mathbb{C})$ are completely reducible.*

Occasionally one has to deal with infinite-dimensional representations of the following type:

Definition 1. We say that a representation of $sl(2,\mathbb{C})$ is rational if it is a sum of finite-dimensional representations.

A way to study special infinite-dimensional representations is through the notion:

Definition 2. We say that an operator $T : V \to V$ is *locally nilpotent* if, given any vector $v \in V$, we have $T^k v = 0$ for some positive k.

Proposition. *The following properties are equivalent for a representation M of $sl(2,\mathbb{C})$:*

(1) M integrates to a rational representation of the group $SL(2,\mathbb{C})$.
(2) M is a direct sum of finite-dimensional irreducible representations of $sl(2,\mathbb{C})$.
(3) M has a basis of eigenvectors for h and the operators e, f are locally nilpotent.

Proof. (1) implies (2), since from Chapter 7, §3.2 $SL(2,\mathbb{C})$ is linearly reductive. (2) implies (1) and (3) from Theorem 2 of 1.2 and the fact, proved in 1.1 that the finite-dimensional irreducible representations of $sl(2,\mathbb{C})$ integrate to rational representations of the group $SL(2,\mathbb{C})$.

Assume (3). Start from a weight vector v for h. Since e is locally nilpotent, we find some nonzero power k for which $w = e^k v \neq 0$ and $ew = 0$. Since f is locally nilpotent, we have again, for some m, that $f^m w \neq 0$, $f^{m+1} w = 0$. Apply

Lemma 1.1 to get that w generates a finite-dimensional irreducible representation of $sl(2, \mathbb{C})$. Now take the sum P of all the finite-dimensional irreducible representations of $sl(2, \mathbb{C})$ contained in M; we claim that $P = M$. If not, by the same discussion we can find a finite-dimensional irreducible $sl(2, \mathbb{C})$ module in M/P. Take vectors $v_i \in M$ which, modulo P, verify the equations 1.1.2. Thus the elements $hv_i - (k - 2i)v_i$, $fv_i - (i + 1)v_{i+1}$, $ev_i - (k - i + 1)v_{i-1}$ lie in P. Clearly, we can construct a finite-dimensional subspace V of P stable under $sl(2, \mathbb{C})$ containing these elements. Therefore, adding to V the vectors v_i we have an $sl(2, \mathbb{C})$ submodule N. Since N is finite dimensional, it is a sum of irreducibles. So $N \subset P$, a contradiction. □

1.3 Semisimple Algebras and Groups

We are now going to take a very long detour into the general theory of semisimple algebras. In particular we want to explain how one classifies irreducible representations in terms of certain objects called dominant weights. The theory we are referring to is part of the theory of representations of complex semisimple Lie algebras and we shall give a short survey of its main features and illustrate it for classical groups.[85]

Semisimple Lie algebras are closely related to linearly reductive algebraic groups and compact groups. We have already seen in Chapter 7 the definition of a semisimple algebraic group as a reductive group with finite center. For compact groups we have a similar definition.

Definition. A connected compact group is called *semisimple* if it has a finite center.

Example. $U(n, \mathbb{C})$ is not semisimple. $SU(n, \mathbb{C})$ is semisimple.

Let L be a complex semisimple Lie algebra. In this chapter we shall explain the following facts:

(a) L is the Lie algebra of a semisimple algebraic group G.
(b) L is the complexification $L = K \otimes_{\mathbb{R}} \mathbb{C}$ of a real Lie algebra K with negative definite Killing form (Chapter 4, §4.4). K is the Lie algebra of a semisimple compact group, maximal compact in G.
(c) L is a direct sum of simple Lie algebras. Simple Lie algebras are completely classified. The key to the classification of Killing–Cartan is the theory of *roots* and *finite reflection groups*.

It is a quite remarkable fact that associated to a continuous group there is a finite group of Euclidean reflections and that the theory of the continuous group can be largely analyzed in terms of the combinatorics of these reflections.

[85] There are many more general results over fields of characteristic 0 or just over the real numbers, but they do not play a specific role in the theory we shall discuss.

1.4 Casimir Element and Semisimplicity

We want to see that the method used to prove the complete reducibility of $sl(2, \mathbb{C})$ works in general for semisimple Lie algebras.

We first need some simple remarks.

Let L be a simple Lie algebra. Given any nontrivial representation $\rho : L \to gl(M)$ of L we can construct its *trace form*, $(a, b) := \mathrm{tr}(\rho(a)\rho(b))$. It is then immediate to verify that this form is *associative* in the sense that $([a, b], c) = (a, [b, c])$. It follows that the kernel of this form is an ideal of L. Hence, unless this form is identically 0, it is nondegenerate.

Remark. The form cannot be identically 0 from Cartan's criterion (Chapter 4, §6.4).

Lemma. *On a simple Lie algebra a nonzero associative form is unique up to scale.*

Proof. We use the bilinear form to establish a linear isomorphism $j : L \to L^*$, through the formula $j(a)(b) = (a, b)$. We have $j([x, a])(b) = ([x, a], b]) = -(a, [x, b]) = -j(a)([x, b])$. Thus j is an isomorphism of L-modules. Since L is irreducible as an L-module, the claim follows from Schur's lemma. □

Remark. In particular, the trace form is a multiple of the Killing form.

Let L be a semisimple Lie algebra, and consider dual bases u_i, u^i for the Killing form. Since the Killing form identifies L with L^*, by general principles the Killing form can be identified with the symmetric tensor $C_L := \sum_i u_i \otimes u^i = \sum_i u^i \otimes u_i$. The associativity property of the form translates into invariance of C_L. C_L is killed by the action of the Lie algebra, i.e.,

$$(1.4.1) \qquad \sum_i ([x, u^i] \otimes u_i + u^i \otimes [x, u_i]) = 0, \quad \forall x \in L.$$

Then by the multiplication map it is best to identify C_L with its image in the enveloping algebra $U(L)$. More concretely:

Theorem 1.

(1) The element $C_L := \sum_i u_i u^i$ does not depend on the dual bases chosen.
(2) The element $C_L := \sum_i u_i u^i$ commutes with all of the elements of the Lie algebra L.
(3) If the Lie algebra L decomposes as a direct sum $L = \bigoplus_i L_i$ of simple algebras, then we have $C_L = \sum_i C_{L_i}$. Each C_{L_i} commutes with L.
(4) If M is an irreducible representation of L each element C_{L_i} acts on M by a scalar. This scalar is 0 if and only if L_i is in the kernel of the representation.

Proof. (1) Let $s_i = \sum_j d_{j,i} u_j$, $s^i = \sum_j e_{j,i} u^j$ be another pair of dual bases. We have

$$\delta_i^j = (s_i, s^j) = \left(\sum_h d_{h,i} u_h, \sum_h e_{h,j} u^h \right) = \sum_h d_{h,i} e_{h,j}.$$

If D is the matrix with entries $d_{i,j}$ and E the one with entries $e_{i,j}$, we thus have $E^tD = 1$, which implies that also $ED^t = 1$. Thus

$$\sum_i s_i s^i = \sum_i \sum_h d_{h,i} u_h \sum_k e_{k,i} u^k = \sum_{h,k} \sum_i d_{h,i} e_{k,i} u_h u^k = \sum_h u_h u^h.$$

(2) Denote by (a, b) the Killing form. If $[c, u_i] = \sum_j a_{j,i} u_j$, $[c, u^j] = \sum_i b_{i,j} u^i$, we have

$$a_{j,i} = ([c, u_i], u^j) = -(u_i, [c, u^j]) = -b_{i,j}. \tag{1.4.2}$$

$$\begin{aligned} \text{Then} \quad [c, C] &= \sum_i [c, u_i] u^i + \sum_i u_i [c, u^i] \\ &= \sum_i \sum_j a_{j,i} u_j u^i + \sum_i u_i \sum_j b_{j,i} u^j \\ &= \sum_i \sum_j a_{j,i} u_j u^i + b_{i,j} u_j u^i = 0. \end{aligned}$$

(3) The ideals L_i are orthogonal under the Killing form, and the Killing form of L restricts to the Killing form of L_i (Chapter 4, §6.2, Theorem 2). The statement is clear, since the L_i commute with each other.

(4) Since C_{L_i} commutes with L and M is irreducible under L, by Schur's lemma, C_{L_i} must act as a scalar. We have to see that it is 0 if and only if L_i acts by 0.

If L_i does not act as 0, the trace form is nondegenerate and is a multiple of the Killing form by a nonzero scalar λ. Then $\operatorname{tr}(\rho(C_{L_i})) = \sum_i \operatorname{tr}(\rho(u_i)\rho(u^i)) = \sum_i \lambda(u_i, u^i) = \lambda \dim L \neq 0$. □

Definition. The element $C_L \in U(L)$ is called the *Casimir element* of the semisimple Lie algebra L.

We can now prove:

Theorem 2. *A finite-dimensional representation of a semisimple Lie algebra L is completely reducible.*

Proof. Apply the method of §1.2. Since $L = [L, L]$, the only 1-dimensional representations of L are trivial. Let M be a module and N an irreducible submodule with M/N 1-dimensional, hence trivial. If N is also trivial, the argument given in 1.2 for $sl(2, \mathbb{C})$ shows that M is trivial. Otherwise, let us compute the value on M of one of the Casimir elements $C_i = C_{L_i}$ which acts on N by a nonzero scalar λ (by the previous theorem). On the quotient M/N the element C_i acts by 0. Therefore C_i on M has eigenvalue λ (with eigenspace N) and 0. There is thus a vector $v \notin N$ for which $C_i v = 0$ such that v spans the 1-dimensional eigenspace of the eigenvalue 0. Since C_i commutes with L, the space generated by v is stable under L, and $Lv = 0$ satisfying the conditions of Theorem 1 of 1.2. □

1.5 Jordan Decomposition

In a Lie algebra L an element x is called *semisimple* if $\mathrm{ad}(x)$ is a semisimple (i.e., diagonalizable) operator.

As for algebraic groups we may ask if the semisimple part of the operator $\mathrm{ad}(x)$ is still of the form $\mathrm{ad}(y)$ for some y, to be called the semisimple part of x. Not all Lie algebras have this property, as simple examples show. The ones which do are called *splittable*.

We need a simple lemma.

Lemma 1. *Let A be any finite-dimensional algebra over $\mathbb{C}$, and D a derivation. Then the semisimple part D_s of D is also a derivation.*

Proof. One can give a direct computational proof (see [Hu1]). Since we have developed some theory of algebraic groups, let us instead follow this path. The group of automorphisms of A is an algebraic group, and for these groups we have seen the Jordan–Chevalley decomposition. Hence, given an automorphism of A, its semisimple part is also an automorphism. D is a derivation if and only if $\exp(tD)$ is a one parameter group of automorphisms (Chapter 3). We can conclude noticing that if $D = D_s + D_n$ is the additive Jordan decomposition, then $\exp(tD) = \exp(tD_s)\exp(tD_n)$ is the multiplicative decomposition. We deduce that $\exp(tD_s)$ is a one parameter group of automorphisms. Hence D_s is a derivation. □

Lemma 2. *Let L be a Lie algebra, and M the Lie algebra of its derivations. The inner derivations* $\mathrm{ad}(L)$ *are an ideal of M and* $[D, \mathrm{ad}(a)] = \mathrm{ad}(D(a))$.

Proof.

$$[D, \mathrm{ad}(a)](b) = D(\mathrm{ad}(a)(b)) - \mathrm{ad}(a)(D(b)) = D[a, b] - [a, D(b)]$$

$$= [D(a), b] + [a, D(b)] - [a, D(b)] = [D(a), b].$$

Thus $\mathrm{ad}(L)$ is an ideal in M. □

Theorem 1. *If L is a semisimple Lie algebra and D is a derivation of L, then D is inner.*

Proof. Let M be the Lie algebra of derivations of L. It contains the inner derivations $\mathrm{ad}(L)$ as an ideal. Since L is semisimple we have a direct sum decomposition $M = \mathrm{ad}(L) \oplus P$ as L modules. Since $\mathrm{ad}(L)$ is an ideal, $[P, \mathrm{ad}(L)] \subset \mathrm{ad}(L)$. Since P is an L module, $[P, \mathrm{ad}(L)] \subset P$. Hence $[P, \mathrm{ad}(L)] = 0$. From the formula $[D, \mathrm{ad}(a)] = \mathrm{ad}(D(a))$, it follows that if $D \in P$, we have $\mathrm{ad}(D(a)) = 0$. Since the center of L is 0, $P = 0$ and $M = \mathrm{ad}(L)$. □

Corollary. *If L is a semisimple Lie algebra, $a \in L$, there exist unique elements $a_s, a_n \in L$ such that*

(1.5.1) $$a = a_s + a_n, \ [a_s, a_n] = 0, \quad \mathrm{ad}(a_s) = \mathrm{ad}(a)_s, \ \mathrm{ad}(a_n) = \mathrm{ad}(a)_n.$$

Proof. By Lemma 1, the semisimple and nilpotent parts of $\mathrm{ad}(a)$ are derivations. By the previous theorem they are inner, hence induced by elements a_s, a_n. Since the map $\mathrm{ad} : L \to \mathrm{ad}(L)$ is an isomorphism the claim follows. □

Finally we want to see that the Jordan decomposition is preserved under any representation.

Theorem 2. *If ρ is any linear representation of a semisimple Lie algebra and $a \in L$, we have $\rho(a_s) = \rho(a)_s$, $\rho(a_n) = \rho(a)_n$.*

Proof. The simplest example is when we take the Lie algebra $sl(V)$ acting on V. In this case we can apply the Lemma of §6.2 of Chapter 4. This lemma shows that the usual Jordan decomposition $a = a_s + a_n$ for a linear operator $a \in sl(V)$ on V induces, under the map $a \mapsto \mathrm{ad}(a)$, a Jordan decomposition $\mathrm{ad}(a) = \mathrm{ad}(a_s) + \mathrm{ad}(a_n)$.

In general it is clear that we can restrict our analysis to simple L, V an irreducible module and $L \subset \mathrm{End}(V)$. Let $M := \{x \in \mathrm{End}(V) \mid [x, L] \subset L\}$. As before M is a Lie algebra and L an ideal of M. Decomposing $M = L \oplus P$ with P an L-module, we must have $[L, P] = 0$. Since the module is irreducible, by Schur's lemma we must have that P reduces to the scalars. Since $L = [L, L]$, the elements of L have all trace 0, hence $L = \{u \in M \mid \mathrm{tr}(u) = 0\}$. Take an element $x \in L$ and decompose it in $\mathrm{End}(V)$ as $x = y_s + y_n$ the semisimple and nilpotent part. By the Lemma of §6.2, Chapter 4 previously recalled, $\mathrm{ad}(x) = \mathrm{ad}(y_s) + \mathrm{ad}(y_n)$ is the Jordan decomposition of operators acting on $\mathrm{End}(V)$. Since $\mathrm{ad}(x)$ preserves L, also $\mathrm{ad}(y_s)$, $\mathrm{ad}(y_n)$ preserve L, hence we must have $y_s, y_n \in M$. Since $\mathrm{tr}(y_n) = 0$ we have $y_n \in L$, hence also $y_s \in L$. By the uniqueness of the Jordan decomposition $x_s = y_s$, $x_n = y_n$. □

There is an immediate connection to algebraic groups.

Theorem 3. *If L is a semisimple Lie algebra, its adjoint group is the connected component of 1 of its automorphism group. It is an algebraic group with Lie algebra L.*

Proof. The automorphism group is clearly algebraic. Its connected component of 1 is generated by the 1-parameter groups $\exp(tD)$ where D is a derivation. Since all derivations are inner, it follows that its Lie algebra is $\mathrm{ad}(L)$. Since L is semisimple, $L = \mathrm{ad}(L)$. □

1.6 Levi Decomposition

Let us first make a general construction. Given a Lie algebra L, let $\mathcal{D}(L)$ be its Lie algebra of derivations. Given a Lie homomorphism ρ of a Lie algebra M into $\mathcal{D}(L)$, we can give to $M \oplus L$ a new Lie algebra structure by the formula (check it):

$$[(m_1, a), (m_2, b)] := ([m_1, m_2], \rho(m_1)(b) - \rho(m_2)(a) + [a, b]). \tag{1.6.1}$$

Definition. $M \oplus L$ with the previous structure is called a *semidirect product* and denoted $M \ltimes L$.

Formula 1.6.1 implies immediately that in $M \ltimes L$ we have that M is a subalgebra and L an ideal. Furthermore, if $m \in M, a \in L$ we have $[m, a] = \rho(m)(a)$.

As an example, take $F \ltimes L$ with $M = F$ 1-dimensional. A homomorphism of F into $D(L)$ is given by specifying a derivation D of L corresponding to 1, so we shall write $FD \ltimes L$ to remind us of the action:

Lemma 1. *(i) If L is solvable, then $FD \ltimes L$ is solvable.*

(ii) If N is a nilpotent Lie algebra and D is a derivation of N, then $FD \ltimes N$ is nilpotent if and only if D is nilpotent.

(iii) If L is semisimple, then $F \ltimes L = L \oplus F$.

Proof. (i) The first part is obvious.

(ii) Assume that $D^m N = 0$ and $N^i = 0$. By formula 4.3.1 of Chapter 4 it is enough to prove that a long enough monomial in elements $\mathrm{ad}(a_i), a_i \in F \ltimes N$ is 0. In fact it is enough to show that such an operator is 0 on N since then a 1-step longer monomial is identically 0. Consider a monomial in the operators $D, \mathrm{ad}(n_j), n_j \in N$. Assume the monomial is of degree $> mi$.

Notice that $D\ \mathrm{ad}(n_i) = \mathrm{ad}(D(n_i)) + \mathrm{ad}(n_i)D$. We can rewrite the monomial as a sum of terms $\mathrm{ad}(D^{h_1}n_1)\,\mathrm{ad}(D^{h_2}n_2)\dots\mathrm{ad}(D^{h_t}n_t)D^{h_{t+1}}$ with $\sum_{k=1}^{t+1} h_k + t > mi$. If $t \geq i-1$, this is 0 by the condition $N^i = 0$. Otherwise, $\sum_{k=1}^{t+1} h_k > (m-1)i$, and since $t < i$ at least one of the exponents h_k must be bigger than $m-1$. So again we get 0 from $D^m = 0$.

Conversely, if D is not nilpotent, then $\mathrm{ad}(D) = D$ on N, hence $\mathrm{ad}(D)^m \neq 0$ for all m, so $FD \ltimes N$ is not nilpotent.

(iii) If L is semisimple, $D = \mathrm{ad}(a)$ is inner, $D - a$ is central and $F \ltimes L = L \oplus F(D-a)$. □

We need a criterion for identifying semidirect products.[86] Given L a Lie algebra and I a Lie ideal, we have

Lemma 2. *If there is a Lie subalgebra A such that as vector spaces $L = A \oplus I$, then $L = A \ltimes I$ where A acts by derivation on I as restriction to I of the inner derivations* $\mathrm{ad}(a)$.

The proof is straightforward.

A trivial example is when L/I is 1-dimensional. Then any choice of an element $a \in L,\ a \notin I$ presents $L = Fa \oplus I$ as a semidirect product.

In the general case, the existence of such an A can be treated by the *cohomological method.*

Let us define $A := L/I,\ p : L \to L/I$ the projection. We want to find a homomorphism $f : A \to L$ with $pf = 1_A$. If such a homomorphism exists it is called a *splitting*. We proceed in two steps. First, choose any linear map $f : A \to L$ with $pf = 1_A$. The condition to be a homomorphism is that the two-variable function $\phi_f(a,b) := f([a,b]) - [f(a), f(b)]$, which takes values in I, must be 0. Given such

[86] As in group theory

an f, if it does not satisfy the homomorphism property we can *correct it* by adding to it a linear mapping $g : A \to I$. Given such a map, the new condition is that

$$\phi_{f+g}(a,b) := f([a,b]) - [f(a), f(b)] + g([a,b]) \\ - [g(a), f(b)] - [f(a), g(b)] - [g(a), g(b)] = 0.$$

In general this is not so easy to handle, but there is a special important case. When I is abelian, then I is naturally an A module. Denoting this module structure by $a.i$, one has $[f(a), g(b)] = a.g(b)$ (independently of f) and the condition becomes: find a g with

$$f([a,b]) - [f(a), f(b)] = a.g(b) - b.g(a) - g([a,b]).$$

Notice that $\phi_f(a,b) = -\phi_f(b,a)$. Given a Lie algebra A, a skew-symmetric two-variable function $\phi(a,b)$ from $A \bigwedge A$ to an A module M of the form $a.g(b) - b.g(a) - g([a,b])$ is called a *2-coboundary*.

The method consists in stressing a property which the element

$$\phi_f(a,b) := f([a,b]) - [f(a), f(b)]$$

shares with 2-coboundaries, deduced from the Jacobi identity:

Lemma 3.

$$a.\phi_f(b,c) - b.\phi_f(a,c) + c.\phi_f(a,b) \\ - \phi_f([a,b],c) + \phi_f([a,c],b) - \phi_f([b,c],a) = 0. \tag{1.6.2}$$

Proof. From the Jacobi identity

$$a.\phi_f(b,c) - b.\phi_f(a,c) + c.\phi_f(a,b) \\ = [f(a), f([b,c])] - [f(b), f([a,c])] + [f(c), f([a,b])]$$

$$\phi_f([a,b],c) - \phi_f([a,c],b) + \phi_f([b,c],a) \\ = -[f([a,b]), f(c)] + [f([a,c]), f(b)] - [f([b,c]), f(a)].$$ □

A skew-symmetric two-variable function $\phi(a,b)$ from $A \bigwedge A$ to an A module M, satisfying 1.6.2 is called a *2-cocycle*. Then one has to understand under which conditions a 2-cocycle is a 2-coboundary.

In general this terminology comes from a cochain complex associated to Lie algebras. We will not need it but give it for reference. The k-cochains are the maps $C^k(A; M) := \hom(\bigwedge^k A, M)$, the coboundary $\delta : C^k(A; M) \to C^{k+1}(A; M)$ is defined by the formula

$$\delta\phi(a_0, \dots, a_k) = \sum_{i=0}^{k} (-1)^i a_i \phi(a_1, \dots, \check{a}_i, \dots, a_k) \\ + \sum_{i<j} (-1)^{i+j} \phi([a_i, a_j], a_0, \dots, \check{a}_i, \dots \check{a}_j, \dots, a_k).$$

By convention the 0-dimensional cochains are identified with M and $\delta(m)(a) := a.m$.

The complex property means that $\delta \circ \delta = 0$ as one can check directly. Then a cocycle is a cochain ϕ with $\delta\phi = 0$, while a coboundary is a cochain $\delta\phi$. For all k, the space of k-cocycles modulo the k-coboundaries is an interesting object called k-cohomology, denoted $H^k(A; M)$. This is part of a wide class of cohomology groups, which appear as measures of obstructions of various possible constructions.

From now on, in this section we assume that the Lie algebras are finite-dimensional over $\mathbb{C}$. In our case we will again use the Casimir element to prove that:

Proposition. *For a semisimple Lie algebra L, every 2-cocycle with values in a nontrivial irreducible module M is a 2-coboundary.*

Proof. If C is the Casimir element of L, C acts by some nonzero scalar λ on M. We compute it in a different way using 1.6.2 (with $c = u_i$):

$$u_i.\phi(a,b) = a.\phi(u_i,b) - b.\phi(u_i,a) + \phi([u_i,a],b) - \phi([u_i,b],a) + \phi([a,b],u_i) \implies$$

$$u^i.u_i.\phi(a,b) = [u^i,a].\phi(u_i,b) + a.u^i.\phi(u_i,b) - [u^i,b].\phi(u_i,a) - b.u^i.\phi(u_i,a)$$

$$+ u^i.\{\phi([u_i,a],b) - \phi([u_i,b],a) - \phi(u_i,[a,b])\}. \tag{1.6.3}$$

Now the identity 1.4.1 implies $\sum_i k([x,u^i],u_i) + k(u^i,[x,u_i]) = 0, \forall x \in L$ and any bilinear map $k(x,y)$. Apply it to the bilinear maps $k(x,y) := x.\phi(y,b)$, and $x.\phi(y,a)$ getting

$$\sum_i([a,u^i].\phi(u_i,b) + u^i.\phi([a,u_i],b)) = \sum_i([b,u^i].\phi(u_i,a) + u^i.\phi([b,u_i],a)) = 0.$$

Now set $h(x) := \sum_i u^i.\phi(u_i,x)$. Summing all terms of 1.6.3 one has

$$\lambda\phi(a,b) = a.h(b) - b.h(a) - h([a,b]). \tag{1.6.4}$$

Dividing by λ one has the required coboundary condition. □

Cohomology in general is a deep theory with many different applications. We mention some as (difficult) exercises:

Exercise (Theorem of Whitehead). Generalize the previous method to show that, under the hypotheses of the previous proposition we have $H^i(L; M)$ for all $i \geq 0$.[87]

[87] For a semisimple Lie algebra the interesting cohomology is the cohomology of the trivial 1-dimensional representation. This can be interpreted topologically as cohomology of the associated Lie group.

Exercise. Given a Lie algebra A and a module M for A one defines an extension as a Lie algebra L with an abelian ideal identified with M such that $L/I = A$ and the induced action of A on M is the given module action. Define the notion of equivalence of extensions and prove that equivalence classes of extensions are classified by $H^2(A; M)$. In this correspondence, the semidirect product corresponds to the 0 class.

When $M = \mathbb{C}$ is the trivial module, cohomology with coefficients in $\mathbb{C}$ is nonzero and has a deep geometric interpretation:

Exercise. Let G be a connected Lie group, with Lie algebra L. In the same way in which we have constructed the Lie algebra as left invariant vector fields we can consider also the space $\Omega^i(G)$ of left invariant differential forms of degree i for each i. Clearly a left invariant form is determined by the value that it takes at 1; thus, as a vector space $\Omega^i(G) = \bigwedge^i(T_1^*(G)) = \bigwedge^i(L^*)$, the space of i-cochains on L with values in the trivial representation.

Observe that the usual differential on forms commutes with the left G action so d maps $\Omega^i(G)$ to $\Omega^{i+1}(G)$. Prove that we obtain exactly the algebraic cochain complex previously described. Show furthermore that the condition $d^2 = 0$ is another formulation of the Jacobi identity.

By a theorem of Cartan, when G is compact: *the cohomology of this complex computes exactly the de Rham cohomology of G as a manifold.*

We return to our main theme and can now prove the:

Theorem (Levi decomposition). *Let L, A be Lie algebras, A semisimple, and $\pi : L \to A$ a surjective homomorphism. Then there is a splitting $i : A \to L$ with $\pi \circ i = 1_A$.*

Proof. Let $K = \operatorname{Ker}(\pi)$ be the kernel; we will proceed by induction on its dimension. If L is semisimple, K is a direct sum of simple algebras and its only ideals are sums of these simple ideals, so the statement is clear. If L is not semisimple, it has an abelian ideal I which is necessarily in K since A has no abelian ideals. By induction $L/I \to A$ has a splitting $j : A \to L/I$; therefore there is a subalgebra $M \supset I$ with $M/I = A$, and we are reduced to the case in which K is a minimal abelian ideal. Since K is abelian, the action of L on K vanishes on K, and K is an A-module. Since the A-submodules are ideals and K is minimal, it is an irreducible module. We have two cases: $K = \mathbb{C}$ is the trivial module or K is a nontrivial irreducible. In the first case we have $[L, K] = 0$, so A acts on L, and K is a submodule. By semisimplicity we must have a stable summand $L = B \oplus K$. B is then an ideal under L and isomorphic under projection to A.

Now, the case K nontrivial. In this case the action of A on K induces a nonzero associative form on A, nondegenerate on the direct sum of the simple components which are not in the kernel of the representation K, and a corresponding Casimir element $C = \sum_i u^i u_i$. Apply now the cohomological method and construct $f : A \to L$ and the function $\phi(a, b) := [f(a), f(b)] - f([a, b])$. We can apply Lemma 3, so f is a cocycle. Then by the previous proposition it is also a coboundary and hence we can modify f so that it is a Lie algebra homomorphism, as required. □

The previous theorem is usually applied to $A := L/R$ where R is the solvable radical of a Lie algebra L (finite-dimensional over $\mathbb{C}$). In this case a splitting $L = A \ltimes R$ is called a *Levi decomposition* of L.

Lemma 4. *Let L be a finite-dimensional Lie algebra over $\mathbb{C}$, I its solvable radical, and N the nilpotent radical of I as a Lie algebra.*

(i) Then N is also the nilpotent radical of L.
(ii) $[I, I] \subset N$.
(iii) If $a \in I$, $a \notin N$, then $\mathrm{ad}(a)$ acts on N by a linear operator with at least one nonzero eigenvalue.

Proof. (i) By definition the nilpotent radical of the Lie algebra I is the maximal nilpotent ideal in I. It is clearly invariant under any automorphism of I. Since we are in characteristic 0, if D is a derivation, N is also invariant under $\exp(tD)$, a 1-parameter group of automorphisms, hence it is invariant under D. In particular it is invariant under the restriction to I of any inner derivation $\mathrm{ad}(a)$, $a \in L$. Thus N is a nilpotent ideal in L. Since conversely the nilpotent radical of L is contained in I, the claim follows.

(ii) From the corollary of Lie's theorem, $[I, I]$ is a nilpotent ideal.

(iii) If $\mathrm{ad}(a)$ acts in a nilpotent way on N, then $\mathbb{C}a \oplus N$ is nilpotent (Lemma 1, (ii)). From (ii) $\mathbb{C}a \oplus N$ is an ideal, and from i) it follows that $\mathbb{C}a \oplus N \subset N$, a contradiction. □

Lemma 5. *Let $L = A \oplus I$ be a Levi decomposition, $N \subset I$ the nilpotent radical of L.*

Since A is semisimple and I, N are ideals, we can decompose $I = B \oplus N$ where B is stable under $\mathrm{ad}(A)$. *Then* $\mathrm{ad}(A)$ *acts trivially on B.*

Proof. Assume that the action of $\mathrm{ad}(A)$ on B is nontrivial. Then there is a semisimple element $a \in A$ such that $\mathrm{ad}(a) \neq 0$ on B. Otherwise, by Theorem 2 of 1.5, $\mathrm{ad}(A)$ on B would act by nilpotent elements and so, by Engel's Theorem, it would be nilpotent, which is absurd since a nonzero quotient of a semisimple algebra is semisimple.

Since $\mathrm{ad}(a)$ is also semisimple (Theorem 2 of 1.5) we can find a nonzero vector $v \in B$ with $\mathrm{ad}(a)(v) = \lambda v$, $\lambda \neq 0$. Consider the solvable Lie algebra $P := \mathbb{C}a \oplus I$. Then $v \in [P, P]$ and $[P, P]$ is a nilpotent ideal of P (cf. Chapter 4, Cor. 6.3). Hence $\mathbb{C}v + [I, I]$ is a nilpotent ideal of I. From Lemma 4 we then have $v \in N$, a contradiction. □

Theorem 2. *Given a Lie algebra L with semisimple part A, we can embed it into a new Lie algebra L' with the following properties:*

(i) L' has the same semisimple part A as L.
(ii) The solvable radical of L' is decomposed as $B' \oplus N'$, where N' is the nilpotent radical of L', B' is an abelian Lie algebra acting by semisimple derivations, and $[A, B'] = 0$.
(iii) $A \oplus B'$ is a subalgebra and $L' = (A \oplus B') \ltimes N'$.

Proof. Using the Levi decomposition we can start decomposing $L = A \ltimes I$ where A is semisimple and I is the solvable radical. Let N be the nilpotent radical of I. By Lemma 4 it is also an ideal of L. By the previous Lemma 5, decompose $I = B \oplus N$ with $[A, B] = 0$. Let $m := \dim(B)$. We work by induction and construct a sequence of Lie algebras $L_i = A \oplus B_i \oplus N_i, i = 1, \dots, m$, with $L_0 = L, \ L_i \subset L_{i+1}$ and with the following properties:

(i) $B_i \oplus N_i$ is the solvable radical, N_i the nilpotent radical of L_i.

(ii) $[A, B_i] = 0$, and B_i has a basis $a_1, \dots, a_i, b_{i+1}, \dots, b_m$ with a_i inducing commuting semisimple derivations of L_i.

(iii) Finally $[a_h, B_i] = 0, h = 1, \dots, i$.

$L' = L_m$ thus satisfies the requirements of the theorem.

Given L_i as before, we construct L_{i+1} as follows. Consider the derivation $\mathrm{ad}(b_{i+1})$ of L_i and denote by a_{i+1} its semisimple part, still a derivation. By hypothesis the linear map $\mathrm{ad}(b_{i+1})$ is 0 on A. $\mathrm{ad}(b_{i+1})$ preserves the ideals I_i, N_i and, since $[B_i, B_i] \subset N_i$ it maps B_i into N_i. Therefore the same properties hold for its semisimple part:

a_{i+1} preserves I_i, N_i. $a_{i+1}(x) = 0, \forall x \in A$. $a_{i+1}(a_h) = 0, h = 1, \dots, i$ and $a_{i+1}(B_i) \subset N_i$.

Construct the Lie algebra $L_{i+1} := \mathbb{C}a_{i+1} \ltimes L_i = A \oplus (\mathbb{C}a_{i+1} \oplus I_i)$. Let $I_{i+1} := \mathbb{C}a_{i+1} \oplus I_i$. Since a_{i+1} commutes with A, I_{i+1} is a solvable ideal. Since $L_{i+1}/I_{i+1} = A$, I_{i+1} is the solvable radical of L_{i+1}.

The element $\mathrm{ad}(b_{i+1} - a_{i+1})$ acts on N_i as the nilpotent part of the derivation $\mathrm{ad}(b_{i+1})$; thus the space $N_{i+1} := \mathbb{C}(a_{i+1} - b_{i+1}) \oplus N_i$ is nilpotent by §1.6, Lemma 1, (ii).

Since $[B_i, B_i] \subset N_i$ we have that N_{i+1} is an ideal in I_{i+1}. By construction we still have $I_{i+1} = B_i \oplus N_{i+1}$. If N_{i+1} is not the nilpotent radical, we can find a nonzero element $c \in B_i$ so that $\mathbb{C} \oplus N_{i+1}$ is a nilpotent algebra. By Lemma 1, this means that c induces a nilpotent derivation in N_{i+1}. This is not possible since it would imply that c also induces a nilpotent derivation in N_i, so that $\mathbb{C} \oplus N_i$ is a nilpotent algebra and an ideal in I_i, contrary to the inductive assumption that N_i is the nilpotent radical of I_i.

The elements $a_h, \ h = 1, \dots, i + 1$, induce commuting semisimple derivations on I_{i+1} which also commute with A. Thus, under the algebra R generated by the operators $\mathrm{ad}(A)$, $\mathrm{ad}(a_h)$, the representation I_{i+1} is completely reducible. Moreover R acts trivially on I_{i+1}/N_{i+1}. Thus we can find an R-stable complement C_{i+1} to $N_{i+1} \bigoplus_{h=1}^{i+1} \mathbb{C}a_h$ in I_{i+1}. By the previous remarks C_{i+1} commutes with the semisimple elements $a_h, \ h = 1, \dots, i + 1$, and with A. Choosing a basis b_j for C_{i+1}, which is $(m - i - 1)$-dimensional, we complete the inductive step.

(iii) This is just a description, in more structural terms, of the properties of the algebra L' stated in (ii). By construction B' is an abelian subalgebra commuting with A thus $A \oplus B'$ is a subalgebra and $L' = (A \oplus B') \ltimes N'$. Furthermore, the adjoint action of $A \oplus B'$ on N' is semisimple. □

The reader should try to understand this construction as follows. First analyze the question: when is a Lie algebra over $\mathbb{C}$ the Lie algebra of an algebraic group?

By the Jordan–Chevalley decomposition this is related to the problem of when is the derivation $\mathrm{ad}(a)_s$ inner for $a \in L$. So our construction just does this: it makes L closed under Jordan decomposition.

Exercise. Prove that the new Lie algebra is the Lie algebra of an algebraic group.

Warning. In order to do this exercise one first needs to understand Ado's Theorem.

1.7 Ado's Theorem

Before we state this theorem let us make a general remark:

Lemma. *Let L be a Lie algebra and D a derivation. Then D extends to a derivation of the universal enveloping algebra U_L.*

Proof.[88] First D induces a derivation on the tensor algebra by setting

$$D(a_1 \otimes a_2 \otimes \cdots \otimes a_m) = \sum_{i=1}^{m} a_1 \otimes a_2 \otimes \cdots \otimes D(a_i) \otimes \cdots \otimes a_m.$$

Given an associative algebra R, a derivation D, and an ideal I, D factors to a derivation of R/I if and only if $D(I) \subset I$ and, to check this, it is enough to do it on a set of generators. In our case:

$$\begin{aligned} D([a, b] - a \otimes b + b \otimes a) &= [D(a), b] + [a, D(b)] - D(a) \otimes b - a \otimes D(b) \\ &\quad + D(b) \otimes a + b \otimes D(a) \\ &= [D(a), b] - D(a) \otimes b + b \otimes D(a) \\ &\quad + [a, D(b)] - a \otimes D(b) + D(b) \otimes a. \end{aligned}$$ □

Ado's Theorem. *A finite-dimensional Lie algebra L can be embedded in matrices.*

The main difficulty in this theorem is the fact that L can have a center; otherwise the adjoint representation of L on L solves the problem (Chapter 4, §4.1.1). Thus it suffices to find a finite-dimensional module M on which the center $Z(L)$ acts faithfully, since then $M \oplus L$ is a faithful module. We will construct one on which the whole nilpotent radical acts faithfully. This is sufficient to solve the problem.

We give the proof in characteristic 0 and for simplicity when the base field is $\mathbb{C}$.[89]

Proof. We split the analysis into three steps.

(1) L is nilpotent, $L^i = 0$. Let U_L be its universal enveloping algebra. By the PBW Theorem we have $L \subset U_L$. Consider U_L as an L-module by multiplication on the left. Let $J := U_L^i$ be the span of all monomials $a_1 \dots a_k$, $k \geq i, a_h \in L, \forall h$. J is a two-sided ideal of U_L and $M := U_L / U_L^i$ is clearly finite dimensional. We claim that M is a faithful L-module. Let $d_k := \dim L^k$, $k = 1, \dots, i-1$, and fix a basis e_i

[88] In part we follow the proof given by Neretin, cf. [Ne].

[89] This restriction can be easily removed.

for L with the property that for each $k < i$ the e_j, $j \leq d_k$, are a basis of L^k. For an element $e \in L$ we define its *weight* $w(e)$ as the number h such that $e \in L^h - L^{h+1}$ if $e \neq 0$ and $w(0) := \infty$. Since $[L^k, L^h] \subset L^{k+h}$ we have for any two elements $w([a, b]) \geq w(a) + w(b)$. Given any monomial $M := e_{i_1} e_{i_2} \dots e_{i_s}$ we define its *weight* as the sum of the weights of the factors: $w(M) := \sum_{j=1}^{s} w(e_{i_j})$.

Now take a monomial and rewrite it as a linear combination of monomials $e_1^{h_1} \dots e_j^{h_j}$. Each time that we have to substitute a product $e_h e_k$ with $e_k e_h + [e_h, e_k]$ we obtain in the sum a monomial of degree 1 less, but its weight does not decrease. Thus, when we write an element of J in the PBW basis, we have a linear combination of elements of weight at least i. Since no monomial of degree 1 can have weight $> i - 1$, we are done.

(2) Assume L has nilpotent radical N and it is a semidirect product $L = R \ltimes N$, $R = L/N$.

Then we argue as follows. The algebra R induces an algebra of derivations on U_N and clearly the span of monomials of degree $\geq k$, for each k, is stable under these derivations.

U_N / U_N^i is thus an R-module, using the action by derivations and an N-module by left multiplication. If $n \in N$ and $D : N \to N$ is a derivation, $D(na) = D(n)a + nD(a)$. In other words, if L_n is the operator $a \mapsto na$ we have $[D, L_n] = L_{D(n)}$. Thus we have that the previously constructed module U_N / U_N^i is also an $R \ltimes N$ module. Since restricted to N this module is faithful, by the initial remark we have solved the problem.

(3) We want to reduce the general case to the previous case. For this it is enough to apply the last theorem of the previous section, embedding $L = A \ltimes I$ into some $(A \oplus B') \ltimes N'$. □

1.8 Toral Subalgebras

The strategy to unravel the structure of semisimple algebras is similar to the strategy followed by Frobenius to understand characters of groups. In each case one tries to understand how the characteristic polynomial of a *generic element* (in one case of the Lie algebra, in the other of the group algebra) decomposes into factors. In other words, once we have that a generic element is semisimple, we study its eigenvalues.

Definition. A toral subalgebra $\mathfrak{t}$ of L is a subalgebra made only of semisimple elements.

Lemma. *A toral subalgebra* $\mathfrak{t}$ *is abelian.*

Proof. If not, there is an element $x \in \mathfrak{t}$ with a nonzero eigenvalue for an eigenvector $y \in \mathfrak{t}$, or $[x, y] = ay$, $a \neq 0$. On the space spanned by x, y the element y acts as $[y, x] = -ay, [y, y] = 0$. On this space the action of y is given by a nonzero nilpotent matrix. Therefore y is not semisimple as assumed. □

It follows from the lemma that the semisimple operators $\mathrm{ad}(x)$, $x \in \mathfrak{t}$ are simultaneously diagonalizable. When we speak of an eigenvector v for $\mathfrak{t}$ we mean a nonzero vector $v \in L$ such that $[h, v] = \alpha(h)v, \forall h \in \mathfrak{t}$. $\alpha : \mathfrak{t} \to \mathbb{C}$ is then a linear form, called the *eigenvalue*.

In a semisimple Lie algebra L a maximal toral subalgebra $\mathfrak{t}$ is called a *Cartan subalgebra*.[90]

We decompose L into the eigenspaces, or weight spaces, relative to $\mathfrak{t}$. The nonzero eigenvalues define a set $\Phi \subset \mathfrak{t}^* - \{0\}$ of nonzero linear forms on $\mathfrak{t}$ called *roots*. If $\alpha \in \Phi$, $L_\alpha := \{x \in L \,|\, [h, x] = \alpha(h)x, \ \forall h \in \mathfrak{t}\}$ is the corresponding *root space*. The nonzero elements of L_α are called *root vectors* relative to α. L_α is also called a *weight space*.

We need a simple remark that we will use in the next proposition.

Consider the following 3-dimensional Lie algebra M with basis a, b, c and multiplication:

$$[a, b] = c, \ [c, a] = [c, b] = 0.$$

The element c is in the center of this Lie algebra which is nilpotent: $[M, [M, M]] = 0$. In any finite-dimensional representation of this Lie algebra, by Lie's theorem, c acts as a nilpotent element.

Proposition. *(1) A Cartan subalgebra* $\mathfrak{t}$ *of a semisimple Lie algebra L is nonzero.*

(2) The Killing form (a, b)*, restricted to* $\mathfrak{t}$*, is nondegenerate.* L_α, L_β *are orthogonal unless* $\alpha + \beta = 0$. $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$.

(3) $\mathfrak{t}$ *equals its 0 weight space.* $\mathfrak{t} = L_0 := \{x \in L \,|\, [h, x] = 0, \ \forall h \in \mathfrak{t}\}$.

(4) We have $L = \mathfrak{t} \bigoplus_{\alpha \in \Phi} L_\alpha$.

From (2) there is a unique element $t_\alpha \in \mathfrak{t}$ with $(h, t_\alpha) = \alpha(h), \forall h \in \mathfrak{t}$. Then,

(5) For each $\alpha \in \Phi, a \in L_\alpha, \ b \in L_{-\alpha}$*, we have* $[a, b] = (a, b)t_\alpha$*. The subspace* $[L_\alpha, L_{-\alpha}] = \mathbb{C}t_\alpha$ *is 1-dimensional.*

(6) $(t_\alpha, t_\alpha) = \alpha(t_\alpha) \neq 0$.

(7) There exist elements $e_\alpha \in L_\alpha$, $f_\alpha \in L_{\alpha}$, $h_\alpha \in \mathfrak{t}$ *which satisfy the standard commutation relations of* $sl(2, \mathbb{C})$.

Proof. (1) Every element has a Jordan decomposition. If all elements were ad nilpotent, L would be nilpotent by Engel's theorem. Hence there are nontrivial semisimple elements.

(2) First let us decompose L into weight spaces for $\mathfrak{t}$, $L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$.

If $a \in L_\alpha, b \in L_\beta, t \in \mathfrak{t}$ we have $\alpha(t)(a, b) = ([t, a], b) = -(a, [t, b]) = -\beta(t)(a, b)$. If $\alpha + \beta \neq 0$, this implies that $(a, b) = 0$. Since the Killing form is nondegenerate we deduce that the Killing form restricted to L_0 is nondegenerate, and the space L_α is orthogonal to all L_β for $\beta \neq -\alpha$ while L_α and $L_{-\alpha}$ are in perfect duality under the Killing form. This will prove (2) once we show that $L_0 = \mathfrak{t}$. $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$ is a simple property of derivations. If $a \in L_\alpha, b \in L_\beta, t \in \mathfrak{t}$, we have $[t, [a, b]] = [[t, a], b] + [a, [t, b]] = \alpha(t)[a, b] + \beta(t)[a, b]$.

(3) This requires a more careful proof.

[90] There is a general notion of Cartan subalgebra for any Lie algebra which we will not use, cf. [J1].

L_0 is a Lie subalgebra (from 2). Let $a \in L_0$ and decompose $a = a_s + a_n$. Since $\mathrm{ad}(a)$ is 0 on $\mathfrak{t}$, we must have that also $a_s, a_n \in L_0$ since they commute with $\mathfrak{t}$. $\mathfrak{t}+\mathbb{C}a_s$ is still toral and by maximality $a_s \in \mathfrak{t}$, so $\mathfrak{t}$ contains all the semisimple parts of the elements of L_0.

Next let us prove that the Killing form restricted to $\mathfrak{t}$ is nondegenerate.

Assume that $a \in \mathfrak{t}$ is in the kernel of the Killing form restricted to $\mathfrak{t}$. Take $c = c_s + c_n \in L_0$. The element $\mathrm{ad}(a)\,\mathrm{ad}(c_n)$ is nilpotent (since the two elements commute), so it has trace 0. The value $(a, c_s) = \mathrm{tr}(\mathrm{ad}(a)\,\mathrm{ad}(c_s)) = 0$ since $c_s \in \mathfrak{t}$, and by assumption $a \in \mathfrak{t}$ is in the kernel of the Killing form restricted to $\mathfrak{t}$. It follows that a is also in the kernel of the Killing form restricted to L_0. This restriction is nondegenerate, so $a = 0$.

Now decompose $L_0 = \mathfrak{t} \oplus \mathfrak{t}^\perp$, where $\mathfrak{t}^\perp$ is the orthogonal complement to $\mathfrak{t}$ with respect to the Killing form. From the same discussion it follows that $\mathfrak{t}^\perp$ is made of ad nilpotent elements. $t^\perp$ is a Lie ideal of L_0, since $a \in \mathfrak{t}, c \in \mathfrak{t}^\perp, b \in L_0 \implies (a, [b, c]) = ([a, b], c) = 0$. Now we claim that $\mathfrak{t}^\perp$ is in the kernel of the Killing form restricted to L_0. In fact since $\mathrm{ad}(\mathfrak{t}^\perp)$ is a Lie algebra made of nilpotent elements, it follows by Engel's theorem that the Killing form on $\mathfrak{t}^\perp$ is 0. Since the Killing form on L_0 is nondegenerate, this implies that $\mathfrak{t}^\perp = 0$ and so $\mathfrak{t} = L_0$.

(4) follows from (2), (3) and the definitions.

Since the Killing form on $\mathfrak{t}$ is nondegenerate, we can use it to identify $\mathfrak{t}$ with its dual $\mathfrak{t}^*$. In particular, for each root α we have an element $t_\alpha \in \mathfrak{t}$ with $(h, t_\alpha) = \alpha(h), h \in \mathfrak{t}$.

(5) By (2) $[L_\alpha, L_{-\alpha}]$ is contained in $\mathfrak{t}$ and, if $h \in \mathfrak{t}, a \in L_\alpha, b \in L_{-\alpha}$ we have $(h, [a, b]) = ([h, a], b) = \alpha(h)(a, b) = (h, (a, b)t_\alpha)$. This means that $[a, b] = (a, b)t_\alpha$ lies in the 1-dimensional space generated by t_α.

(6) Since $L_\alpha, L_{-\alpha}$ are paired by the Killing form, we can find $a \in L_\alpha, b \in L_{-\alpha}$ with $(a, b) = 1$, and hence $[a, b] = t_\alpha$. We have $[t_\alpha, a] = \alpha(t_\alpha)a, [t_\alpha, b] = -\alpha(t_\alpha)b$. If $\alpha(t_\alpha) = 0$ we are in the setting of the remark preceding the proposition: a, b, t_α span a solvable Lie algebra, and in any representation t_α is nilpotent. Since $t_\alpha \in \mathfrak{t}$, it is semisimple in every representation, in particular in the adjoint representation. We deduce that $\mathrm{ad}(t_\alpha) = 0$, which is a contradiction.

(7) We are claiming that one can choose nonzero elements $e_\alpha \in L_\alpha,\ f_\alpha \in L_{-\alpha}$ such that, setting $h_\alpha := [e_\alpha, f_\alpha]$, we have the canonical commutation relations of $sl(2, \mathbb{C})$:

$$h_\alpha := [e_\alpha, f_\alpha],\ [h_\alpha, e_\alpha] = 2e_\alpha,\ [h_\alpha, f_\alpha] = -2f_\alpha.$$

In fact let $e_\alpha \in L_\alpha, f_\alpha \in L_{-\alpha}$ with $(e_\alpha, f_\alpha) = 2/(t_\alpha, t_\alpha)$ and let $h_\alpha := [e_\alpha, f_\alpha] = 2/(t_\alpha, t_\alpha)t_\alpha$. We have $[h_\alpha, e_\alpha] = \alpha(2/(t_\alpha, t_\alpha)t_\alpha)e_\alpha = 2e_\alpha$. The computation for f_α is similar. □

One usually identifies $\mathfrak{t}$ with its dual $\mathfrak{t}^*$ using the Killing form. In this identification t_α corresponds to α. One can transport the Killing form to $\mathfrak{t}^*$. In particular we have for two roots α, β that

$$(1.8.1) \qquad (\alpha, \beta) = (t_\alpha, t_\beta) = \alpha(t_\beta) = \beta(t_\alpha) = \frac{(\alpha, \alpha)}{2}\beta(h_\alpha).$$

1.9 Root Spaces

At this point we can make use of the powerful results we have about the representation theory of $sl(2, \mathbb{C})$.

Lemma 1. *Given a root α and the algebra $sl_\alpha(2, \mathbb{C})$ spanned by $e_\alpha, f_\alpha, h_\alpha$, we can decompose L as a direct sum of irreducible representations of $sl_\alpha(2, \mathbb{C})$ in which the highest weight vectors are weight vectors also for $\mathfrak{t}$.*

Proof. The space of highest weight vectors is $U := \{a \in L \mid [e_\alpha, a] = 0\}$. If $h \in \mathfrak{t}$ and $[e_\alpha, a] = 0$, we have $[e_\alpha, [h, a]] = [[e_\alpha, h], a] = -\alpha(h)[e_\alpha, a] = 0$, so U is stable under $\mathfrak{t}$. Since $\mathfrak{t}$ is diagonalizable we have a basis of weight vectors. □

We have two possible types of highest weight vectors, either root vectors, or elements $h \in \mathfrak{t}$ with $\alpha(h) = 0$. The latter are the trivial representations of $sl_\alpha(2, \mathbb{C})$. For the others, if e_β is a highest weight vector and a root vector relative to the root β, we have that it generates under $sl_\alpha(2, \mathbb{C})$ an irreducible representation of dimension $k + 1$ where k is the weight of e_β under h_α, i.e., $\beta(h_\alpha)$. The elements $\mathrm{ad}(f_\alpha)^i(e_\beta)$, $i = 0, \dots, k$, are all nonzero root vectors of weights $\beta - i\alpha$. These roots by definition form an *α-string.*

One of these irreducible representations is the Lie algebra $sl_\alpha(2, \mathbb{C})$ itself.

We can next use the fact that all these representations are also representations of the group $SL(2, \mathbb{C})$. In particular, let us see how the element $s_\alpha := \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$ acts. If $h \in \mathfrak{t}$ is in the kernel of α, we have seen that h is killed by $sl_\alpha(2, \mathbb{C})$ and so it is fixed by $SL_\alpha(2, \mathbb{C})$. Instead $s_\alpha(h_\alpha) = -h_\alpha$. We thus see that s_α induces on $\mathfrak{t}$ the orthogonal reflection relative to the *root hyperplane* $H_\alpha := \{x \in \mathfrak{t} \mid \alpha(x) = 0\}$.

Lemma 2. *The group $SL_\alpha(2, \mathbb{C})$ acts by automorphisms of the Lie algebra. Given two roots α, β we have that $s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ is a root, $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ is an integer, and $s_\alpha(L_\beta) = L_{s_\alpha(\beta)}$.*

Proof. Since $sl_\alpha(2, \mathbb{C})$ acts by derivations, its exponentials, which generate $SL_\alpha(2, \mathbb{C})$, act by automorphisms. If $a \in L_\beta$, $h \in \mathfrak{t}$, we have $[h, s_\alpha(a)] = [s_\alpha^{-1}(h), a] = \beta(s_\alpha^{-1}(h))a = s_\alpha(\beta)(h)a$. The roots come in α strings $\beta - i\alpha$ with $\beta(h_\alpha)$ a positive integer. We have $2(\beta - i\alpha, \alpha)/(\alpha, \alpha) = 2(\beta, \alpha)/(\alpha, \alpha) - 2i = 2\beta(t_\alpha)/(\alpha, \alpha) - 2i = \beta(h_\alpha) - 2i$. □

Proposition. *(1) For every root α we have* $\dim L_\alpha = 1$.

(2) If $\alpha \in \Phi$ we have $c\alpha \in \Phi$ if and only if $c = \pm 1$.

(3) If $\alpha, \beta, \alpha + \beta$ are roots, $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.

Proof. We want to take advantage of the fact that in each irreducible representation of $sl_\alpha(2, \mathbb{C})$ there is a unique weight vector (up to scalars) of weight either 0 or 1 (Remark 1.1).

Let us first take the sum

$$M_\alpha = \mathfrak{t} + \bigoplus_{\beta \in \Phi,\ \beta = c\alpha} L_\beta$$

of all the irreducible representations in which the weights are multiples of α. We claim that this sum coincides with $\mathfrak{t} + sl_\alpha(2, \mathbb{C})$. In M_α the 0 weights for h_α are also 0 weights for $\mathfrak{t}$ by definition. By (3) of Proposition 1.8 the zero weight space of $\mathfrak{t}$ is $\mathfrak{t}$. Hence in M_α there are no other even representations apart from $\mathfrak{t} + sl_\alpha(2, \mathbb{C})$. This already implies that $\dim L_\alpha = 1$ and that no even multiple of a root is a root. If there were a weight vector of weight 1 for h_α, this would correspond to the weight $\alpha/2$ on $\mathfrak{t}$. This is not a root; otherwise, we contradict the previous statement since $\alpha = 2(\alpha/2)$ is a root. This proves finally that $M_\alpha = \mathfrak{t} + sl_\alpha(2, \mathbb{C})$ which implies (1) and (2).

For (3), given a root β let us consider all possible roots of type $\beta + i\alpha$ with i any integer. The sum P_β of the corresponding root spaces is again a representation of $sl_\alpha(2, \mathbb{C})$. The weight under h_α of a root vector in $L_{\beta+i\alpha}$ is $\beta(h_\alpha) + 2i$. Since these numbers are all distinct and all of the same parity, from the structure of representations of $sl_\alpha(2, \mathbb{C})$, we have that P_β is irreducible. In an irreducible representation if u is a weight vector for h_α of weight λ and $\lambda + \alpha$ is a weight, we have that $e_\alpha u$ is a nonzero weight vector of weight $\lambda + \alpha$. This proves that if $\alpha + \beta$ is a root, $[e_\alpha, L_\beta] = L_{\alpha+\beta}$. □

The integer $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ appears over and over in the theory; it deserves a name and a symbol. It is called a *Cartan integer* and denoted by $\langle \beta \mid \alpha \rangle := \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. Formula 1.8.1 becomes

$$\beta(h_\alpha) = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} = \langle \beta \mid \alpha \rangle. \tag{1.9.1}$$

Warning. The symbol $\langle \beta \mid \alpha \rangle$ is linear in β but *not* in α.

The next fact is that:

Theorem. *(1) The rational subspace $V := \sum_{\alpha \in \Phi} \mathbb{Q}\alpha$ is such that $\mathfrak{t}^* = V \otimes_{\mathbb{Q}} \mathbb{C}$.*

(2) For the Killing form (h, k), $h, k \in \mathfrak{t}$, we have

$$(h, k) = \operatorname{tr}(\operatorname{ad}(h)\operatorname{ad}(k)) = \sum_{\alpha \in \Phi} \alpha(h)\alpha(k) = 2 \sum_{\alpha \in \Phi^+} \alpha(h)\alpha(k). \tag{1.9.2}$$

(3) The dual of the Killing form, restricted to V, is rational and positive definite.

Proof. (1) We have already seen that the numbers $2(\beta, \alpha)/(\alpha, \alpha)$ are integers. First, we have that the roots α span $\mathfrak{t}^*$; otherwise there would be a nonzero element $h \in \mathfrak{t}$ in the center of L. Let α_i, $i = 1, \ldots, n$, be a basis of $\mathfrak{t}^*$ extracted from the roots. If α is any other root, write $\alpha = \sum_i a_i \alpha_i$. It is enough to show that the coefficients a_i are rationals so that the α_i are a $\mathbb{Q}$-basis for V. In order to compute the coefficients a_i we may take the scalar products and get $(\alpha, \alpha_j) = \sum_i a_i(\alpha_i, \alpha_j)$. We can then solve this using Cramer's rule. To see that the solution is rational we can multiply it by $2/(\alpha_j, \alpha_j)$ and rewrite the system with integer coefficients $\langle \alpha \mid \alpha_j \rangle = \sum_i a_i \langle \alpha_i \mid \alpha_j \rangle$.

(2) If h, k are in the Cartan subalgebra, the linear operators $\operatorname{ad}(h)$, $\operatorname{ad}(k)$ commute and are diagonal with simultaneous eigenvalues $\alpha(h), \alpha(k)$, $\alpha \in \Phi$. Therefore the formula follows.

(3) For a root β apply 1.9.2 to t_β and get

$$(\beta, \beta) = 2 \sum_{\alpha \in \Phi^+} (\alpha, \beta)^2.$$

This implies $2/(\beta, \beta) = \sum_{\alpha \in \Phi^+} \langle \alpha \mid \beta \rangle^2 \in \mathbb{N}$. From 1.8.1 and the fact that $\langle \beta \mid \alpha \rangle$ is an integer, it follows that $\beta(t_\alpha)$ is rational. Thus, if h is in the rational space W spanned by the t_α, the numbers $\alpha(h)$ are rational, so on this space the Killing form is rational and positive definite.

By duality W is identified with V since t_α corresponds to α. □

Remark. As we will see, in all of the important formulas the Killing form appears only through the Cartan integers which are invariant under changes of scale.

The way in which Φ sits in the Euclidean space $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$ can be axiomatized giving rise to the abstract notion of *root system*. This is the topic of the next section.

One has to understand that the root system is independent of the chosen toral subalgebra. One basic theorem states that all Cartan subalgebras are conjugate under the group of inner automorphisms. Thus the dimension of each of them is a well-defined number called the *rank* of L. The root systems are also isomorphic (see §2.8).

2 Root Systems

2.1 Axioms for Root Systems

Root systems can be viewed in several ways. In our setting they give an axiomatized approach to the properties of the roots of a semisimple Lie algebras, but one can also think more geometrically of their connection to *reflection groups*.

In a Euclidean space E the reflection with respect to a hyperplane H is the orthogonal transformation which fixes H and sends each vector v, orthogonal to H, to $-v$. It is explicitly computed by the formula (cf. Chapter 5, §3.9):

(2.1.1) $$r_v : x \mapsto x - \frac{2(x, v)}{(v, v)} v.$$

Lemma. *If X is an orthogonal transformation and v a vector,*

(2.1.2) $$X r_v X^{-1} = r_{X(v)}.$$

A finite reflection group is a finite subgroup of the orthogonal group of E, generated by reflections.[91] Among finite reflection groups a special role is played by *crystallographic groups*, the groups that preserve some lattice of integral points.

Roots are a way to construct the most important crystallographic groups, the *Weyl groups*.

[91] There are of course many other reflection groups, infinite and defined on non-Euclidean spaces, which produce rather interesting geometry but play no role in this book.

Definition 1. Given a Euclidean space E (with positive scalar product (u, v)) and a finite set Φ of nonzero vectors in E, we say that Φ is a *reduced root system* if:

(1) the elements of Φ span E;
(2) $\forall\alpha \in \Phi, c \in \mathbb{R}$, we have $c\alpha \in \Phi$ if and only if $c = \pm 1$;
(3) the numbers

$$\langle \alpha \mid \beta \rangle := \frac{2(\alpha, \beta)}{(\beta, \beta)}$$

are integers (called Cartan integers);
(4) For every $\alpha \in \Phi$ consider the reflection $r_\alpha : x \to x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha$. Then $r_\alpha(\Phi) = \Phi$.

The dimension of E is also called the *rank* of the root system.

The theory developed in the previous section implies that the roots $\Phi \in \mathfrak{t}^*$ arising from a semisimple Lie algebra form a root system in the real space which they span, considered as a Euclidean space under the restriction of the dual of the Killing form.

Axiom (4) implies that the subgroup generated by the orthogonal reflections r_α is a finite group (identified with the group of permutations that it induces on Φ). This group is usually denoted by W and called the *Weyl group*. It is a basic group of symmetries similar to the symmetric group with its canonical permutation representation.

When the root system arises from a Lie algebra, it is in fact possible to describe W also in terms of the associated algebraic or compact group as N_T/T where N_T is the normalizer of the associated maximal torus (cf. 6.7 and 7.3).

Definition 2. A root system Φ is called reducible if we can divide it as $\Phi = \Phi^1 \cup \Phi^2$, into mutually orthogonal subsets; otherwise, it is irreducible.

An isomorphism between two root systems Φ_1, Φ_2 is a 1-1 correspondence between the two sets of roots which preserves the Cartan integers.

Exercise. It can be easily verified that any isomorphism between two irreducible root systems is induced by the composition of an isometry of the ambient Euclidean spaces and a homothety (i.e., a multiplication by a nonzero scalar).

First examples In Euclidean space $\mathbb{R}^n$ the standard basis is denoted by $\{e_1, \dots, e_n\}$.

Type A_n The root system is in the subspace E of $\mathbb{R}^{n+1}$ where the sum of the coordinates is 0 and the roots are the vectors $e_i - e_j$, $i \neq j$. The Weyl group is the symmetric group S_{n+1}, which permutes the basis elements. Notice that the Weyl group permutes transitively the roots. The roots are the integral vectors of V with Euclidean norm 2.

Type B_n Euclidean space $\mathbb{R}^n$, roots:

(2.1.3) $$\pm e_i, \quad e_i - e_j,\ e_i + e_j,\ -e_i - e_j,\ i \neq j \leq n.$$

The Weyl group is the semidirect product $S_n \ltimes Z/(2)^n$ of the symmetric group S_n, which permutes the coordinates, with the *sign group* $\mathbb{Z}/(2)^n := (\pm 1, \pm 1, \dots, \pm 1)$, which changes the signs of the coordinates.

Notice that in this case we have two types of roots, with Euclidean norm 2 or 1. The Weyl group has two orbits and permutes transitively the two types of roots.

Type C_n Euclidean space $\mathbb{R}^n$, roots:

(2.1.4)
$$e_i - e_j,\ e_i + e_j,\ -e_i - e_j,\ i \neq j, \quad \pm 2e_i,\ i = 1, \dots, n.$$

The Weyl group is the same as for B_n. Again we have two types of roots, with Euclidean norm 2 or 4, and the Weyl group permutes transitively the two types of roots.

From the previous analysis it is in fact clear that there is a duality between roots of type B_n and of type C_n, obtained formally by passing from roots e to *coroots*, $e^\vee := \frac{2e}{(e,e)}$.

Type D_n Euclidean space $\mathbb{R}^n$, roots:

(2.1.5)
$$e_i - e_j,\ e_i + e_j,\ -e_i - e_j,\ i \neq j \leq n.$$

The Weyl group is a semidirect product $S_n \ltimes S$. S_n is the symmetric group permuting the coordinates. S is the subgroup (of index 2), of the *sign group* $\mathbb{Z}/(2)^n := (\pm 1, \pm 1, \dots, \pm 1)$ which changes the signs of the coordinates, formed by only even number of sign changes.[92]

The Weyl group permutes transitively the roots which all have Euclidean norm 2.

Exercise. Verify the previous statements.

2.2 Regular Vectors

When dealing with root systems one should think geometrically. The hyperplanes $H_\alpha := \{v \in E \mid (v, \alpha) = 0\}$ orthogonal to the roots are the reflection hyperplanes for the reflections s_α called *root hyperplanes*.

Definition 1. The complement of the union of the root hyperplanes H_α is called the set of *regular vectors*, denoted E^{reg}. It consists of several open connected components called *Weyl chambers*.

Examples. In type A_n the regular vectors are the ones with distinct coordinates $x_i \neq x_j$. For B_n, C_n we have the further conditions $x_i \neq \pm x_j, x_i \neq 0$, while for D_n we have only the condition $x_i \neq \pm x_j$.

It is convenient to fix one chamber once and for all and call it the *fundamental chamber* C. In the examples we can choose:

---

[92] Contrary to type B_n, not all sign changes of the coordinates are possible.

$$A_n : \qquad C := \{(x_1, x_2, \dots, x_{n+1}) \mid x_1 > x_2 \dots > x_{n+1}, \ \sum_i x_i = 0\},$$

$$B_n, C_n : \qquad C := \{(x_1, x_2, \dots, x_n) \mid x_1 > x_2 \dots > x_n > 0\},$$

$$D_n : \qquad C := \{(x_1, x_2, \dots, x_n) \mid x_1 > x_2 \dots > x_n, \ x_{n-1} + x_n > 0\}.$$

For every regular vector v we can decompose Φ into two parts

$$\Phi_v^+ := \{\alpha \in \Phi | (v, \alpha) > 0\}, \ \Phi_v^- := \{\alpha \in \Phi | (v, \alpha) < 0\}.$$

Weyl chambers are clearly convex cones.[93] Clearly this decomposition depends only on the chamber in which v lies. When we have chosen a fundamental chamber we drop the symbol v and write simply $\Phi^+, \ \Phi^-$.

From the definition of regular vector it follows that $\Phi^- = -\Phi^+, \ \Phi = \Phi^+ \cup \Phi^-$. One calls $\Phi^+, \ \Phi^-$ the *positive* and *negative roots.*

Notation We write $\alpha \succ 0$, $\alpha \prec 0$, to mean that α is a positive, resp. a negative root.

To understand root systems one should first of all understand the 2-dimensional case.

From axiom (3) and the Schwarz inequality one has

$$\text{(2.2.1)} \qquad \langle \alpha \mid \beta \rangle \langle \beta \mid \alpha \rangle = \frac{4(\alpha, \beta)^2}{(\beta, \beta)(\alpha, \alpha)} = 4\cos(\theta)^2 \leq 4,$$

where θ is the angle between the two roots. In 2.2.1 the equality is possible if and only if the two roots are proportional or $\alpha = \pm\beta$ by axiom 2. It follows that the only possible convex angles between two roots are $0, \ \pi/6, \ \pi/4, \ \pi/3, \ \pi/2, \ 2\pi/3, \ 3\pi/4, \ 5\pi/6, \pi$.

Exercise. At this point the reader should be able to prove, by simple arguments of Euclidean geometry, that the only possibilities are those shown in the illustration on p. 318, of which one is reducible.

From the picture, it follows that:

Lemma. *If two roots α, β form an obtuse angle, i.e., $(\alpha, \beta) < 0$, then $\alpha + \beta$ is a root.*

If two roots α, β form an acute angle, i.e., $(\alpha, \beta) > 0$, then $\alpha - \beta$ is a root.

The theory of roots is built on the two notions of decomposable and simple roots:

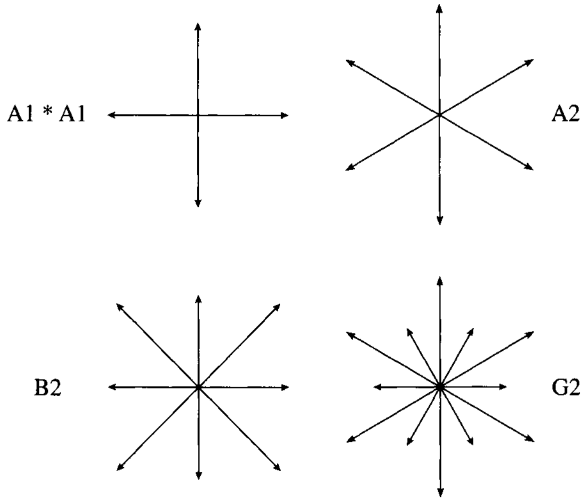
Definition 2. We say that a root $\alpha \in \Phi^+$ is *decomposable* if $\alpha = \beta + \gamma, \ \beta, \gamma \in \Phi^+$.

An indecomposable positive root is also called a *simple* root.

We denote by Δ the set of simple roots in Φ^+. The basic construction is given by the following:

---

[93] A cone is a set S of Euclidean space with the property that if $x \in S$ and $r > 0$ is a positive real number, then $rx \in S$.



Theorem. *(i) Every element of* Φ^+ *is a linear combination of elements of* Δ, *with nonnegative integer coefficients.*

(ii) If $\alpha, \beta \in \Delta$ *we have* $(\alpha\ ,\ \beta) \leq 0$.

(iii) The set Δ *of indecomposable roots in* Φ^+ *is a basis of* E. $\Delta = \{\alpha_1, \dots, \alpha_n\}$ *is called the set, or basis, of* ***simple roots*** *(associated to* Φ^+*).*

Proof. (i) We order the positive roots so that if $(v, \alpha) > (v, \beta)$ we have $\alpha > \beta$. Since $(v, \beta + \gamma) = (v, \beta) + (v, \gamma)$ is a sum of two positive numbers, by induction every positive root is a linear combination of elements of Δ with nonnegative integer coefficients.

(ii) If $\alpha, \beta \in \Delta$, we must show that $\gamma := \alpha - \beta$ is not a root, hence $(\alpha\ ,\ \beta) \leq 0$ (by the previous lemma). Suppose by contradiction that γ is a positive root; then $\alpha = \gamma + \beta$ is decomposable, contrary to the definition. If γ is negative we argue in a similar way, reversing the role of the two roots.

(iii)) Since Φ^+ spans E by assumption, it remains only to see that the elements of Δ are linearly independent. Suppose not. Then, up to reordering, we have a relation $\sum_{i=1}^{k} a_i\alpha_i = \sum_{j=k+1}^{m} b_j\alpha_j$ with the $a_i, b_j \geq 0$. We deduce that

$$0 \leq \left(\sum_{i=1}^{k} a_i\alpha_i, \sum_{i=1}^{k} a_i\alpha_i\right) = \left(\sum_{i=1}^{k} a_i\alpha_i, \sum_{j=k+1}^{m} b_j\alpha_j\right) = \sum_{i,j} a_i b_j (\alpha_i, \alpha_j) \leq 0,$$

which implies $\sum_{i=1}^{k} a_i\alpha_i = 0$. Now $0 = (v, \sum_{i=1}^{k} a_i\alpha_i) = \sum_{i=1}^{k} a_i(v, \alpha_i)$. Since all the $(v, \alpha_i) > 0$, this finally implies all $a_i = 0$, for all i. In a similar way all $b_j = 0$. □

Now assume we are given a root system Φ of rank n. Choose a set of positive roots Φ^+ and the associated simple roots Δ.

The corresponding *fundamental chamber* is

$$C := \{x \in E \mid (x, \alpha_i) > 0, \ \forall \alpha_i \in \Delta\}.$$

In fact we could reverse the procedure, i.e., we could define a *basis* Δ as a subset of the roots with the property that each root is written in a unique way as a linear combination of elements of Δ with the coefficients either all nonnegative or all non-positive integers.

Exercise. It is easy to realize that bases and chambers are in 1-1 correspondence.

Notice that the root hyperplanes H_{α_i} intersect the closure of C in a domain containing the open (in H_{α_i}) set $U_i := \{x \in H_{\alpha_i} \mid (x, \alpha_j) > 0, \ \forall j \neq i\}$.

The hyperplanes H_{α_i} are called the *walls* of the chamber. We write H_i instead of H_{α_i}.

2.3 Reduced Expressions

We set $s_i := s_{\alpha_i}$ and call it a *simple reflection.*

Lemma. *s_i permutes the positive roots $\Phi^+ - \{\alpha_i\}$.*

Proof. Let $\alpha \in \Phi^+$, so that $s_i(\alpha) = \alpha - \langle\alpha|\alpha_i\rangle\alpha_i$. Let $\alpha = \sum_j n_j\alpha_j$ with n_j positive coefficients. Passing to $s_i(\alpha)$, only the coefficient of α_i is modified. Hence if $\alpha \in \Phi^+ - \{\alpha_i\}$, then $s_i(\alpha)$, as a linear combination of the α_i, has at least one positive coefficient. A root can have only positive or only negative coefficients, so $s_i(\alpha)$ is positive. □

Consider an element $w \in W$, written as a product $w = s_{i_1}s_{i_2}\dots s_{i_k}$ of simple reflections.

Definition. We say that $s_{i_1}s_{i_2}\dots s_{i_k}$ is a *reduced* expression of w, if w cannot be written as a product of less than k simple reflections. In this case we write $k = \ell(w)$ and say that w has *length k*.

Remark. Assume that $s_{i_1}s_{i_2}\dots s_{i_k}$ is a reduced expression for w.

(1) $s_{i_k}s_{i_{k-1}}\dots s_{i_1}$ is a reduced expression for w^{-1}.
(2) For any $1 \leq a < b \leq k$, $s_{i_a}s_{i_{a+1}}\dots s_{i_b}$ is also a reduced expression.

Proposition 1 (Exchange property). *For $\alpha_i \in \Delta$, suppose that $s_{i_1}s_{i_2}\dots s_{i_k}(\alpha_i)$ is a negative root; then for some $h \leq k$ we have*

$$s_{i_1}s_{i_2}\dots s_{i_k}s_i = s_{i_1}s_{i_2}\dots s_{i_{h-1}}s_{i_{h+1}}s_{i_{h+2}}\dots s_{i_k}. \tag{2.3.1}$$

Proof. Consider the sequence of roots $\beta_h := s_{i_h}s_{i_{h+1}}s_{i_{h+2}}\dots s_{i_k}(\alpha_i) = s_{i_h}(\beta_{h+1})$. Since β_1 is negative and $\beta_{k+1} = \alpha_i$ is positive, there is a maximum h so that β_{h+1} is positive and β_h is negative. By the previous lemma it must be $\beta_{h+1} = \alpha_{i_h}$ and, by 2.1.2,

$$s_{i_h} = s_{i_{h+1}}s_{i_{h+2}}\dots s_{i_k}s_i(s_{i_{h+1}}s_{i_{h+2}}\dots s_{i_k})^{-1} \quad \text{or}$$
$$s_{i_h}s_{i_{h+1}}s_{i_{h+2}}\dots s_{i_k} = s_{i_{h+1}}s_{i_{h+2}}\dots s_{i_k}s_i,$$

which is equivalent to 2.3.1. □

Notice that the expression on the left-hand side of 2.3.1 is a product of $k+1$ reflections, so it is *not* reduced, since the one on the right-hand side consists of $k-1$ reflections.

Corollary. *If $s_{i_1}s_{i_2}\dots s_{i_k}$ is a reduced expression, then $s_{i_1}s_{i_2}\dots s_{i_k}(\alpha_{i_k}) \prec 0$.*

Proof. $s_{i_1}s_{i_2}\dots s_{i_k}(\alpha_{i_k}) = -s_{i_1}s_{i_2}\dots s_{i_{k-1}}(\alpha_{i_k})$. Since $s_{i_1}s_{i_2}\dots s_{i_k}$ is reduced, the previous proposition implies that $s_{i_1}s_{i_2}\dots s_{i_{k-1}}(\alpha_{i_k}) \succ 0$. □

Proposition 2. *If $w = s_{i_1}s_{i_2}\dots s_{i_k}$ is a reduced expression, then k is the number of positive roots α such that $w(\alpha) \prec 0$.*

The set of positive roots sent into negative roots by $w = s_{i_1}s_{i_2}\dots s_{i_k}$ is the set of elements $\beta_h := s_{i_k}\dots s_{i_{h+1}}(\alpha_{i_h})$, $h = 1, \dots, k$.

Proof. Since $s_{i_k}\dots s_{i_{h+1}}s_{i_h}$ is reduced, the previous proposition implies that β_h is positive. Since $s_{i_1}s_{i_2}\dots s_{i_h}$ is reduced, $w(\beta_h) = s_{i_1}s_{i_2}\dots s_{i_h}(\alpha_{i_h}) \prec 0$. Conversely, if $s_{i_1}s_{i_2}\dots s_{i_k}(\beta) \prec 0$, arguing as in the previous proposition, for some h we must have $s_{i_{h+1}}s_{i_{h+2}}\dots s_{i_k}(\beta) = \alpha_{i_h}$, i.e., $\beta = \beta_h$. □

Exercise. Let $w = s_{i_1}s_{i_2}\dots s_{i_k}$, $A_w := \{\beta \in \Phi^+ \mid w(\beta) \prec 0\}$. If $\alpha, \beta \in A_w$ and $\alpha+\beta$ is a root, then $\alpha+\beta \in A_w$. In the ordered list of the elements β_i, $\alpha+\beta$ always occurs in-between α and β. This is called a *convex ordering.*

Conversely, any convex ordering of A_w is obtained from a reduced expression of w. This is a possible combinatorial device to compute the reduced expressions of an element w.

A set $L \subset \Phi^+$ is *closed* if $\alpha, \beta \in L$ and $\alpha+\beta$ is a root implies $\alpha+\beta \in L$. Prove that $L = A_w$ for some w if and only if L, $\Phi^+ - L$ are closed.

Hint: Find a simple root $\alpha \in L$. Next consider $s_\alpha(L - \{\alpha\})$.

Remark. A permutation σ sends a positive root $\alpha_i - \alpha_j$ to a negative root if and only if $\sigma(i) > \sigma(j)$. We thus say that in the pair $i < j$, the permutation has an *inversion.* Thus, the length of a permutation counts the number of inversions. We say instead that σ has a *descent* in i if $\sigma(i) > \sigma(i+1)$. Descents clearly correspond to simple roots in A_σ.[94]

One of our goals is to prove that W is generated by the simple reflections s_i. Let us temporarily define W' to be the subgroup of W generated by the s_i. We first prove some basic statements for W' and only afterwards we prove that $W = W'$.

Lemma. *(1) W' acts in a transitive way on the chambers.*

(2) W' acts in a transitive way on the bases of simple roots.

(3) If $\alpha \in \Phi$ is a root, there is an element $w \in W'$ with $w(\alpha) \in \Delta$.

[94] There is an extensive literature on counting functions on permutations. In the literature of combinatorics these functions are usually referred to as *statistics*.

Proof. Since bases and chambers are in 1-1 correspondence, (1) and (2) are equivalent.

(1) Let $v \in C$ be in the fundamental chamber and x a regular vector in some chamber C'. Take, in the W' orbit of x, a vector $y = wx$ with the scalar product (y, v) as big as possible (so the angle between y, v is as small as possible). We claim that $y \in C$, this proves that $w(C') = C$ as required.

If $y \notin C$, for some i we have $(y, \alpha_i) < 0$. This gives a contradiction, since $(v, s_i w) = (v, y - 2(y, \alpha_i)/(\alpha_i, \alpha_i)\alpha_i) = (v, y) - 2(y, \alpha_i)/(\alpha_i, \alpha_i)(v, \alpha_i) > (v, y)$.

(3) Take a root α. In the root hyperplane H_α choose a *relatively regular* vector u, i.e., a vector u with the property that $(u, \alpha) = 0,\ (u, \beta) \neq 0,\ \forall \beta \in \Phi,\ \beta \neq \pm\alpha$. Next take a regular vector $y \in E$ with $(\alpha, y) > 0$. If $\epsilon > 0$ is sufficiently small, since the regular vectors are an open cone, $u + \epsilon y$ is regular. $(\beta, u) + \epsilon(\beta, y)$ can be made as close to (β, u) as we wish. At this point we see that α is a positive root for the regular vector $u + \epsilon y$ and, provided we take ϵ sufficiently small, the scalar product $(\alpha, u + \epsilon y)$ is strictly less than the scalar product of $u + \epsilon y$ with any other positive root. This implies readily that α is indecomposable and thus a simple root for the basis Δ' determined by the vector $u + \epsilon y$. Since we have already proved that W' acts transitively on bases we can find $w \in W'$ with $w(\Delta') = \Delta$, hence $w(\alpha) \in \Delta$. □

The next theorem collects most of the basic geometry:

Theorem. *(1) The group W is generated by the simple reflections s_i.*

(2) W acts in a simply transitive way on the chambers.

(3) W acts in a simply transitive way on the bases of simple roots.

(4) Every vector in E is in the W-orbit of a unique vector in the closure $\overline{C}$ of the fundamental chamber.

(5) The stabilizer of a vector x in $\overline{C}$ is generated by the simple reflections s_i with respect to the walls which contain x.

Proof. (1) We have to prove that $W = W'$. Since W is generated by the reflections s_α for the roots α, it is sufficient to see that $s_\alpha \in W'$ for every root α. From the previous lemma, there is a $w \in W'$ and a simple root α_i with $\alpha = w(\alpha_i)$. From 2.1.2 we have $s_\alpha = w s_i w^{-1} \in W' \implies W = W'$.

(2), (3) Suppose now that an element w fixes the fundamental chamber C and its associated basis. Write $w = s_{i_1} s_{i_2} \dots s_{i_k}$ as a reduced expression. If $w \neq 1$, we have $k \geq 1$ and $w(\alpha_{i_k}) \prec 0$ by the previous corollary, a contradiction. So the action is simply transitive.

(4), (5) Since every vector is in the closure of some chamber, from (2) it follows that every vector in Euclidean space is in the orbit of a vector in $\overline{C}$. We prove (4), (5) at the same time. Take $x, y \in \overline{C}$ and $w \in W$ with $wx = y$. We want to show that $x = y$ and w is a product of the simple reflections s_i with respect to the walls H_i containing x. Write w as a reduced product $s_{i_1} s_{i_2} \dots s_{i_k}$ and work by induction on k. If $k > 0$, by the previous propositions w^{-1} maps some simple root α_i to a negative root, hence $0 \leq (y, \alpha_i) = (wx, \alpha_i) = (x, w^{-1}\alpha_i) \leq 0$ implies that x, wx are in the wall H_i. By the exchange property 2.3.1, $\ell(ws_i) < \ell(w)$ and $x = s_i x, y = ws_i x$. We can now apply induction since ws_i has shorter length.

We deduce that $x = y$ and w is generated by the given reflections. □

Remark. If C is the fundamental chamber, then also $-C$ is a chamber with corresponding basis $-\Delta$. We thus have a unique element $w_0 \in W$ with the property that $w_0(\Phi^+) = \Phi^-$.

w_0 is the *longest element of* W, its length is $N = |\Phi^+|$, the number of positive roots.

Let us choose a reduced expression $w_0 = s_{i_1} s_{i_2} \dots s_{i_N}$.

Proposition. *We obtain the list of positive roots in the form*

$$\beta_h := s_{i_N} \dots s_{i_{h+1}}(\alpha_{i_h}), \; h = 1, \dots, N. \tag{2.3.2}$$

Proof. Apply Proposition 2 of §2.3 to w_0. □

Example. In S_n the longest element is the permutation reversing the order $w_0(i) := n + 1 - i$.

The previous theorem implies that every element $w \in W$ has a reduced expression as a product of simple reflections, hence a length $l(w)$.

Exercise. Given $w \in W$ and a simple reflection s_i, prove that $l(ws_i) = l(w) \pm 1$. Moreover $l(ws_i) = l(w) + 1$ if and only if $w(\alpha_i) \succ 0$.

Given any vector $v \in E$, its stabilizer in W is generated by the reflections s_α for the roots α which satisfy $\alpha(v) = 0$.

2.4 Weights

First, let us discuss the *coroots* $\Phi^\vee$ where $e^\vee := \frac{2e}{(e,e)}$.

Proposition 1. *(1) $\Phi^\vee$ is a root system in E having the same regular vectors as Φ.*

(2) If Δ is a basis of simple roots for Φ, then $\Delta^\vee$ is a basis of simple roots for $\Phi^\vee$.

Proof. (1) is clear. For (2) we need a simple remark. Given a basis $e_1, \dots, e_n$ of a real space and the quadrant $C := \{\sum_i a_i e_i, a_i \geq 0\}$, we have that the elements $ae_i \in C$ are characterized as those vectors in C which cannot be written as sum of two linearly independent vectors in C.

If C is the fundamental Weyl chamber, a vector $v \in E$ is such that $(x, v) \geq 0$, $\forall v \in C$, if and only if x is a linear combination with positive coefficients of the elements $\Delta^\vee$ or of the elements in Δ', the simple roots for $\Phi^\vee$. The previous remark implies that the elements of $\Delta^\vee$ are multiples of those of Δ', hence $\Delta^\vee = \Delta'$ from axiom (2). □

Given the set of simple roots $\alpha_1, \dots, \alpha_n$ (of rank n), the matrix $C := (c_{ij})$ with entries the Cartan integers $c_{ij} := \langle \alpha_i \mid \alpha_j \rangle$ is called the *Cartan matrix* of the root system.

One can also characterize root systems by the properties of the Cartan matrix. This is an integral matrix A, which satisfies the following properties:

(1) $c_{ii} = 2,\ c_{ij} \leq 0$, if $i \neq j$. If $c_{ij} = 0$, then $c_{ji} = 0$.
(2) A is symmetrizable, i.e., there is a diagonal matrix D with positive integers entries d_i, $(= (\alpha_i, \alpha_i))$ such that $A := CD$ is symmetric.
(3) $A := CD$ is positive definite.

If the root system is irreducible, we have a corresponding irreducibility property for C: one cannot reorder the rows and columns (with the same permutation) and make C in block form $C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$.

We introduce now, for a given root system, an associated lattice called the *weight lattice*. Its introduction has a full justification in the representation theory of semisimple Lie algebras (see §5). For now the *weight lattice* Λ will be introduced geometrically, as follows:

(2.4.1) $$\Lambda := \{\lambda \in E \,|\, \langle \lambda | \alpha \rangle \in \mathbb{Z},\ \forall \alpha \in \Phi\}.$$

From proposition (1) it follows that we also have

$$\Lambda := \{\lambda \in E \,|\, \langle \lambda | \alpha \rangle \in \mathbb{Z},\ \forall \alpha \in \Delta\}.$$

One also sets

$$\Lambda^+ := \{\lambda \in \Lambda | \langle \lambda, \alpha \rangle \geq 0,\ \forall \alpha \in \Phi^+\}.$$

Λ^+ is called the *set of dominant weights*.

In particular we have the elements $\omega_i \in \Lambda$ with $(\omega_i \,|\, \alpha_j^\vee) = \langle \omega_i \,|\, \alpha_j \rangle = \delta_{ij}$, $\forall \alpha_j \in \Delta$, which are the basis dual to $\Delta^\vee$. Then if $\lambda = \sum_i m_i \omega_i$ we have

(2.4.2) $$m_i = \langle \lambda \,|\, \alpha_i \rangle, \quad \lambda = \sum_i \langle \lambda \,|\, \alpha_i \rangle \omega_i, \qquad \Lambda^+ = \Big\{ \sum_{i=1}^n m_i \omega_i,\ m_i \in \mathbb{N} \Big\}.$$

$(\omega_1, \omega_2, \ldots, \omega_n)$ is called the *set of fundamental weights* (relative to Φ^+).

Exercise. Draw the picture of the simple roots and the fundamental weights for root systems of rank 2.

Let us define the *root lattice* to be the abelian group generated by the roots. We have:

Proposition 2. *The root lattice is a sublattice of the weight lattice of index the determinant of the Cartan matrix.*

Proof. By the theory developed, a basis of the root lattice is given by the simple roots α_i. From the formula 2.4.2, it follows that $\alpha_j = \sum_i \langle \alpha_j \,|\, \alpha_i \rangle \omega_i = \sum_i c_{j,i} \omega_i$. Since the $c_{j,i}$ are integers, the root lattice is a sublattice. Since the $c_{j,i}$ express a base of the root lattice in terms of a basis of the weight lattice, from the theory of elementary divisors the index is the absolute value of the determinant of the base change. In our case the determinant is positive. □

Theorem. *(a) Λ is stable under the Weyl group.*

(b) Every element of Λ is conjugate to a unique element in Λ^+.

(c) The stabilizer of a dominant weight $\sum_{i=1}^n m_i\omega_i$ is generated by the reflections s_i for the values of i such that $m_i = 0$.

Proof. Take $\lambda \in \Lambda$, $w \in W$. We have $\langle w(\lambda) \,|\, \alpha\rangle = \langle \lambda \,|\, w^{-1}(\alpha)\rangle \in \mathbb{Z}$, $\forall \alpha \in \Phi$, proving (a).

Λ^+ is the intersection of λ with the closure of the fundamental Weyl chamber, and a dominant weight $\lambda = \sum_{i=1}^n m_i\omega_i$ has $m_i = 0$ if and only if $\lambda \in H_i$, from the formula 2.4.2. Hence (b), (c) follow from (6) and (5) of Theorem 2.3. □

It is also finally useful to introduce

$$\Lambda^{++} := \{\lambda \in \Lambda | \langle \lambda \,|\, \alpha\rangle > 0, \ \forall \alpha \in \Phi^+\}. \tag{2.4.3}$$

Λ^{++} is called the *set of regular dominant weights* or *strongly dominant*. It is the intersection of Λ with the (open) Weyl chamber. We have $\Lambda^{++} = \Lambda^+ + \rho$, where $\rho := \sum_i \omega_i$ is the *smallest* element of Λ^{++}. ρ plays a special role in the theory, as we will see when we discuss the Weyl character formula. Let us remark:

Proposition 3. *We have $\langle \rho \,|\, \alpha_i\rangle = 1$, $\forall i$. Then $\rho = 1/2\left(\sum_{\alpha\in\Phi^+}\alpha\right)$ and, for any simple reflection s_i, we have $s_i(\rho) = \rho - \alpha_i$.*

Proof. $\langle \rho \,|\, \alpha_i\rangle = 1$, $\forall i$ follows from the definition. We have $s_i(\rho) = \rho - \langle \rho \,|\, \alpha_i\rangle\alpha_i = \rho - \alpha_i$. Also the element $1/2\left(\sum_{\alpha\in\Phi^+}\alpha\right)$ satisfies the same property with respect to a simple reflection s_i, since such a reflection permutes all positive roots different from α_i sending α_i to $-\alpha_i$. Hence $\rho - 1/2\left(\sum_{\alpha\in\Phi^+}\alpha\right)$ is fixed under all simple reflections and the Weyl group. An element fixed by the Weyl group is in all the root hyperplanes, hence it is 0, and we have the claim. □

For examples see Section 5.1.

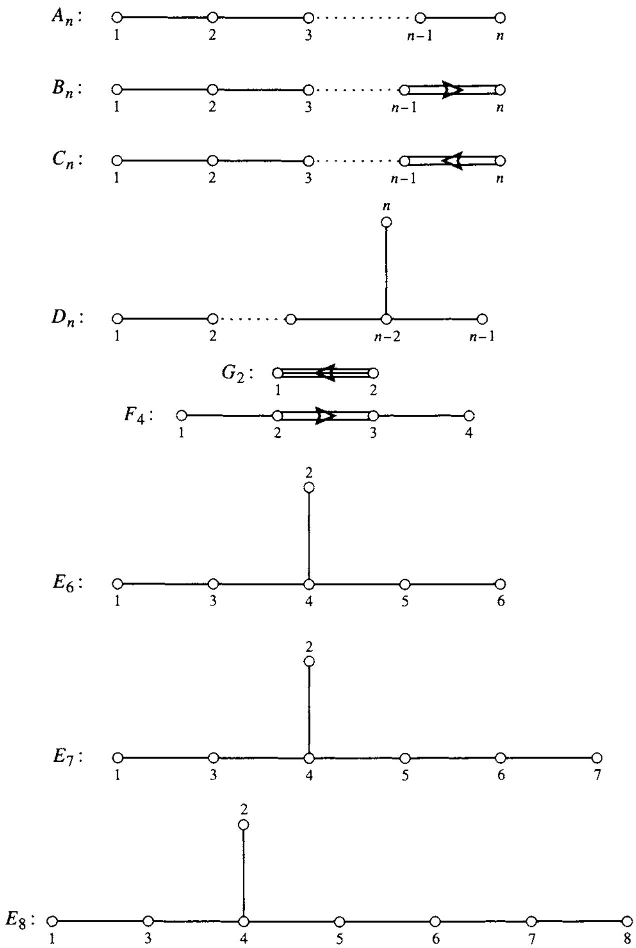
2.5 Classification

The basic result of classification is that it is equivalent to classify simple Lie algebras or irreducible root systems up to isomorphism, or irreducible Cartan matrices up to reordering rows and columns.

The final result is usually expressed by Dynkin diagrams of the types illustrated on p. 325 (see [Hu],[B2]).

The Dynkin diagram is constructed by assigning to each simple root a node $\circ$ and joining two nodes, corresponding to two simple roots α, β, with $\langle \alpha \mid \beta\rangle\langle \beta, \alpha\rangle$ edges. Finally the arrow points towards the *shorter root*. The classification is in two steps. First we see that the only possible Dynkin diagrams are the ones exhibited. Next we see that each of them corresponds to a uniquely determined root system.[95]

[95] There are by now several generalizations of this theory, first to characteristic $p > 0$, then to infinite-dimensional Lie algebras as Kac–Moody algebras. For these one considers Cartan matrices which satisfy only the first property: there is a rich theory which we shall not discuss.



Proposition. *A connected Dynkin diagram determines the root system up to scale.*

Proof. The Dynkin diagram determines the Cartan integers. If we fix the length of one of the simple roots, the other lengths are determined for all other nodes connected to the chosen one. In fact, if α, β are connected by an edge we can use the formula $(\beta, \beta) = \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \beta \rangle}(\alpha, \alpha)$. Since the scalar products of the simple roots are expressed by the Cartan integers and the lengths, the Euclidean space structure on the span of the simple roots is determined.

Next the Cartan integers determine the simple reflections, which generate the Weyl group. Hence the statement follows from Theorem 2.3. □

Let us thus start formally from an irreducible Cartan matrix $C = (c_{i,j}),\ i, j = 1, \dots, n$, (we do not know yet that it is associated to a root system). We can define for C the associated Dynkin diagram as before, with n nodes i and i, j connected by $c_{i,j}c_{j,i}$ edges. If C is irreducible we have a connected diagram.

By assumption $A := CD$ is a positive symmetric matrix and set $a_{i,j} = c_{i,j}d_j$ to be the entries of A. We next construct a vector space E with basis $w_i,\ i = 1, \dots, n$, and scalar product given by A in this basis. Thus E is a Euclidean space.

Theorem. *The Dynkin diagrams associated to irreducible Cartan matrices are those of the list $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$ displayed as above.*

Proof. It is convenient to pass to a simplified form of the Dynkin diagram in which the lengths of the roots play no role. So we replace the vectors w_i of norm $2d_i$, by the vectors $v_i := w_i/|w_i|$ of norm 1. Thus $(v_i, v_j) = \frac{c_{i,j}d_j}{|w_i||w_j|} = \frac{c_{j,i}d_i}{|w_i||w_j|}$. Of the conditions we still have that the v_i are linearly independent, $(v_i, v_j) \leq 0,\ c_{i,j}c_{j,i} := 4(v_i, v_j)^2 = 0, 1, 2, 3$ for distinct vectors. Moreover, by assumption the quadratic form $\sum_{i,j} a_i a_j (v_i, v_j)$ is positive. Call such a list of vectors admissible. The Dynkin diagram is the same as before except that we have no arrow. In particular the diagram is connected. We deduce various restrictions on the diagram, observing that *any subset of an admissible set is admissible.*

1. *The diagram has no loops.* In fact, consider k vectors v_{i_j} out of our list. We have

$$0 < \left(\sum_{j=1}^{k} v_{i_j}, \sum_{j=1}^{k} v_{i_j} \right) = k + \sum_{h,s} 2a_{i_h, j_s}.$$

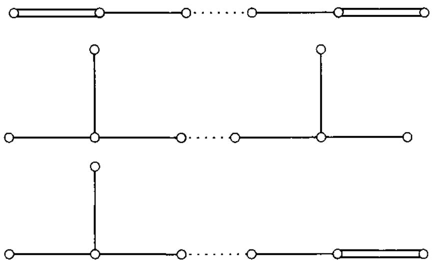
Since for all $a_{i,j} \neq 0$, we have $2a_{i,j} \leq -1$, we must have that the total number of $a_{i,j}$ present is strictly less than k. This implies that the v_i cannot be in a loop.

2. *No more than 3 edges can exit a vertex.* In fact, if $v_1, \dots, v_s$ are vertices connected to v_0, we have that they are not connected to each other since there are no loops. Since they are also linearly independent, we can find a unit vector w orthogonal to the $v_i, i = 1, \dots, s$, in the span of $v_0, v_1, \dots, v_s$. We have that $w, v_i,\ i = 1, \dots, s$, are orthonormal, so $v_0 = (v_0, w)w + \sum_{i=1}^{s}(w, v_i)v_i$ and $1 = (v_0, w)^2 + \sum_{i=1}^{s}(w, v_i)^2$. Since $(v_0, w) \neq 0$, we have $\sum_i (v_0, v_i)^2 < 1 \implies 4\sum_i (v_0, v_i)^2 < 4$, which gives the desired inequality.

3. *If we have a triple edge, then the system is G_2.* Otherwise one of the two nodes of this subgraph is connected to another node. Then out of this at least 4 edges originate.

Suppose that some vectors $v_1, \dots, v_k$ in the diagram form a *simple chain* as in type A_n; in other words $(v_i, v_{i+1}) = -1/2,\ i = 1, \dots, k-1$ (i.e., they are linked by a single edge) and no v_i, $1 < i < k$ is linked to any other node. Then:

4. *Replacing all these vectors by $v = \sum_i v_i$, creates a new admissible list.* In fact, first of all $(v, v) = k-(k-1) = 1$ is a unit vector. Next, if v_j is a vector different from the given ones, it can connect only to v_1 or v_k. In the first case $(v_j, v) = (v_j, v_1)$, and similarly for the second. The diagram associated to this new list is the one in which the simple chain has been contracted to a node. We deduce then that the diagram cannot contain any of the following subdiagrams; otherwise contracting a simple chain we obtain a node to which 4 edges are connected:



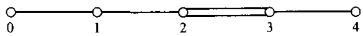
5. *We are thus left with the possible following types:*

(i) A single simple chain, this is type A_n.

(ii) Two nodes a, b connected by two edges, from each one starts a simple chain $a = a_0, a_1, \dots, a_k;\ b = b_0, b_1, \dots, b_h$.

(iii) One node from which three simple chains start.

In case (ii) we must show that it is not possible that both $h, k > 1$. In other words,



is not admissible. In fact consider $\epsilon := v_0 + 2v_1 + 3v_2 + 2\sqrt{2}v_3 + \sqrt{2}v_4$. Computing we have $(\epsilon, \epsilon) = 0$, which is impossible.

In the last case assume we have the three simple chains

$$a_1, \dots, a_{p-1}, a_p = d;\ b_1, \dots, b_{q-1}, b_q = d;\ c_1, \dots, c_{r-1}, c_r = d,$$

from the node d. Consider the three orthogonal vectors

$$x := \sum_{i=1}^{p-1} i a_i, \quad y := \sum_{i=1}^{q-1} i b_i, \quad z := \sum_{i=1}^{r-1} i c_i.$$

d is not in their span and $(x, x) = p(p-1)/2,\ (y, y) = q(q-1)/2,\ (z, z) = r(r-1)/2$.

Expanding d in an orthonormal basis of the space $\langle d, x, y, z\rangle$ we have $(d, x)^2/(x, x) + (d, y)^2/(y, y) + (d, z)^2/(z, z) < 1$. We deduce that

$$1 > \frac{(p-1)^2}{4}\frac{2}{p(p-1)} + \frac{(q-1)^2}{4}\frac{2}{q(q-1)}$$

$$+ \frac{(r-1)^2}{4}\frac{2}{r(r-1)} = \frac{1}{2}\left(3 - \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)\right). \tag{2.5.1}$$

It remains for us to discuss the basic inequality 2.5.1, which is just $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. We can assume $p \le q \le r$. We cannot have $p > 2$ since otherwise the three terms are $\le 1/3$. So $p = 2$ and we have $\frac{1}{q} + \frac{1}{r} > 1/2$. We must have $q \le 3$. If $q = 2$, r can be arbitrary and we have the diagram of type D_n. Otherwise if $q = 3$, we still have $1/r > 1/6$ or $r \le 5$.

For $r = 3, 4, 5$, we have E_6, E_7, E_8. □

We will presently exhibit for each $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$ a corresponding root system. For the moment let us make explicit the corresponding Cartan matrices (see the table on p. 328).

TABLE OF CARTAN MATRICES

$$A_n := \begin{pmatrix} 2 & -1 & 0 & & \dots & & & 0 \\ -1 & 2 & -1 & 0 & \dots & & & 0 \\ 0 & -1 & 2 & -1 & & & & 0 \\ & & & \dots & \dots & & & \\ 0 & 0 & 0 & & & 0 & -1 & 2 \end{pmatrix}$$

$$B_n := \begin{pmatrix} 2 & -1 & 0 & & \dots & & & 0 \\ -1 & 2 & -1 & 0 & \dots & & & 0 \\ 0 & \dots & & \dots & & & & 0 \\ & & & \dots & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & & & 0 & -1 & 2 \end{pmatrix}$$

$$C_n := \begin{pmatrix} 2 & -1 & 0 & & \dots & & & 0 \\ -1 & 2 & -1 & 0 & \dots & & & 0 \\ 0 & \dots & & \dots & & & & 0 \\ & & & \dots & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & & & 0 & -2 & 2 \end{pmatrix}$$

$$D_n := \begin{pmatrix} 2 & -1 & 0 & & & \dots & & & 0 \\ -1 & 2 & -1 & 0 & & \dots & & & 0 \\ 0 & \dots & & \dots & & & & & 0 \\ 0 & \dots & & \dots & -1 & 2 & -1 & 0 & 0 \\ 0 & & \dots & & 0 & -1 & 2 & -1 & -1 \\ & & & \dots & & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & & & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$E_6 := \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \quad E_7 := \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_8 := \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \quad F_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$G_2 := \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

2.6 Existence of Root Systems

The *classical* root systems of types A, B, C, D have already been exhibited as well as G_2. We leave to the reader the simple exercise:

Exercise. Given a root system Φ with Dynkin diagram Δ and a subset S of the simple roots, let E_S be the subspace spanned by S. Then $\Phi \cap E$ is a root system, with simple roots S and Dynkin diagram the subdiagram with nodes S.

Given the previous exercise, we start by exhibiting F_4, E_8 and deduce E_6, E_7 from E_8.

F_4. Consider the 4-dimensional Euclidean space with the usual basis e_1, e_2, e_3, e_4. Let $a := (e_1 + e_2 + e_3 + e_4)/2$ and let Λ be the lattice of vectors of type $\sum_{i=1}^4 n_i e_i + ma$, $n_i, m \in \mathbb{Z}$. Let $\Phi := \{u \in \Lambda \mid (u, u) = 1, \text{ or } (u, u) = 2\}$. We easily see that Φ consists of the 24 vectors $\pm e_i \pm e_j$ of norm 2 and the 24 vectors of norm 1: $\pm e_i$, $(\pm e_1 \pm e_2 \pm e_3 \pm e_4)/2$.

One verifies directly that the numbers $\langle \alpha \mid \beta \rangle$ are integers $\forall \alpha, \beta \in \Phi$. Hence Λ is stable under the reflections s_α, $\alpha \in \Phi$, so clearly Φ must also be stable under the Weyl group. The remaining properties of root systems are also clear.

It is easy to see that a choice of simple roots is

$$e_2 - e_3, e_3 - e_4, e_4, (e_1 - e_2 - e_3 - e_4)/2.$$

E_8. We start, as for F_4, from the 8-dimensional space and the vector $a = \left(\sum_{i=1}^8 e_i\right)/2$. Now it turns out that we cannot just take the lattice

$$\sum_{i=1}^{8} n_i e_i + ma, \quad n_i, m \in \mathbb{Z}$$

but we have to impose a further constraint. For this, we remark that although the expression of an element of Λ as $\sum_{i=1}^8 n_i e_i + ma$ is not unique, the value of $\sum_i n_i$ is unique mod 2. In fact, $\sum_{i=1}^8 n_i e_i + ma = 0$ implies $m = 2k$ even, $n_i = -k, \forall i$ and $\sum_i n_i \equiv 0$, mod 2.

Since the map of Λ to $\mathbb{Z}/(2)$ given by $\sum_i n_i$ is a homomorphism, its kernel Λ_0 is a sublattice of Λ of index 2. We define the root system E_8 as the vectors Φ in Λ_0 of norm 2.

It is now easy to verify that the set Φ consists of the 112 vectors $\pm e_i \pm e_j$, and the 128 vectors $\sum_{i=1}^8 \pm e_i/2 = \sum_{i \in P} e_i - a$, with P the set of indices where the signs are positive. The number of positive signs must be even.

From our discussion is clear that the only point which needs to be verified is that the numbers $\langle \alpha \mid \beta \rangle$ are integers. In our case this is equivalent to proving that the scalar products between two of these vectors is an integer. The only case requiring some discussion is when both α, β are of the type $\sum_{i=1}^8 \pm e_i/2$. In this case the scalar product is of the form $(a - b)/4$, where a counts the number of contributions of 1, while b is the number of contributions of -1 in the scalar product of the two numerators. By definition $a + b = 8$, so it suffices to prove that b is even. The

8 terms ± 1 appearing in the scalar product can be divided as follows. We have t minus signs in α which pair with a plus sign in β, then u minus signs in α which pair with a minus sign in β. Finally we have r plus signs in α which pair with a minus sign in β. By the choice of Λ, the numbers $t+u, r+u$ are even, while $b = t+r$, hence $b \equiv t+u+r+u \equiv 0, \mod 2$.

For the set Δ_8, of simple roots in E_8, we take

$$1/2\Big(e_1 + e_8 - \sum_{i=2}^{7} e_i\Big),\ e_2 + e_1,\ e_2 - e_1,\ e_3 - e_2,\ e_4 - e_3,$$

$$e_5 - e_4,\ e_6 - e_5,\ e_7 - e_6. \tag{2.6.1}$$

E_7, E_6. Although these root systems are implicitly constructed from E_8, it is useful to extract some of their properties. We call $x_i,\ i = 1, \dots, 8$, the coordinates in $\mathbb{R}^8$. The first 7 roots in Δ_8 span the subspace E of $\mathbb{R}^8$ in which $x_7 + x_8 = 0$. Intersecting E with the roots of E_8 we see that of the 112 vectors $\pm e_i \pm e_j$, only those with $i, j \neq 7, 8$ and $\pm(e_7 - e_8)$ appear, a total of 62. Of the 128 vectors $\sum_{i=1}^{8} \pm e_i/2$ we have 64 in which the signs of e_7, e_8 do not coincide, a total of 126 roots.

For E_6 the first 6 roots in Δ_8 span the subspace F in which $x_6 = x_7 = -x_8$. Intersecting F with the roots of E_7 we find the 40 elements $\pm e_i \pm e_j, 1 \leq i < j \leq 5$ and the 32 elements $\pm 1/2(e_6 + e_7 - e_8 + \sum_{i=1}^{5} \pm e_i)$ with an even number of minus signs, a total of 72 roots.

2.7 Coxeter Groups

We have seen that the Weyl group is a reflection group, generated by the simple reflections s_i. There are further important points to this construction. First, the Dynkin diagram also determines defining relations for the generators s_i of W. Recall that if s, t are two reflections whose axes form a (convex) angle θ, then st is a rotation of angle 2θ. Apply this remark to the case of two simple reflections s_i, s_j $(i \neq j)$ in a Weyl group. Then $s_i s_j$ is a rotation of $\frac{2\pi}{m_{i,j}}$ with $m_{i,j} = 2, 3, 4, 6$ according to whether i and j are connected by 0, 1, 2, 3 edges in the Dynkin diagram. In particular,

$$\big(s_i s_j\big)^{m_{i,j}} = 1.$$

It turns out that these relations, together with $s_i^2 = 1$, afford a presentation of W.

Theorem 1. [Hu3, 1.9] *The elements (called Coxeter relations):*

$$s_i^2, \qquad \big(s_i s_j\big)^{m_{i,j}} \qquad \text{Coxeter relations} \tag{2.7.1}$$

generate the normal subgroup of defining relations for W.

In general, one defines a *Coxeter system* (W, S) as the group W with generators S and defining relations

$$(st)^{m_{s,t}} = 1, \quad s, t \in S,$$

where $m_{s,s} = 1$ and $m_{s,t} = m_{t,s} \geq 2$ for $s \neq t$. If there is no relation between s and t, we set $m_{s,t} = \infty$.

The presentation can be completely encoded by a weighted graph (the *Coxeter graph*) Γ. The vertices of Γ are the elements of S, and $s, t \in S$ are connected by an edge (with weight $m_{s,t}$) if $m_{s,t} > 2$, i.e., if they do not commute. For instance, the group corresponding to the graph consisting of two vertices and one edge labeled by ∞ is the free product of two cyclic groups of order 2 (hence it is infinite). There is a natural notion of irreducibility for Coxeter groups which corresponds exactly to the connectedness of the associated Coxeter graphs.

When drawing Coxeter graphs, it is customary to draw an edge with no label if $m_{s,t} = 3$. With this notation, to obtain the Coxeter graph from the Dynkin diagram of a Weyl group just forget about the arrows and replace a double (resp. triple) edge by an edge labeled by 4 (resp. 6).

Several questions arise naturally at this point: classify finite Coxeter groups, and single out the relationships between finite Coxeter groups, finite reflection groups and Weyl groups. In the following we give a brief outline of the answer to the previous problems, referring the reader to [Hu3, Chapters 1, 2] and [B1], [B2], [B3] for a thorough treatment of the theory.

We start from the latter problem. For any Coxeter group G, one builds up the space V generated by vectors α_s, $s \in S$ and the symmetric bilinear form

$$B_G(\alpha_s, \alpha_t) = -\cos \frac{\pi}{m_{s,t}}.$$

We can now define a "reflection", setting

$$\sigma(s)(\lambda) = \lambda - 2B_G(\lambda, \alpha_s)\alpha_s.$$

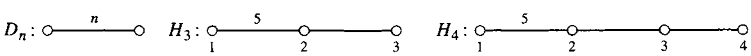
Proposition. [Ti] *The map $s \mapsto \sigma_s$ extends uniquely to a representation $\sigma : G \to GL(V)$. $\sigma(G)$ preserves the form B_G on V. The order of st in G is exactly $m(s, t)$.*

Hence Coxeter groups admit a "reflection representation" (note however that V is not in general a Euclidean space). The main result is the following:

Theorem 2. [Hu3, Theorem 6.4] *The following conditions are equivalent:*

(1) G is a finite Coxeter group.
(2) B_G is positive definite.
(3) G is a finite reflection group.

The classification problem can be reduced to determining the Coxeter graphs for which the form B_G is positive definite. Finally, the graphs of the irreducible Coxeter groups are those obtained from the Weyl groups and three more cases: the *dihedral groups* D_n, of symmetries of a regular n-gon, and two reflection groups, H_3 and H_4, which are 3-dimensional and 4-dimensional, respectively, given by the Coxeter graphs



An explicit construction of the reflection group for the latter cases can be found in [Gb] or [Hu3, 2.13]. Finally, remark that Weyl groups are exactly the finite Coxeter groups for which $m_{s,t} \in \{2, 3, 4, 6\}$. This condition can be shown to be equivalent to the following: G stabilizes a lattice in V.

3 Construction of Semisimple Lie Algebras

3.1 Existence of Lie Algebras

We now pass to the applications to Lie algebras. Let L be a simple Lie algebra, $\mathfrak{t}$ a Cartan subalgebra, Φ the associated root system, and choose the simple roots $\alpha_1, \dots, \alpha_n$. We have seen in Proposition 1.8 (7) that one can choose $e_i \in L_{\alpha_i}$, $f_i \in L_{-\alpha_i}$ so that $e_i, f_i, h_i := [e_i, f_i]$ are $sl(2, \mathbb{C})$ triples, and $h_i = 2/(t_{\alpha_i}, t_{\alpha_i})t_{\alpha_i}$. Call $sl_i(2, \mathbb{C})$ and $SL_i(2, \mathbb{C})$ the corresponding Lie algebra and group.

The previous generators are normalized so that, for each element λ of $\mathfrak{t}^*$,

$$\lambda(h_i) = \lambda(2/(t_{\alpha_i}, t_{\alpha_i})t_{\alpha_i}) = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} = \langle \lambda \mid \alpha_i \rangle. \tag{3.1.1}$$

Then one obtains $[h_i, e_j] := \langle \alpha_j \mid \alpha_i \rangle e_j$, $[h_i, f_j] := -\langle \alpha_i \mid \alpha_j \rangle f_j$ and one can deduce a fundamental theorem of Chevalley and Serre (using the notation $a_{ij} := \langle \alpha_j \mid \alpha_i \rangle$). Before stating it, let us make some remarks. From Proposition 1.2 it follows that, for each i, we can integrate the adjoint action of $sl_i(2, \mathbb{C})$ on L to a rational action of $SL_i(2, \mathbb{C})$. From Chapter 4, §1.5, since the adjoint action is made of derivations, these groups $SL_i(2, \mathbb{C})$ act as automorphisms of the Lie algebra. In particular, let us look at how the element $s_i := \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \in SL_i(2, \mathbb{C})$, acts. We intentionally use the same notation as for the simple reflections.

Lemma. *s_i preserves the Cartan subalgebra and, under the identification with the dual, acts as the simple reflection s_i. $s_i(L_\alpha) = L_{s_i(\alpha)}$.*

Proof. If $h \in \mathfrak{t}$ is such that $\alpha_i(h) = 0$, we have that h commutes with e_i, h_i, f_i, and so it is fixed by the entire group $SL_i(2, \mathbb{C})$. On the other hand, $s_i(h_i) = -h_i$, hence the first part.

Since s_i acts by automorphisms, if u is a root vector for the root α, we have

$$[h, s_i u] = [s_i^2 h, s_i u] = s_i[s_i h, u] = s_i(\alpha(s_i h)u) = s_i(\alpha)(h)s_i u. \qquad \square$$

Exercise. Consider the group $\tilde{W}$ of automorphisms of L generated by the s_i. We have a homomorphism $\pi : \tilde{W} \to W$. Its kernel is a finite group acting on each L_α with ± 1.

Theorem 1. *The Lie algebra L is generated by the $3n$ elements e_i, f_i, h_i, $i = 1, \dots, n$, called **Chevalley generators**. They satisfy the **Serre relations**:*

$$[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j \quad [h_i, f_j] = -a_{ij} f_j \quad [e_i, f_j] = \delta_{ij} h_i \tag{3.1.2}$$

$$\operatorname{ad}(e_i)^{1-a_{ij}}(e_j) = 0, \qquad \operatorname{ad}(f_i)^{1-a_{ij}}(f_j) = 0. \tag{3.1.3}$$

Proof. Consider the Lie subalgebra L^0 generated by the given elements. It is a subrepresentation of each of the groups $SL_i(2, \mathbb{C})$; in particular, it is stable under all the s_i and the group they generate. Given any root α, there is a product w of s_i which sends α to one of the simple roots α_i; hence under the inverse w^{-1} the element e_i is mapped into L_α. This implies that $L_\alpha \subset L$ and hence $L = L^0$.

Let us see why these relations are valid. The first (3) are the definition and normalization. If $i \neq j$, the element $[e_i, f_j]$ has weight $\alpha_i - \alpha_j$. Since this is not a root $[e_i, f_j]$ must vanish.

This implies that f_j is a highest weight vector for a representation of $sl_i(2, \mathbb{C})$ of highest weight $[h_i, f_j] = -a_{i,j} f_j$. This representation thus has dimension $-a_{i,j} + 1$ and each time we apply $\mathrm{ad}(f_i)$ starting from f_j, we descend by 2 in the weight. Thus $\mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0$. The others are similar. □

More difficult to prove is the converse. Given an irreducible root system Φ:

Theorem 2 (Serre). *Let L be the quotient of the free Lie algebra in the generators e_i, f_i, h_i modulo the Lie ideal generated by the relations* 3.1.2 *and* 3.1.3.

L is a simple finite-dimensional Lie algebra. The h_i are a basis of a Cartan subalgebra of L and the associated root system is Φ.

Proof. First, some notation: on the vector space with basis the h_i, we define α_i to be the linear form given by $\alpha_i(h_j) := a_{ij} = \langle \alpha_j \mid \alpha_i \rangle$.

We proceed in two steps. First, consider in the free Lie algebra the relations of type 3.1.2. Call L_0 the resulting Lie algebra. For L_0 we prove the following statements.

(1) In L_0 the images of the $3n$ elements e_i, f_i, h_i remain linearly independent.
(2) $L_0 = \mathfrak{u}_0^- \oplus \mathfrak{h} \oplus \mathfrak{u}_0^+$, where $\mathfrak{h}$ is the abelian Lie algebra with basis $h_1, \dots, h_n$, $\mathfrak{u}_0^-$ is the Lie subalgebra generated by the classes of the f_i, and $\mathfrak{u}_0^+$ is the Lie subalgebra generated by the classes of the e_i.
(3) $\mathfrak{u}_0^+$ (resp. $\mathfrak{u}_0^-$) has a basis of eigenvectors for the commuting operators h_i with eigenvalues $\sum_{i=1}^n m_i\alpha_i$ (resp. $-\sum_{i=1}^n m_i\alpha_i$) with m_i nonnegative integers.

The proof starts by noticing that by applying the commutation relations 3.1.2, one obtains that $L_0 = \mathfrak{u}_0^- + \mathfrak{h} + \mathfrak{u}_+$, where $\mathfrak{h}$ is abelian. Since, for a derivation of an algebra, the two formulas $D(a) = \alpha a$, $D(b) = \beta b$ imply $D(ab) = (\alpha + \beta)ab$, an easy induction proves (3). This implies (2), except for the independence of the h_i, since the three spaces of the decomposition belong to positive, 0, and negative eigenvalues for $\mathfrak{h}$. It remains to prove (1) and in particular exclude the possibility that these relations define a trivial algebra.

Consider the free associative algebra $M := \mathbb{C}\langle e_1, \dots, e_n \rangle$. We define linear operators on M which we call e_i, f_i, h_i and prove that they satisfy the commutation relations 3.1.2.

Set e_i to be left multiplication by e_i. Set h_i to be the semisimple operator which, on a tensor $u := e_{i_1}e_{i_2}\dots e_{i_k}$, has eigenvalue $\sum_{j=1}^k \alpha_{i_j}(h_i)$. Set for simplicity $\sum_{j=1}^k \alpha_{i_j} := \alpha_u$. Define f_i inductively as a map which decreases the degree of tensors by 1 and $f_i 1 = 0$, $f_i(e_j u) := e_j f_i(u) - \delta_i^j \alpha_u(h_i)u$. It is clear that these $3n$ operators are linearly independent. It suffices to prove that they satisfy the relations 3.1.2, since then they produce a representation of L_0 on M.

By definition, the elements h_i commute and

$$(h_j e_i - e_i h_j)u = (\alpha_i(h_j) + \alpha_u(h_j))e_i u - \alpha_u(h_j)e_i u = \alpha_i(h_j)e_i u.$$

For the last relations

$$(f_i e_j - e_j f_i)u = f_i e_j u - e_j f_i u = -\delta_i^j \alpha_u(h_i)u = -\delta_i^j h_i u$$

and $(f_i h_j - h_j f_i)u = \alpha_u(h_j) f_i u - \alpha_{f_i u}(h_j) f_i u$. So it suffices to remark that by the defining formula, f_i maps a vector u of weight α_u into a vector of weight $\alpha_u - \alpha_i$.

Now we can present L as the quotient of L_0 modulo the ideal I generated in L_0 by the unused relations 3.1.3. This ideal can be described as follows. Let I^+ be the ideal of $\mathfrak{u}_+$ generated by the elements $\mathrm{ad}(e_i)^{1-a_{ij}}(e_j)$, and I^- the ideal of $\mathfrak{u}_0^-$ generated by the elements $\mathrm{ad}(f_i)^{1-a_{ij}}(f_j)$. If we prove that I^+, I^- are ideals in L_0, it follows that $I = I^+ + I^-$ and that $L_0/I = \mathfrak{u}_0^+/I^+ \oplus \mathfrak{h} \oplus \mathfrak{u}_0^-/I^-$. We prove this statement in the case of the f's; the case of the e's is identical. Set $R_{i,j} := \mathrm{ad}(f_i)^{1-a_{ij}}(f_j)$. Observe first that $\mathrm{ad}(f_i)^{1-a_{ij}}(f_j)$ is a weight vector under $\mathfrak{h}$ of weight $-(\alpha_j + (1 - a_{ij})\alpha_i)$. By the commutation formulas on the elements, it is clearly enough to prove that $[e_l, \mathrm{ad}(f_i)^{1-a_{ij}}(f_j)] = \mathrm{ad}(e_l)\,\mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0$ for all l. If l is different from i, $\mathrm{ad}(e_l)$ commutes with $\mathrm{ad}(f_i)$ and $\mathrm{ad}(f_i)^{1-a_{ij}}\,\mathrm{ad}(e_l)(f_j) = \delta_l^j\,\mathrm{ad}(f_i)^{1-a_{ij}} h_j$. If $a_{ij} = 0$, f_i commutes with h_j. If $1 - a_{ij} > 1$, we have $[f_i, [f_i, h_j]] = 0$. In either case $\mathrm{ad}(e_l)R_{ij} = 0$. We are left with the case $l = i$. In this case we use the fact that e_i, f_i, h_i generate an $sl(2, \mathbb{C})$. The element f_j is killed by e_i and is of weight $-\langle \alpha_j \,|\, \alpha_i \rangle = -a_{ij}$. Lemma 1.1 applied to $v = f_j$ implies that for all s, $\mathrm{ad}(e_i)\,\mathrm{ad}(f_i)^s f_j = s(-a_{ij} - s + 1)\,\mathrm{ad}(f_i)^{s-1} f_j$. For $s = 1 - a_{ij}$, we indeed get $\mathrm{ad}(e_i)\,\mathrm{ad}(f_i)^s f_j = 0$.

At this point of the analysis we obtain that the algebra L defined by the Chevalley generators and Serre's relations is decomposed in the form $L = \mathfrak{u}^+ \oplus \mathfrak{h} \oplus \mathfrak{u}^-$. $\mathfrak{h}$ has as a basis the elements h_i, $\mathfrak{u}^+$ (resp. $\mathfrak{u}^-$) has a basis of eigenvectors for the commuting operators h_i with eigenvalues $\sum_{i=1}^n m_i \alpha_i$ (resp. $-\sum_{i=1}^n m_i \alpha_i$), with m_i nonnegative integers.

The next step consists of proving that the elements $\mathrm{ad}(e_i)$, $\mathrm{ad}(f_i)$ are *locally nilpotent* (cf. §1.2). Observe for this that, given an algebra L and a derivation D, the set of $u \in L$ killed by a power of D is a subalgebra since

$$D^k(ab) = \sum_{i=0}^{k} \binom{k}{i} D^i(a) D^{k-i}(b)$$

Since clearly for L the elements e_i, h_i, f_i belong to this subalgebra for $\mathrm{ad}(e_j)$, $\mathrm{ad}(f_j)$, this claim is proved.

From Proposition 1.2, for each i, L is a direct sum of finite-dimensional irreducible representations of $SL_i(2, \mathbb{C})$. So we can find, for each i, an element

$s_i = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$ as in §1.9 Lemma 2, which on the roots acts as the simple reflection associated to α_i. Arguing as in 1.9 we see that s_i transforms the subspace L_γ relative to a weight γ into the subspace $L_{s_i(\gamma)}$. In particular if two weights can be transformed into each other by an element of W, the corresponding weight spaces have the same dimension. Remark that, by construction, the space L_{α_i} is 1-dimensional with basis e_i, and similarly for $-\alpha_i$. We already know that the weights are of type $\sum_i m_i\alpha_i$, with the m_i all of the same sign. We deduce, using Theorem 2.3 (4), that if α is a root, $\dim L_\alpha = 1$. Suppose now α is not a root. We want to show that $L_\alpha = 0$. Let us look, for instance, at the case of positive weights, elements of type $\mathrm{ad}(e_{i_1})\,\mathrm{ad}(e_{i_2})\ldots\mathrm{ad}(e_{i_{k-1}})e_{i_k}$. If this monomial has degree > 1, the indices i_{k-1}, i_k must be different (or we have 0), so a multiple of a simple root never appears as a weight. By conjugation the same happens for any root. Finally, assume α is not a multiple of a root. Let us show that conjugating it with an element of W, we obtain a linear combination $\sum_i m_i\alpha_i$ in which two indices m_i have strictly opposite signs. Observe that if α is not a multiple of any root, there is a regular vector v in the hyperplane orthogonal to α. We can then find an element $w \in W$ so that wv is in the fundamental chamber. Since $(w\alpha, wv) = 0$, writing $w\alpha = \sum_i m_i\alpha_i$ we have $\sum_i m_i(\alpha_i, wv) = 0$. Since all $(\alpha_i, wv) > 0$, the claim follows.

Now we know that $w\alpha$ is not a possible weight, so also α is not a weight.

At this point we are very close to the end. We have shown that $L = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$ and that $\dim L_\alpha = 1$. Clearly, $\mathfrak{h}$ is a maximal toral subalgebra and Φ its root system. We only have to show that L is semisimple or, assuming Φ irreducible, that L is simple. Let I be a nonzero ideal of L. Let us first show that $I \supset L_\alpha$ for some root α. Since I is stable under $\mathrm{ad}(\mathfrak{h})$, it contains a nonzero weight vector v. If v is a root vector we have achieved our first step; otherwise $v \in \mathfrak{h}$. Using a root α with $\alpha(v) \neq 0$ we see that $L_\alpha = [v, L_\alpha] \subset I$. Since I is an ideal it is stable under all the $sl(2, \mathbb{C})$ and the corresponding groups. In particular, it is stable under all the s_i, so we deduce that some $e_i, f_i, h_i \in I$. If $a_{ij} \neq 0$, we have $a_{i,j}e_j = [h_i, e_j] \in I$. We thus get all the $e_j \in I$ for all the nodes j of the Dynkin diagram connected to i. Since the Dynkin diagram is connected we obtain in this way all the elements e_j, and similarly for the f_j, h_j. Once we have all the generators, $I = L$. □

Therefore, given a root system, we have constructed an associated semisimple Lie algebra, using these generators and relations, thus proving the *existence theorem.*

This theorem, although quite satisfactory, leaves in the dark the explicit multiplication structure of the corresponding Lie algebra. In fact with some effort one can prove.

Theorem 3. *One can choose nonzero elements e_α in each of the root spaces L_α so that, if $\alpha, \beta, \alpha+\beta$ are roots, one has $[e_\alpha, e_\beta] = \pm e_{\alpha+\beta}$. These signs can be explicitly determined.*

Sketch of proof. Take the longest element of the Weyl group w_0 and write it as a reduced expression $w_0 = s_{i_1}s_{i_2}\ldots s_{i_n}$. For $\alpha = s_{i_1}s_{i_2}\cdots s_{i_{h-1}}\alpha_{i_h}$, define $e_\alpha := s_{i_1}s_{i_2}\cdots s_{i_{h-1}}e_{i_h}$. Next use Exercise 3.1 on $\tilde{W}$, which easily implies the claim. □

The determination of the explicit signs needs a long computation. We refer to Tits [Ti].

3.2 Uniqueness Theorem

We need to prove that the root system of a semisimple Lie algebra L is uniquely determined. We will use the theory of *regular elements.*

Definition. An element $h \in L$ is said to be regular semisimple if h is semisimple and the centralizer of h is a maximal toral subalgebra.

Given a maximal toral subalgebra $\mathfrak{t}$ and an element $h \in \mathfrak{t}$, we see that its centralizer is $\mathfrak{t} \bigoplus_{\alpha \in \Phi \,|\, \alpha(h)=0} L_\alpha$. So we have:

Lemma. *An element $h \in \mathfrak{t}$ is regular if and only if $h \notin \cup_{\alpha \in \Phi} H_\alpha$, $H_\alpha := \{h \in \mathfrak{t} \,|\, \alpha(h) = 0\}$.*[96]

In particular the regular elements of $\mathfrak{t}$ form an open dense set. We can now show:

Theorem. *Two maximal toral subalgebras $\mathfrak{t}_1, \mathfrak{t}_2$ are conjugate under the adjoint group.*

Proof. Let G denote the adjoint group and $\mathfrak{t}_1^{reg}, \mathfrak{t}_2^{reg}$ the regular elements in the two toral subalgebras. We claim that $G\mathfrak{t}_1^{reg}, G\mathfrak{t}_2^{reg}$ contain two Zariski open sets of L and hence have nonempty intersection. In fact, compute at the point $(1, h)$, $h \in \mathfrak{t}_1^{reg}$ the differential of the map $\pi : G \times \mathfrak{t}_1^{reg} \to L$, $\pi(g, h) := gh$. It is $L \times \mathfrak{t}_1 \to [L, h] + \mathfrak{t}_1$. Since h is regular, $[L, h] = \bigoplus_{\alpha \in \Phi} L_\alpha$. This implies that $L = [L, h] + \mathfrak{t}_1$ and the map is dominant.

Once we have found an element $g_1 h_1 = g_2 h_2$, $h_1 \in \mathfrak{t}_1^{reg}$, $h_2 \in \mathfrak{t}_2^{reg}$, we have that $g_2^{-1} g_1 h_1$ is a regular element of $\mathfrak{t}_2$, from which it follows that for the centralizers, $\mathfrak{t}_2 = g_2^{-1} g_1(\mathfrak{t}_1)$. □

Together with the existence we now have:

Classification. *The simple Lie algebras over $\mathbb{C}$ are classified by the Dynkin diagrams.*

Proof. Serre's Theorem shows that the Lie algebra is canonically determined by the Dynkin diagram. The previous result shows that the Dynkin diagram is identified (up to isomorphism) by the Lie algebra and is independent of the toral subalgebra. □

[96] By abuse of notation we use the symbol H_α not only for the hyperplane in the real reflection representation, but also as a hyperplane in $\mathfrak{t}$.

4 Classical Lie Algebras

4.1 Classical Lie Algebras

We want to illustrate the concepts of roots (simple and positive) and the Weyl group for classical groups. In these examples we can be very explicit, and the reader can verify all the statements directly.

We have seen that an associative form on a simple Lie algebra is unique up to scale. If we are interested in the Killing form only up to scale, we can compute the form $\mathrm{tr}(\rho(a)\rho(b))$ for any linear representation of L, not necessarily the adjoint one.

This is in particular true for the classical Lie algebras which are presented from the beginning as algebras of matrices.

In the examples we have that:

(1) $sl(n+1,\mathbb{C}) = A_n$: A Cartan algebra is formed by the space of diagonal matrices $h := \sum_{i=1}^{n+1} \alpha_i e_{ii}$, and $\sum_i \alpha_i = 0$. The spaces L_α are the 1-dimensional spaces generated by the root vectors e_{ij}, $i \neq j$ and $[h, e_{ij}] = (\alpha_i - \alpha_j)e_{ij}$. Thus the linear forms $\sum_{i=1}^{n+1} \alpha_i e_{ii} \to \alpha_i - \alpha_j$ are the roots of $sl(n+1,\mathbb{C})$. We can consider the α_i as an orthonormal basis of a real Euclidean space $\mathbb{R}^{n+1}$. We have the root system A_n.
The positive roots are the elements $\alpha_i - \alpha_j$, $i < j$. The corresponding root vectors e_{ij}, $i < j$ span the Lie subalgebra of strictly upper triangular matrices, and similarly for negative roots

(4.1.1) $$\mathfrak{u}^+ := \bigoplus_{i<j} \mathbb{C}e_{ij}, \ \mathfrak{u}^- := \bigoplus_{i>j} \mathbb{C}e_{ij}, \ \mathfrak{t} := \bigoplus_{i=1}^{n} \mathbb{C}(e_{i,i} - e_{i+1,i+1}).$$

The simple roots and the root vectors associated to simple roots are

(4.1.2) $$\alpha_i - \alpha_{i+1}, \ e_{i,i+1}$$

The Chevalley generators are

(4.1.3) $$e_i := e_{i,i+1}, \quad f_i := e_{i+1,i}, \quad h_i := e_{i,i} - e_{i+1,i+1}.$$

As for the Killing form let us apply 1.9.2 to a diagonal matrix with entries x_i, $i = 1, \dots, n+1$, $\sum x_i = 0$ to get

$$\sum_{i \neq j} (x_i - x_j)^2 = 2(n+1) \sum_{i=1}^{n+1} x_i^2.$$

Using the remarks after 1.9.2 and in §1.4, and the fact that $sl(n+1,\mathbb{C})$ is simple, we see that for any two matrices A, B the Killing form is $2(n+1)\,\mathrm{tr}(AB)$.

(2) $so(2n+1,\mathbb{C}) = B_n$: In block form a matrix $A := \begin{pmatrix} a & b & e \\ c & d & f \\ m & n & p \end{pmatrix}$ satisfies $A^t I_{2n+1} = -I_{2n+1} A$ if and only if $d = -a^t$, b, c are skew symmetric, $p = 0$, $n = -e^t$, $m = -f^t$.

A Cartan subalgebra is formed by the diagonal matrices

$$h := \sum_{i=1}^{n} \alpha_i (e_{ii} - e_{n+i,n+i}).$$

Root vectors are

$$e_{ij} - e_{n+j,n+i},\ i \neq j \leq n,\ e_{i,n+j} - e_{j,n+i},$$
$$i \neq j \leq n,\ e_{n+i,j} - e_{n+j,i}\ i \neq j \leq n$$
$$e_{i,2n+1} - e_{2n+1,i+n}, e_{n+i,2n+1} - e_{2n+1,i}, \quad i = 1, \dots, n$$

with roots

(4.1.4) $$\alpha_i - \alpha_j,\ \alpha_i + \alpha_j,\ -\alpha_i - \alpha_j,\ i \neq j \leq n, \quad \pm\alpha_i,\ i = 1, \dots, n.$$

We have the root system of type B_n. For $so(2n+1, \mathbb{C})$ we set

(4.1.5) $$\Phi^+ := \alpha_i - \alpha_j,\ \alpha_i + \alpha_j,\ i < j \leq n,\ \alpha_i.$$

The simple roots and the root vectors associated to the simple roots are

(4.1.6) $$\alpha_i - \alpha_{i+1},\ \alpha_n; \quad e_{i,i+1} - e_{n+i+1,n+i},\ e_{n,2n+1} - e_{2n+1,2n}.$$

The Chevalley generators are

(4.1.7) $$e_i := e_{i,i+1} - e_{n+i+1,n+i},\ e_n := e_{n,2n+1} - e_{2n+1,2n}.$$

(4.1.8) $$f_i := e_{i+1,i} - e_{n+i,n+i+1},\ f_n := e_{2n,2n+1} - e_{2n+1,n}.$$

$$h_i := e_{i,i} - a_{i+1,i+1} - e_{n+i,n+i} + e_{n+i+1,n+i+1},$$

(4.1.9) $$h_n := e_{n,n} - e_{2n,2n}.$$

As for the Killing form, we apply 1.9.2 to a diagonal matrix with entries $x_i,\ i = 1, \dots, n$, and $-x_i, i = n+1, \dots, 2n$ and get (using 4.1.4):

$$\sum_{i \neq j} [(x_i - x_j)^2 + (x_i + x_j)^2 + 2x_i^2] = 2(n+1) \sum_{i=1}^{n} x_i^2.$$

Using the remark after 1.9.2 and the fact that $so(2n+1, \mathbb{C})$ is simple, we see that for any two matrices A, B the Killing form is $(n+1)\ \mathrm{tr}(AB)$.

$$\mathfrak{u}^+ := \bigoplus_{i<j} \mathbb{C}(e_{ij} - e_{n+j,n+i}) \bigoplus_{i \neq j} \mathbb{C}(e_{i,n+j} - e_{j,n+i})$$

(4.1.10) $$\bigoplus_{i=1}^{n} \mathbb{C}(e_{i,2n+1} - e_{2n+1,i}),$$

$$\mathfrak{u}^- := \bigoplus_{i>j} \mathbb{C}(e_{ij} - e_{n+j,n+i}) \bigoplus_{i \neq j} \mathbb{C}(e_{n+i,j} - e_{n+j,i})$$

(4.1.11) $$\bigoplus_{i=n+1}^{2n} \mathbb{C}(e_{i,2n+1} - e_{2n+1,i}).$$

(3) $sp(2n, \mathbb{C}) = C_n$ In block form a matrix $A := \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ satisfies $A^t J_{2n} = -J_{2n} A$ if and only if $d = -a^t$ and b, c are symmetric.

A Cartan subalgebra is formed by the diagonal matrices

$$h := \sum_{i=1}^{n} \alpha_i (e_{ii} - e_{n+i,n+i}).$$

Root vectors are the elements

$$e_{ij} - e_{n+j,n+i},\ i \neq j \leq n,\ e_{i,n+j} + e_{j,n+i},\ \ i, j \leq n,\ e_{n+i,j} + e_{n+j,i}\ i, j \leq n.$$

with roots

(4.1.12)
$$\alpha_i - \alpha_j,\ \alpha_i + \alpha_j,\ -\alpha_i - \alpha_j,\ i \neq j,\ \pm 2\alpha_i,\ i = 1, \dots, n.$$

We have a root system of type C_n.

For $sp(2n, \mathbb{C})$ we set

(4.1.13)
$$\Phi^+ := \alpha_i - \alpha_j,\ \alpha_i + \alpha_j,\ i < j, 2\alpha_i.$$

The simple roots and the root vectors associated to simple roots are

(4.1.14)
$$\alpha_i - \alpha_{i+1}, 2\alpha_n, \quad e_{i,i+1} - e_{n+i+1,n+i},\ e_{n,2n}.$$

The Chevalley generators are

$$e_i := e_{i,i+1} - e_{n+i+1,n+i},\ i = 1, \dots, n-1,\ e_n := e_{n,2n}$$

$$f_i := e_{i+1,i} - e_{n+i,n+i+1},\ i = 1, \dots, n-1,\ f_n := e_{2n,n}$$

$$h_i := e_{i,i} - e_{i+1,i+1} + e_{n+i+1,n+i+1} - e_{n+i,n+i},\ i < n,$$

$$h_n := e_{n,n} - e_{2n,2n}.$$

As for the Killing form, we apply 1.9.2 to a diagonal matrix with entries x_i, $i = 1, \dots, n, -x_i,\ i = n+1, \dots, 2n$, and get (using 4.1.14):

$$\sum_{i \neq j} [(x_i - x_j)^2 + (x_i + x_j)^2 + 8x_i^2] = 2(n+4) \sum_{i=1}^{n} x_i^2.$$

Using the usual remarks and the fact that $sp(2n, \mathbb{C})$ is simple we see that for any two matrices A, B the Killing form is $(n+4)\ \mathrm{tr}(AB)$. We have $(i, j \leq n)$

(4.1.15)
$$\mathfrak{u}^+ := \bigoplus_{i<j} \mathbb{C}(e_{ij} - e_{n+j,n+i}) \bigoplus_{i \neq j, \leq n} \mathbb{C}(e_{i,n+j} - e_{j,n+i}),$$

(4.1.16)
$$\mathfrak{u}^+ := \bigoplus_{i<j} \mathbb{C}(e_{ij} - e_{n+j,n+i}) \bigoplus_{i,j \leq n} \mathbb{C}(e_{i,n+j} + e_{j,n+i}),$$

(4.1.17)
$$\mathfrak{u}^- := \bigoplus_{i>j} \mathbb{C}(e_{ij} - e_{n+j,n+i}) \bigoplus_{i \neq j} \mathbb{C}(e_{n+i,j} + e_{n+j,i}).$$

In this case the Lie algebra $\mathfrak{u}^+$ in block matrix form is the matrices $\begin{vmatrix} a & b \\ 0 & -a^t \end{vmatrix}$ with a strictly upper triangular and b symmetric.

(4) $so(2n, \mathbb{C}) = D_n$: In block form a matrix $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies $A^t I_{2n} = -I_{2n} A$ if and only if $d = -a^t$ and b, c are skew symmetric.
A Cartan subalgebra is formed by the diagonal matrices

$$h := \sum_{i=1}^{n} \alpha_i (e_{ii} - e_{n+i,n+i}).$$

Root vectors are the elements

$$e_{ij} - e_{n+j,n+i}, \ i \neq j \leq n, \ e_{i,n+j} - e_{j,n+i}, \\ i \neq j \leq n, \ e_{n+i,j} - e_{n+j,i} \ i \neq j \leq n$$

with roots

(4.1.18) $$\alpha_i - \alpha_j, \ \alpha_i + \alpha_j, \ -\alpha_i - \alpha_j, \ i \neq j \leq n.$$

We have the root system D_n.
For $so(2n, \mathbb{C})$ we set

(4.1.19) $$\Phi^+ := \alpha_i - \alpha_j, i < j, \ \alpha_i + \alpha_j, \ i \neq j.$$

The simple roots and the root vectors associated to simple roots are

(4.1.20) $$\alpha_i - \alpha_{i+1}, \ \alpha_{n-1} + \alpha_n, \ e_{i,i+1} - e_{n+i+1,n+i}, \ e_{n-1,2n} - e_{n,2n-1}.$$

The Chevalley generators are

(4.1.21) $$\begin{aligned} e_i &:= e_{i,i+1} - e_{n+i+1,n+i}, \ i = 1, \dots, n-1, \ e_n := e_{n-1,2n} - e_{n,2n-1} \\ f_i &:= e_{i+1,i} - e_{n+i,n+i+1}, \ i = 1, \dots, n-1, \ f_n := e_{2n,n-1} - e_{2n-1,n} \\ h_i &:= e_{i,i} - e_{i+1,i+1} + e_{n+i+1,n+i+1} - e_{n+i,n+i}, \ i < n, \\ h_n &:= e_{n-1,n-1} + e_{n,n} - e_{2n-1,2n-1} - e_{2n,2n}. \end{aligned}$$

As for the Killing form, we apply 1.9.2 to a diagonal matrix with entries $x_i, \ i = 1, \dots, n, -x_i, \ i = 1, \dots, n$, and get (using 4.1.19):

$$\sum_{i \neq j} [(x_i - x_j)^2 + (x_i + x_j)^2] = 2n \sum_{i=1}^{n} x_i^2.$$

Using Remark 1.4 and the fact that $so(2n, \mathbb{C})$ is simple (at least if $n \geq 4$), we see that for any two matrices A, B the Killing form is $n \ \mathrm{tr}(AB)$. We have

(4.1.22) $$\mathfrak{u}^+ := \bigoplus_{i<j} \mathbb{C}(e_{ij} - e_{n+j,n+i}) \bigoplus_{i \neq j, \leq n} \mathbb{C}(e_{i,n+j} - e_{j,n+i}),$$

(4.1.23) $$\mathfrak{u}^- := \bigoplus_{i>j} \mathbb{C}(e_{ij} - e_{n+j,n+i}) \bigoplus_{i \neq j, \leq n} \mathbb{C}(e_{n+i,j} - e_{n+j,i}).$$

In this case the Lie algebra $\mathfrak{u}^+$ in block matrix form is the matrices $\begin{vmatrix} a & b \\ 0 & -a^t \end{vmatrix}$ with a strictly upper triangular and b skew symmetric.

4.2 Borel Subalgebras

For all these (as well as for all semisimple) Lie algebras we have the direct sum decomposition (as vector space)

$$L = \mathfrak{u}^+ \oplus \mathfrak{t} \oplus \mathfrak{u}^-.$$

One sets

$$\mathfrak{b}^+ := \mathfrak{u}^+ \oplus \mathfrak{t}, \ \mathfrak{b}^+ := \mathfrak{u}^- \oplus \mathfrak{t};$$

these are called two opposite Borel subalgebras.

Theorem. *The fundamental property of the Borel subalgebras is that they are maximal solvable.*

Proof. To see that they are solvable we repeat a remark used previously. Suppose that a, b are two root vectors; so for $t \in \mathfrak{t}$ we have $[t, a] = \alpha(t)a, \ [t, b] = \beta(t)b$. Then $[t, [a, b]] = [[t, a], b] + [a, [t, b]] = (\alpha(t) + \beta(t))[a, b]$. In other words, $[a, b]$ is a weight vector (maybe 0) of weight $(\alpha(t) + \beta(t))$.

The next remark is that a positive root $\alpha = \sum_i n_i \alpha_i$ (the α_i simple) has a positive *height* $ht(\alpha) = \sum_i n_i$. For instance, in the case of A_n, with simple roots $\delta_i = \alpha_i - \alpha_{i+1}$, the positive root $\alpha_i - \alpha_j = \sum_{h=1}^{j-i} \delta_{i+h}$. Hence $ht(\alpha_i - \alpha_j) = j - i$.

So let $\mathfrak{b}_k$ be the subspace of $\mathfrak{b}^+$ spanned by the root vectors relative to roots of height $\geq k$ (visualize it for the classical groups). We get that $[\mathfrak{b}_k, \mathfrak{b}_h] \subset \mathfrak{b}_{k+h}$. Moreover, $\mathfrak{b}_1 = \mathfrak{u}^+$ and $[\mathfrak{b}^+, \mathfrak{b}^+] = [\mathfrak{u}^+ \oplus \mathfrak{t}, \mathfrak{u}^+ \oplus \mathfrak{t}] = [\mathfrak{u}^+, \mathfrak{u}^+] + [\mathfrak{t}, \mathfrak{u}^+] = \mathfrak{u}^+ = \mathfrak{u}_1$. From these two facts it follows inductively that the k^{th} term of the derived series is contained in $\mathfrak{b}_k$, and so the algebra is solvable.

To see that it is maximal solvable, consider a proper subalgebra $\mathfrak{a} \supset \mathfrak{b}^+$. Since $\mathfrak{a}$ is stable under $\mathrm{ad}(t), \ t \in \mathfrak{t}$, $\mathfrak{a}$ must contain a root vector f_α for a negative root α. But then $\mathfrak{a}$ contains the subalgebra generated by f_α, e_α which is certainly not solvable, being isomorphic to $sl(2, \mathbb{C})$.

5 Highest Weight Theory

In this section we complete the work and classify the finite-dimensional irreducible representations of a semisimple Lie algebra, proving that they are in 1-1 correspondence with *dominant weights*.

5.1 Weights in Representations, Highest Weight Theory

Let L be a semisimple Lie algebra. Theorem 2, 1.4 tells us that all finite-dimensional representations of L are completely reducible.

Theorem 2, 1.5 implies that any finite-dimensional representation M of L has a basis formed by weight vectors under the Cartan subalgebra $\mathfrak{t}$.

Lemma. *The weights that may appear are exactly the weights defined abstractly for the corresponding root system.*

Proof. By the representation theory of $sl(2, \mathbb{C})$, each h_i acts in a semisimple way on M. So, since the h_i commute, M has a basis of weight vectors for $\mathfrak{t}$. Moreover, we know that if u is a weight vector for h_i, then we have $h_i u = n_i u,\ n_i \in \mathbb{Z}$. Therefore, if u is a weight vector of weight χ we have $\chi(h_i) \in \mathbb{Z},\ \forall i$. Recall (1.8.1) that $\chi(h_i) = \frac{2(\chi,\alpha_i)}{(\alpha_i,\alpha_i)} = \langle \chi \mid \alpha_i \rangle$. By Definition 2.4.1 we have $\chi \in \Lambda$. □

We should make some basic remarks in the case of the classical groups, relative to weights for the Cartan subalgebra and for the maximal torus.

Start with $GL(n, \mathbb{C})$: its maximal torus T consists of diagonal matrices with nonzero entries x_i, with Lie algebra the diagonal matrices with entries a_i. Using the exponential and setting $x_i = e^{a_i}$, we then have that given a rational representation of $GL(n, \mathbb{C})$, a vector v is a weight vector for T if and only if it is a weight vector for $\mathfrak{t}$ and the two weights are related, as they are $\prod_{i=1}^n x_i^{m_i}$, $\sum_{i=1}^n m_i a_i$.

We now treat all the classical groups as matrix groups, using the notations of 4.1, and make explicit the fundamental weights and the special weight $\rho := \sum_i \omega_i$.

For $sl(n+1)$,

$$\omega_k := \sum_{i \le k} \alpha_i, \quad \prod_{i \le k} x_i, \quad i \le n, \quad \rho = \sum_{i=1}^{n} (n+1-i)\alpha_i, \quad \prod_{i=1}^{n} x_i^{n+1-i}.$$

For $so(2n)$ we again consider diagonal matrices with entries $\alpha_i, -\alpha_i$. The weights are

(5.1.1)
$$\omega_k := \sum_{i \le k} \alpha_i,\ i \le n-2,\ s_\pm := \frac{1}{2}\Big(\sum_{i=1}^{n-1} \alpha_i \pm \alpha_n\Big),$$

$$\rho = \sum_{i=1}^{n-1} (n-i-1/2)\alpha_i, \quad \prod_{i=1}^{n-1} x_i^{n-1/2-i}.$$

This shows already that the last two weights do not exponentiate to weights of the maximal torus of $SO(2n, \mathbb{C})$. The reason is that there is a double covering of this group, the spin group, which possesses these two representations called *half spin representations* which do not factor through $SO(2n, \mathbb{C})$. We study them in detail in Chapter 11, §7.2.

For $so(2n+1)$, the fundamental weights and ρ are

$$\omega_k := \sum_{i \le k} \alpha_i,\ i \le n-1,\ s := \frac{1}{2}\Big(\sum_{i=1}^{n} \alpha_i\Big),$$

(5.1.2)
$$\rho = \sum_{i=1}^{n} (n-i+1/2)\alpha_i, \quad \prod_{i=1}^{n} x_i^{n+1/2-i}.$$

The discussion of the spin group is similar (Chapter 11, §7.1) .

For $sp(2n)$ the fundamental weights and ρ are

$$(5.1.3) \qquad \omega_k := \sum_{i\leq k}\alpha_i,\ i \leq n; \quad \rho = \sum_{i=1}^{n}(n+1-i)\alpha_i, \quad \prod_{i=1}^{n} x_i^{n+1-i}.$$

5.2 Highest Weight Theory

Highest weight theory is a way of determining a *leading term* in the character of a representation. For this, it is convenient to introduce the *dominance order* of weights.

Definition 1. Given two weights λ, μ we say that $\lambda \prec \mu$ if $\lambda - \mu$ is a linear combination of simple roots with nonnegative coefficients.

Notice that $\lambda \prec \mu$, called *dominance order*, is a partial order on weights.

Proposition 1. *Given a finite-dimensional irreducible representation M of a semisimple Lie algebra L.*

(1) The space of vectors $M^+ := \{m \in M \mid \mathfrak{u}^+ m = 0\}$ is 1-dimensional and a weight space under $\mathfrak{t}$ of some weight λ. λ is called the highest weight of M. A nonzero vector $v \in M^+$ is called a highest weight vector and denoted v_λ.
(2) M^+ is the unique 1-dimensional subspace of M stable under the subalgebra $\mathfrak{b}^+$.
(3) λ is a dominant weight.
(4) M is spanned by the vectors obtained from M^+ applying elements from $\mathfrak{u}_-$.
(5) All the other weights are strictly less of λ in the dominance order.

Proof. From the theorem of Lie it follows that there is a nonzero eigenvector v, of some weight λ, for the solvable Lie algebra $\mathfrak{b}^+$. Consider the subspace of M spanned by the vectors $f_{i_1} f_{i_2} \dots f_{i_k} v$ obtained from v by acting repeatedly with the elements f_i (of weight the negative simple roots $-\alpha_i$). From the commutation relations, if $h \in \mathfrak{t}$:

$$hf_{i_1} f_{i_2} \cdots f_{i_k} v = \sum_{j=1}^{k} f_{i_1} f_{i_2} \dots [h, f_{i_j}] \dots f_{i_k} v + f_{i_1} f_{i_2} \dots f_{i_k} hv$$

$$= \left(\lambda - \sum_{j=1}^{k} \alpha_{i_j}\right)(h) f_{i_1} f_{i_2} \dots f_{i_k} v.$$

$$e_i f_{i_1} f_{i_2} \dots f_{i_k} v = \sum_{j=1}^{k} f_{i_1} f_{i_2} \dots [e_i, f_{i_j}] \dots f_{i_k} v + f_{i_1} f_{i_2} \dots f_{i_k} e_i v$$

$$= \sum_{j=1}^{k} f_{i_1} f_{i_2} \dots \delta_{i_j}^{i} h_i \dots f_{i_k} v.$$

We see that the vectors $f_{i_1} f_{i_2} \dots f_{i_k} v$ are weight vectors and span a stable submodule. Hence by the irreducibility of M, they span the whole of M. The weights we have computed are all strictly less than λ in the dominance order, except for the weight of v which is λ.

The set of vectors $\{u \in M \mid \mathfrak{u}^+ m = 0\}$ is clearly stable under $\mathfrak{t}$, and since M has a basis of eigenvectors for $\mathfrak{t}$, so must this subspace. If there were another vector u with this property, then there would be one which is also an eigenvector (under $\mathfrak{t}$) of some eigenvalue μ. The same argument shows that $\lambda \prec \mu$. Hence $\lambda = \mu$ and $u \in \mathbb{C}v$. This proves all points except for three. Then let v be a highest weight vector of weight λ, in particular $e_i v = 0$ for all the e_i associated to the simple roots. This means that v is a highest weight vector for each $sl(2, \mathbb{C})$ of type e_i, h_i, f_i. By the theory for $sl(2, \mathbb{C})$, §1.1, it follows that $h_i v = k_i v$ for some $k_i \in \mathbb{N}$ a nonnegative integer, or $\lambda(h_i) = k_i$ for all i. This is the condition of dominance. In fact, Formula 1.9.1 states that if α_i is the simple root corresponding to the given e_i, we have $\lambda(h_i) = \frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)} = \langle \lambda \mid \alpha_i \rangle$. □

The classification of finite-dimensional irreducible modules states that each dominant weight corresponds to one and only one irreducible representation. The existence can be shown in several different ways. One can take an algebraic point of view as in [Hu1] and define the module by generators and relations in the same spirit as in the proof of existence of semisimple Lie algebras through Serre's relations.

The other approach is to relate representations of semisimple Lie algebras with that of compact groups, and use the Weyl character formula, as in [A], [Ze]. In our work on classical groups we will in fact exhibit all the finite-dimensional irreducible representations of classical groups explicitly in tensor spaces.

The uniqueness is much simpler. In general, for any representation M, a vector $m \in M$ is called a *highest weight vector* if $\mathfrak{u}^+ m = 0$ and m is a weight vector under $\mathfrak{t}$.

Proposition 2. *Let M be a finite-dimensional representation of a semisimple Lie algebra L and u a highest weight vector (of weight λ). The L-submodule generated by u is irreducible.*

Proof. We can assume without loss of generality that the L-submodule generated by u is M. The same argument given in the proof of the previous theorem shows that u is the only weight vector of weight λ. Decompose $M = \bigoplus_i N_i$ into irreducible representations. Each irreducible decomposes into weight spaces, but from the previous remark, u must be contained in one of the summands N_i. Since M is the minimal submodule containing u, we have $M = N_i$ is irreducible. □

We can now prove the uniqueness of an irreducible module with a given highest weight.

Theorem. *Two finite-dimensional irreducible representations of a semisimple Lie algebra L are isomorphic if and only if they have the same highest weight.*

Proof. Suppose we have given two such modules N_1, N_2 with highest weight λ and highest weight vectors u_1, u_2, respectively.

In $N_1 \oplus N_2$, consider the vector (u_1, u_2); it is clearly a highest weight vector, and so it generates an irreducible submodule N. Now projecting to the two summands we see that N is isomorphic to both N_1 and N_2, which are therefore isomorphic. □

It is quite important to observe that given two finite-dimensional representations M, N, we have the following:

Proposition 3. *If $u \in M$, $v \in N$ are two highest weight vectors of weight λ, μ respectively, then $u \otimes v$ is a highest weight vector of weight $\lambda + \mu$. If all other weights in M (resp. N) are strictly less than λ (resp. μ) in the dominance order, then all other weights in $M \otimes N$ are strictly less than $\lambda + \mu$ in the dominance order.*[97]

Proof. By definition we have $e(u \otimes v) = eu \otimes v + u \otimes ev$ for every element e of the Lie algebra. From this the claim follows easily. □

In particular, we will use this fact in the following forms.

(1) *Cartan multiplication* Given two dominant weights λ, μ we have $M_\lambda \otimes M_\mu = M_{\lambda+\mu} + M'$, where M' is a sum of irreducibles with highest weight strictly less than $\lambda+\mu$. In particular we have the canonical projection $\pi : M_\lambda \otimes M_\mu \to M_{\lambda+\mu}$ with kernel M'. The composition $M_\lambda \times M_\mu \to M_\lambda \otimes M_\mu \xrightarrow{\pi} M_{\lambda+\mu}$, $(m, n) \mapsto \pi(m \otimes n)$ is called *Cartan multiplication.*
(2) Take an irreducible representation V_λ with highest weight λ and highest weight vector v_λ and consider the second symmetric power $S^2(V_\lambda)$.

Corollary. *$S^2(V_\lambda)$ contains the irreducible representation $V_{2\lambda}$ with multiplicity* 1.

Proof. $v_\lambda \otimes v_\lambda$ is a highest weight vector. By the previous proposition it generates the irreducible representation $V_{2\lambda}$. It is a symmetric tensor, so $V_{2\lambda} \subset S^2(V_\lambda)$. Finally since all other weights are strictly less than 2λ, the representation $V_{2\lambda}$ appears with multiplicity 1. □

5.3 Existence of Irreducible Modules

We arrive now at the final existence theorem. It is better to use the language of associative algebras, and present irreducible L-modules as cyclic modules over the enveloping algebra U_L. The PBW theorem and the decomposition $L = \mathfrak{u}^- \oplus \mathfrak{t} \oplus \mathfrak{u}^+$ imply that U_L as a vector space is $U_L = U_{\mathfrak{u}^-} \otimes U_{\mathfrak{t}} \otimes U_{\mathfrak{u}^+}$. This allows us to perform the following construction. If $\lambda \in \mathfrak{t}^*$, consider the 1-dimensional representation $\mathbb{C}_\lambda := \mathbb{C}u_\lambda$ of $\mathfrak{t} \oplus \mathfrak{u}^+$, with basis a vector u_λ, given by $hu_\lambda = \lambda(h)u_\lambda$, $\forall h \in \mathfrak{t}$, $e_i u_\lambda = 0$. $\mathbb{C}_\lambda$ induces a module over U_L called $V(\lambda) := U_L \otimes_{U_{\mathfrak{t}} \otimes U_{\mathfrak{u}^+}} \mathbb{C}u_\lambda$, called the *Verma module*. Equivalently, it is the cyclic left U_L-module subject to the defining relations

$$hu_\lambda = \lambda(h)u_\lambda, \ \forall h \in \mathfrak{t}, \quad e_i u_\lambda = 0, \ \forall i. \tag{5.3.1}$$

We remark then that given any module M and a vector $v \in M$ subject to 5.3.1 we have a map $j : V(\lambda) \to M$ mapping u_λ to v. Such a v is also called a *singular vector.* It is easily seen from the PBW theorem that the map $a \mapsto au_\lambda$, $a \in U_{\mathfrak{u}^-}$ establishes a linear isomorphism between $U_{\mathfrak{u}^-}$ and $V(\lambda)$. Of course the extra L-module structure on $V(\lambda)$ depends on λ. The module $V(\lambda)$ shares some basic properties of irreducible modules, although in general it is not irreducible and it is always infinite dimensional.

[97] We do not assume irreducibility.

(1) It is generated by a unique vector which is a weight vector under $\mathfrak{t}$.
(2) It has a basis of weight vectors and the weights are all less than λ in the dominance order.
(3) Moreover, each weight space is finite dimensional. For a weight γ the dimension of its weight space is the number of ways we can write $\gamma = \lambda - \sum_{\alpha \in \Phi^+} m_\alpha \alpha$.

It follows in particular that the sum of all proper submodules is a proper submodule and the quotient of $V(\lambda)$ by the maximal proper submodule is an irreducible module, denoted by $L(\lambda)$. By construction, $L(\lambda)$ has a unique singular vector, the image of u_λ.

Theorem. *$L(\lambda)$ is finite dimensional if and only if λ is dominant.*

The set of highest weights coincides with the set of dominant weights.

The finite-dimensional irreducible representations of a semisimple Lie algebra L are the modules $L(\lambda)$ parameterized by the dominant weights.

When λ is dominant, and a weight μ appears in $L(\lambda)$, then μ is in the W-orbit of the finite set of dominant weights $\mu \preceq \lambda$.

Proof. First, let N_λ be a finite-dimensional irreducible module with highest weight vector v_λ and highest weight λ. By Proposition 1 of §5.2, λ is dominant. We clearly have a map of $V(\lambda)$ to N_λ mapping u_λ to v_λ. Clearly this map induces an isomorphism between $L(\lambda)$ and N_λ. Thus, the theorem is proved if we see that if λ is dominant, then $L(\lambda)$ is finite dimensional. We compute now in $L(\lambda)$. Call v_λ the class of u_λ. The first statement is that $f_i^{\lambda(h_i)+1} v_\lambda = 0$. For this it suffices to see that $f_i^{\lambda(h_i)+1} u_\lambda$ is a singular vector in $V(\lambda)$. $f_i^{\lambda(h_i)+1} u_\lambda$ is certainly a weight vector under $\mathfrak{t}$, so we need to show that $e_j f_i^{\lambda(h_i)+1} u_\lambda$ for all j. If $i \neq j$, e_j commutes with f_i and $e_j f_i^{\lambda(h_i)+1} u_\lambda = f_i^{\lambda(h_i)+1} e_j u_\lambda = 0$. Otherwise, the argument of Lemma 1.1, which we already used in Serre's existence theorem, shows that $e_i f_i^{\lambda(h_i)+1} u_\lambda = 0$. The argument of Lemma 1.1 shows that v_λ generates, under the $sl_i(2, \mathbb{C})$ given by e_i, h_i, f_i, an irreducible module of dimension $\lambda(h_i) + 1$.

The next statement we prove is:

For each i, $L(\lambda)$ is a direct sum of finite-dimensional irreducible $sl_i(2, \mathbb{C})$ modules.

To prove this let M be the sum of all such irreducibles. $M \neq 0$ from what we just proved. It is enough to see that M is an L-submodule or that $aM \subset M$, $\forall a \in L$. If $N \subset M$ is a finite-dimensional $sl_i(2, \mathbb{C})$ submodule, consider $N' := \sum_{a \in L} aN$. This is clearly a finite-dimensional subspace and we claim that it is also an $sl_i(2, \mathbb{C})$ submodule. In fact, if $u \in sl_i(2, \mathbb{C})$, we have $uaN \subset [u, a]N + auN \subset [u, a]N + aN \subset N'$. Thus M is an L-submodule.

Having established the previous statement we can integrate each $sl_i(2, \mathbb{C})$ action to an action of the group $SL_i(2, \mathbb{C})$. As usual we find an action of the elements s_i which permutes weight spaces. From our constraint on weights, it follows that the only weights which can appear are those γ such that $w(\gamma) \prec \lambda, \forall w \in W$. We know (Theorem 2.4) that each weight is W-conjugate to a dominant weight. Even if the simple roots are not a basis of the weight lattice, we can still write a weight

$\lambda = \sum_i m_i\alpha_i$ where m_i are rational numbers with denominator a fixed integer d, for instance the index of the root lattice in the weight lattice. If $\lambda = \sum_i m_i\alpha_i$, $\mu = \sum_i p_i\alpha_i$ are two dominant weights, the condition $\mu \prec \lambda$ means that $\sum_i (m_i - n_i)\alpha_i$ is a positive linear combination of positive roots. This implies that $n_i \leq m_i$ for all i. Since also $0 \leq n_i$ and dn_i are integers, we have that the set of dominant weights satisfying $\mu \prec \lambda$ is finite. We can finally deduce that $L(\lambda)$ is finite dimensional, since it is the sum of its weight spaces, each of finite dimension. The weights appearing are in the W-orbits of the finite set of dominant weights $\mu \prec \lambda$. □

Modules under a Lie algebra can be composed by tensor product and also dualized. In general, a tensor product $L(\lambda) \otimes L(\mu)$ is not irreducible. To determine its decomposition into irreducibles is a rather difficult task and the known answers are just algorithms. Nevertheless by Proposition 3 of §5.2, we know that $L(\lambda) \otimes L(\mu)$ contains the *leading term* $L(\lambda + \mu)$. Duality is a much easier issue:

Proposition. $L(\lambda)^* = L(-w_0(\lambda))$, *where* w_0 *is the longest element of* W.

Proof. The dual of an irreducible representation is also irreducible, so $L(\lambda)^* = L(\mu)$ for some dominant weight μ to be determined. Let u_i be a basis of weight vectors for $L(\lambda)$ with weights μ_i. The dual basis u^i, by the basic definition of dual action, is a basis of weight vectors for $L(\lambda)^*$ with weights $-\mu_i$. Thus the dual of the highest weight vector is a *lowest weight vector* with weight $-\lambda$ in $L(\lambda)^*$. The weights of $L(\lambda)^*$ are stable under the action of the Weyl group. The longest element w_0 of W (2.3) maps negative roots into positive roots, and hence reverses the dominance order. We deduce that $-w_0(\lambda)$ is the highest weight for $L(\lambda)^*$. □

6 Semisimple Groups

6.1 Dual Hopf Algebras

At this point there is one important fact which needs clarification. We have classified semisimple Lie algebras and their representations and we have proved that the adjoint group G_L associated to such a Lie algebra is an algebraic group.

It is not true (not even for $sl(2, \mathbb{C})$) that a representation of the Lie algebra integrates to a representation of G_L. We can see this in two ways that have to be put into correspondence. The first is by inspecting the weights. We know that in general the weight lattice is bigger than the root lattice. On the other hand, it is clear that the weights of the maximal torus of the adjoint group are generated by the roots. Thus, whenever in a representation we have a highest weight which is not in the root lattice, this representation cannot be integrated to the adjoint group. The second approach comes from the fact that in any case a representation of the Lie algebra can be integrated to the simply connected universal covering. One has to understand what this simply connected group is.

A possible construction is via the method of Hopf algebras, Chapter 8, §7.2. We can define the simply connected group as the spectrum of its *Hopf algebra of matrix coefficients*.

The axioms §7.2 of that chapter have a formal duality exchanging multiplication and comultiplication. This suggests that, given a Hopf algebra A, we can define a *dual Hopf algebra* structure on the dual A^* exchanging multiplication and comultiplication. This procedure works perfectly if A is finite dimensional, but in general we encounter the difficulty that $(A \otimes A)^*$ is much bigger than $A^* \otimes A^*$. The standard way to overcome this difficulty is to restrict the dual to:

Definition. The finite dual A^f of an algebra A, over a field F, is the space of linear forms $\phi : A \to F$ such that the kernel of ϕ contains a left ideal of finite codimension.

On A^f we can define multiplication, comultiplication, antipode, unit and counit as dual maps of comultiplication, multiplication, antipode, counit and unit in A.

Remark. If J is a left ideal and $\dim A/J < \infty$, the homomorphism $A \to \mathrm{End}(A/J)$ has as kernel a two-sided ideal I contained in J. So the condition for the finite dual could also be replaced by the condition that the kernel of ϕ contains a two-sided ideal of finite codimension.

Again we can consider the elements of the finite dual as *matrix coefficients* for finite-dimensional modules. In fact, given a left ideal J of finite codimension, one has that A/J is a finite-dimensional cyclic A-module (generated by the class $\overline{1}$ of 1), and given a linear form Φ on A vanishing on J, this induces a linear form ϕ on A/J, $\phi(a\overline{1}) = \Phi(a)$. For $a \in A$ we have the formal matrix coefficient $\Phi(a) = \langle \phi \mid a\overline{1} \rangle$.

Conversely, let M be a finite-dimensional module, $\phi \in M^*$, $u \in M$, and consider the linear form $c_{\phi,u}(a) := \langle \phi \mid au \rangle$. This form vanishes on the left ideal $J := \{a \in A \mid au = 0\}$ and we call it a *matrix coefficient.*

Exercise. The reader should verify that on the finite dual the Hopf algebra structure dualizes.[98]

Let us at least remark how one performs multiplication of matrix coefficients. By definition if $\Phi, \Psi \in A^f$, we have by duality $\Phi\Psi(a) := \langle \Phi \otimes \Psi \mid \Delta(a) \rangle$. In other words, if $\Phi = c_{\phi,u}$ is a matrix coefficient for a finite-dimensional module M and $\Psi = c_{\psi,v}$ a matrix coefficient for a finite-dimensional module N, we have

$$\Phi\Psi(a) = \langle \phi \otimes \psi \mid \Delta(a)u \otimes v \rangle.$$

The formula $\Delta(a)u \otimes v$ is the definition of the tensor product action; thus we have the basic formula

$$c_{\phi,u}c_{\psi,v} = c_{\phi\otimes\psi,u\otimes v}. \tag{6.1.1}$$

Since comultiplication is coassociative, A^f is associative as an algebra. It is commutative if Δ is also cocommutative, as for enveloping algebras of Lie algebras.

[98] Nevertheless it may not be that A is the dual of A^f.

As far as comultiplication is concerned, it is the dual of multiplication, and thus $\langle \delta(\Phi), a \otimes b \rangle = \langle \Phi, ab \rangle$. When $\Phi = c_{\phi,u}$, is a matrix coefficient for a finite-dimensional module M, choose a basis u_i of M and let u^i be the dual basis. The identity $1_M = \sum_i u_i \otimes u^i$ and

$$\langle c_{\phi,u}, ab \rangle = \langle \phi, a1_M bu \rangle = \langle \phi, a \sum_i u_i \otimes u^i bu \rangle$$

$$= \sum_i \langle \phi, au_i \rangle \langle u^i, bu \rangle = \sum_i c_{\phi,u_i}(a) c_{u^i,u}(b).$$

In other words,

$$\Delta(c_{\phi,u}) = \sum_i c_{\phi,u_i} \otimes c_{u^i,u}. \tag{6.1.2}$$

The unit element of A^f is the counit $\eta : A \to F$ of A, which is also a matrix coefficient for the trivial representation. Finally for the antipode and counit we have $S(c_{\phi,u})(a) = (c_{\phi,u})(S(a))$, $\eta(c_{\phi,u}) = c_{\phi,u}(1) = \langle \phi \mid u \rangle$.

Let us apply this construction to $A = U_L$, the universal enveloping algebra of a semisimple Lie algebra.

Recall that an enveloping algebra U_L is a *cocommutative* Hopf algebra with $\Delta(a) = a \otimes 1 + 1 \otimes a$, $S(a) = -a$, $\eta(a) = 0$, $\forall a \in L$. As a consequence we have a theorem whose proof mimics the formal properties of functions on a group.

Proposition 1. *(i) We have the Peter–Weyl decomposition, indexed by the dominant weights* Λ^+*:*

$$U_L^f = \bigoplus\nolimits_{\lambda \in \Lambda^+} \mathrm{End}(V_\lambda)^* = \bigoplus\nolimits_{\lambda \in \Lambda^+} V_\lambda \otimes V_\lambda^*. \tag{6.1.3}$$

(ii) U_L^f *is a finitely generated commutative Hopf algebra. One can choose as generators the matrix coefficients of* $V := \bigoplus_i V_{\omega_i}$, *where the* ω_i *are the fundamental weights.*

Proof. (i) If I is a two-sided ideal of finite codimension, U_L/I is a finite-dimensional representation of L; hence it is completely reducible and it is the sum of some irreducibles V_{λ_i}, for finitely many distinct dominant weights $\lambda_i \in \Lambda^+$. In other words, U_L/I is the regular representation of a semisimple algebra. From the results of Chapter 6, §2 it follows that the mapping $U_L \to \bigoplus_i \mathrm{End}(V_{\lambda_i})$ is surjective with kernel I. Thus for any finite set of distinct dominant weights the dual $\bigoplus_i \mathrm{End}(V_{\lambda_i})^*$ maps injectively into the space of matrix coefficients and any matrix coefficient is sum of elements of $\mathrm{End}(V_\lambda)^*$ as $\lambda \in \Lambda^+$.

(ii) We can use Cartan's multiplication (5.2) and formula 6.1.1 to see that $\mathrm{End}(V_{\lambda+\mu})^* \subset \mathrm{End}(V_\lambda)^* \, \mathrm{End}(V_\mu)^*$. Since any dominant weight is a sum of fundamental weights, the statement follows. □

From this proposition it follows that U_L^f is the coordinate ring of an algebraic group G_s. Our next task is to identify G_s and prove that it is semisimple and simply connected with Lie algebra L. We begin with:

Lemma. *Any finite-dimensional representation V of L integrates to an action of a semisimple algebraic group G_V, whose irreducible representations appear all in the tensor powers $V^{\otimes m}$.*

Proof. In fact let us use the decomposition $L = \mathfrak{u}^- \oplus \mathfrak{t} \oplus \mathfrak{u}^+$. Both $\mathfrak{u}^-$ and $\mathfrak{u}^+$ act as Lie algebras of nilpotent elements. Therefore (Chapter 7, Theorem 3.4) the map exp establishes a polynomial isomorphism (as varieties) of $\mathfrak{u}^-$ and $\mathfrak{u}^+$ to two unipotent algebraic groups U^-, U^+, of which they are the Lie algebras, acting on V. As for $\mathfrak{t}$ we know that V is a sum of weight spaces. Set Λ' to be the lattice spanned by these weights. Thus the action of $\mathfrak{t}$ integrates to an algebraic action of the torus T' having Λ' as a group of characters. Even more explicitly, if we fix a basis e_i of V of weight vectors with the property that if λ_i, λ_j are the weights of e_i, e_j, respectively, and $\lambda_i \prec \lambda_j$, then $i > j$ we see that U^- is a closed subgroup of the group of strictly lower triangular matrices, U^+ is a closed subgroup of the group of strictly upper triangular matrices, and T' is a closed subgroup of the group of diagonal matrices. Then the multiplication map embeds $U^- \times T' \times U^+$ as a closed subvariety of the open set of matrices which are the product of a lower triangular diagonal and an upper triangular matrix. On the other hand, $U^-T'U^+$ is contained in the Lie group $G_V \subset GL(V)$ which integrates L and is open in G_V. It follows (Chapter 4, Criterion 3.3) that G_V coincides with the closure of U^-TU^+ and it is therefore algebraic. Since L is semisimple, $V^{\otimes m}$ is a semisimple representation of G_V for all m. Since $L = [L, L]$ we have that G_V is contained in $SL(V)$. We can therefore apply Theorem 1.4 of Chapter 7 to deduce all the statements of the lemma. □

Theorem. *U_L^f is the coordinate ring of a linearly reductive semisimple algebraic group $G_s(L)$, with Lie algebra L, whose irreducible representations coincide with those of the Lie algebra L.*

Proof. Consider the representation $V := \bigoplus_i V_{\omega_i}$ of L, with the sum running over all fundamental weights. We want to see that the coordinate Hopf algebra of the group $G_s(L) := G_V$ (constructed in the previous lemma) is U_L^f. Now (by Cartan's multiplication) any irreducible representation of L appears in a tensor power of V, and thus every finite-dimensional representation of L integrates to an algebraic representation of $G_s(L)$. We claim that U_L^f, as a Hopf algebra, is identified with the coordinate ring $\mathbb{C}[G_s(L)]$ of $G_s(L)$. In fact, the map is the one which identifies, for any dominant weight λ, the space $\mathrm{End}(V_\lambda)^*$ as a space of matrix coefficients either of L or of $G_s(L)$. We have only to prove that the Hopf algebra operations are the same. There are two simple ideas to follow.

First, the algebra of regular functions on $G_s(L)$ is completely determined by its restriction to the dense set of elements $\exp(a),\ a \in L$.

Second, although the element $\exp(a)$, being given by an infinite series, is not in the algebra U_L, nevertheless, any matrix coefficient $c_{\phi,u}$ of U_L extends by continuity to a function $\langle \phi \mid \exp(a)u \rangle$ on the elements $\exp(a)$ which, being the corresponding representation of the $G_s(L)$ algebraic group, is the restriction of a regular algebraic function.

In this way we start identifying the algebra of matrix coefficients U_L^f with the coordinate ring of $G_s(L)$. At least as vector spaces they are both $\bigoplus_{\lambda \in \Lambda^+} \mathrm{End}(V_\lambda)^*$.

Next, we have to verify some identities in order to prove that we have an isomorphism as Hopf algebras. To verify an algebraic identity on functions on $G_s(L)$, it is enough to verify it on the dense set of elements $\exp(a)$, $a \in L$. By 6.1.1, 6.1.2. and the definition of matrix coefficient in the two cases, it follows that we have an isomorphism of algebras and coalgebras. Also the isomorphism respects unit and counit, as one sees easily. For the antipode, one has only to recall that in U_L, for $a \in L$ we have $S(a) = -a$ so that $S(e^a) = e^{-a}$.

Since for functions on $G_s(L)$ the antipode is $Sf(g) = f(g^{-1})$, we have the compatibility expressed by $Sf(e^a) = f((e^a)^{-1}) = f(e^{-a}) = f(S(e^a))$. We have thus an isomorphism between U_L^f and $\mathbb{C}[G_s(L)]$ as Hopf algebras. □

It is still necessary to prove that the group $G_s(L)$, the spectrum of U_L^f, is simply connected and a finite universal cover of the adjoint group of L. Before doing this we draw some consequences of what we have already proved. Nevertheless we refer to $G_s(L)$ as the simply connected group. In any case we can prove:

Proposition 2. *Let G be an algebraic group with Lie algebra a semisimple Lie algebra L. Then G is a quotient of $G_s(L)$ by a finite group.*

Proof. Any representation of L which integrates to G integrates also to a representation of $G_s(L)$; thus we have the induced homomorphism. Since this induces an isomorphism of Lie algebras its kernel is discrete, and since it is also algebraic, it is finite. □

Let us make a final remark.

Proposition 3. *Given two semisimple Lie algebras L_1, L_2, we have*

$$G_s(L_1 \oplus L_2) = G_s(L_1) \times G_s(L_2).$$

Proof. The irreducible representations of $L_1 \oplus L_2$ are the tensor products $M \otimes N$ of irreducibles for the two Lie algebras; hence $U_{L_1 \oplus L_2}^f = U_{L_1}^f \otimes U_{L_2}^f$ form which the claim follows. □

6.2 Parabolic Subgroups

Let L be a simple Lie algebra, $\mathfrak{t}$ a maximal toral subalgebra, Φ^+ a system of positive roots and $\mathfrak{b}^+$ the corresponding Borel subalgebra, and $G_s(L)$ the group constructed in the previous section. Let ω_i be the corresponding fundamental weights and V_i the corresponding fundamental representations with highest weight vectors v_i. We consider a dominant weight $\lambda = \sum_{i \in J} m_i \omega_i$, $m_i > 0$. Here J is the set of indices i which appear with nonnegative coefficient. We want now to perform a construction which in algebraic geometry is known as the *Veronese embedding*. Consider now the tensor product $M := \otimes_{i \in J} V_i^{\otimes n_i}$. It is a representation of L (not irreducible) with highest weight vector $\otimes_{i \in J} v_i^{\otimes n_i}$. This vector generates inside M the irreducible representation V_λ.

We have an induced Veronese map of projective spaces (say $k = |J|$):

$$\pi_\lambda := \prod_{i \in J} \mathbb{P}(V_i) \to \mathbb{P}(M),$$
$$\pi(\mathbb{C}a_1, \dots, \mathbb{C}a_k) := \mathbb{C}a_1^{\otimes m_1} \otimes \dots \otimes a_k^{\otimes m_k}, \quad 0 \neq a_i \in V_i.$$

This map is an embedding and it is equivariant with respect to the algebraic group $G_s(L)$ (which integrates to the Lie algebra action). In particular we deduce that:

Proposition 1. *The stabilizer in $G_s(L)$ of the line through the highest weight vector v_λ is the intersection of the stabilizers H_i of the lines through the highest weight vectors v_i for the fundamental weights. We set $H_J := \cap_{i \in J} H_i$.*

Proposition 2. *Let B be the Borel subgroup, with Lie algebra $\mathfrak{b}^+$. If $H \supset B$ is an algebraic subgroup, then $B = H_J$ for a subset J of the nodes of the Dynkin diagram.*

Proof. Let H be any algebraic subgroup containing B. From Theorem 1, §2.1 Chapter 7, there is a representation M of G and a line ℓ in M such that H is the stabilizer of ℓ. Since $B \subset H$, ℓ is stabilized by B. The unipotent part of B (or of its Lie algebra $\mathfrak{u}^+$) must act trivially. Hence ℓ is generated by a highest weight vector in an irreducible representation of G (Proposition 2 of §5.2). Hence by Proposition 1, we have that H must be equal to one of the groups H_J. □

To identify all these groups let us first look at the Lie algebras. Given a set $J \subset \Delta$ of nodes of the Dynkin diagram,[99] let Φ_J denote the root system generated by the simple roots not in this set, i.e., the set of roots in Φ which are linear combinations of the α_i, $i \notin J$.

Remark. The elements α_i, $i \notin J$, form a system of simple roots for the root system Φ_J.

We can easily verify that

(6.2.1) $$\mathfrak{p}_J := \mathfrak{b}^+ \bigoplus\nolimits_{\alpha \in \Phi^+ \cap \Phi_J} L_{-\alpha},$$

is a Lie algebra. Moreover one easily has that $\mathfrak{p}_A \cap \mathfrak{p}_B = \mathfrak{p}_{A \cup B}$ and that $\mathfrak{p}_J$ is generated by $\mathfrak{b}^+$ and the elements f_i for $i \notin J$. We set $\mathfrak{p}_i := \mathfrak{p}_{\{i\}}$ so that $\mathfrak{p}_J = \cap_{i \in J} \mathfrak{p}_i$. Let B, P_J, P_i be the connected groups in $G_s(L)$ with Lie algebras $\mathfrak{b}^+$, $\mathfrak{p}_J$, $\mathfrak{p}_i$.

Lemma. *The Lie algebra of H_J is $\mathfrak{p}_J$.*

Proof. Since $f_i v_\lambda = 0$ if and only if $\langle \lambda, \alpha_i \rangle = 0$, we have that f_i is in the Lie algebra of the stabilizer of v_λ if and only if $m_i = 0$, i.e., $i \notin J$. Thus $\mathfrak{p}_J$ is contained in the Lie algebra of H_J. Since these Lie algebras are all distinct and the H_J exhaust the list of all groups containing B, the claim follows. □

[99] We can identify the nodes with the simple roots.

Theorem. *$P_J = H_J$, in particular H_J, is connected.*

Proof. We have that P_J stabilizes ℓ_λ and contains B, so it must coincide with one of the groups H and it can be only H_J. □

Remark. Since in Proposition 2 we have seen that the subgroups H_J exhaust all the algebraic subgroups containing B, we have in particular that all the algebraic subgroups containing B are connected.[100]

It is important to understand the Levi decomposition for these groups and algebras. Decompose $\mathfrak{t} = \mathfrak{t}_J \oplus \mathfrak{t}_J^\perp$, where $\mathfrak{t}_J$ is spanned by the elements $h_i,\ i \notin J$ and $\mathfrak{t}_J^\perp$ is the orthogonal, i.e., $\mathfrak{t}_J^\perp = \{h \in \mathfrak{t} \mid \alpha_i(h) = 0,\ \forall i \notin J\}$.

Proposition 3. *(i) The algebra $\mathfrak{l}_J := \mathfrak{t}_J \oplus_{\alpha\in\Phi_J} L_\alpha$ is the Lie algebra of the (not necessarily irreducible) root system Φ_J.*

(ii) The algebra $\mathfrak{s}_J := \mathfrak{t}_J^\perp \oplus_{\alpha\in\Phi^+-\Phi_J} L_\alpha$ is the solvable radical of $\mathfrak{p}_J$.

(iii) $\mathfrak{p}_J = \mathfrak{l}_J \oplus \mathfrak{s}_J$ is a Levi decomposition.

Proof. (i) The elements $e_i, f_i, h_i,\ i \notin J$ are in $\mathfrak{l}_J$ and satisfy Serre's relations for the root system Φ_J. By Serre's theorem we thus have a homomorphism from the Lie algebra associated to Φ_J to $\mathfrak{l}_J$. This map sends a basis to a basis, so it is an isomorphism.

(ii) $\mathfrak{p}_J$ is contained in the Borel subalgebra of L, so it is solvable. It is easily seen to be an ideal of $\mathfrak{p}_J$. Since $\mathfrak{p}_J/\mathfrak{s}_J = \mathfrak{l}_J$ is semisimple, it is the solvable radical.

(iii) Follows from (i), (ii). □

We finally need to understand the weight lattice, and dominant and fundamental weights associated to Φ_J. The main remark is that if $i \notin J$ and ω_i is the corresponding fundamental weight for $\mathfrak{t}$, since the elements $h_j,\ j \notin J$ span $\mathfrak{t}_J$ we have that ω_i restricted to $\mathfrak{t}_J$ coincides with the fundamental weight dual to $\check{\alpha}_i$.

Let $L_J \subset P_J \subset G_s(L)$ be the corresponding semisimple group.

Proposition 4. *L_J is simply connected.*

Proof. To prove that a semisimple group is simply connected (according to our provisional definition), it suffices to prove that all the representations of its Lie algebra integrate to representations of the group. Now if we restrict an irreducible representation V_λ of $G_s(L)$ to L_J, it will not remain irreducible, but its highest weight vector v_λ is still a highest weight vector for the corresponding Borel subalgebra $\mathfrak{t}_J \bigoplus_{\alpha\in\Phi_J^+} L_\alpha$ of L_J. From the previous discussion, all dominant weights appear in this way. □

The subgroup $L := L_J T$ has Lie algebra $\mathfrak{l}_J \oplus \mathfrak{t}^\perp$ and it is called a *Levi factor* of P_J, L_J is the semisimple part of the Levi factor. L is a connected reductive group. If U_J denotes the unipotent radical of P_J, i.e., the unipotent group with Lie algebra $\mathfrak{u}_J := \bigoplus_{\alpha\in\Phi^+-\Phi_J} L_\alpha$, we have that $L \cap U_J = 1$. This follows from the fact that a unipotent group with trivial Lie algebra is trivial. It gives the *Levi decomposition* for P:

[100] With a more careful analysis in fact one can drop from the hypotheses the requirement to be algebraic.

Theorem (Levi decomposition). *The multiplication map* $m : L \times U_J \to P$ *is an isomorphism of varieties and* $P = L \ltimes U_J$ *is a semidirect product.*

Proof. Since L normalizes U_J and the map m is injective with invertible Jacobian, it follows that the image of m is an open subgroup, hence equal to P_J since this group is connected and clearly gives a semidirect product. □

In fact a similar argument (see [Bor], [Hu2], [OV]) shows in general that if G is any connected algebraic group with unipotent radical G_u, one can find (using the Levi decomposition for Lie algebras and the fact that the Lie algebra is algebraic) a reductive subgroup L with $G = L \ltimes G_u$ a semidirect product.

Remark. There is a certain abuse in the expression "Levi decomposition." For Lie algebras we used this term to find a presentation of a Lie algebra as a semidirect product of a semisimple and a solvable Lie algebra. Then, in order to prove Ado's theorem we corrected the decomposition so that it was really a presentation of a Lie algebra as a semidirect product of a reductive and a nilpotent Lie algebra. This is the type of decomposition which we are now stressing for P.

6.3 Borel Subgroups

We can now complete the analysis of Borel subgroups.

Let G be a connected algebraic group.

Definition. A maximal connected solvable subgroup of G is called a *Borel subgroup.*

A subgroup H with the property that G/H is projective (i.e., compact) is called a *parabolic subgroup*.

Before we start the main discussion let us make a few preliminary remarks.

Lemma 1. *(i) A Borel subgroup B of a connected algebraic group G contains the solvable radical of G.*

(ii) If $G = G_1 \times G_2$ is a product, a Borel subgroup B of G is a product $B_1 \times B_2$ of Borel subgroups in the two factors.

Proof. (i) Let R be the solvable radical, a normal subgroup. Thus BR is a subgroup; it is connected since it is the image under multiplication of $B \times R$. Finally BR is clearly solvable, hence $BR = B$.

(ii) If B is connected solvable, the two projections B_1, B_2 on the two factors are connected solvable, hence $B \subset B_1 \times B_2$. If B is maximal, we have then $B = B_1 \times B_2$. □

In this way one reduces the study of Borel subgroups to the case of semisimple groups.

Lemma 2. *Let B be connected solvable and H a parabolic subgroup of G. Then B is contained in a conjugate of H.*

Proof. By Borel's fixed point theorem, B fixes some point $aH \in G/H$, hence $B \subset aHa^{-1}$. □

Theorem. *For an algebraic subgroup $H \subset G$ the following two conditions are equivalent:*

1. *H is maximal connected solvable (a Borel subgroup).*
2. *H is minimal parabolic.*

Proof. We claim that it suffices to prove that if B is a suitably chosen Borel subgroup, then G/B is projective, hence B is parabolic. In fact, assume for a moment that this has been proved and let H be minimal parabolic. First, $B \subset aHa^{-1}$ for some a by the previous lemma. Since aHa^{-1} is minimal parabolic, we must have $B = aHa^{-1}$. Given any other Borel subgroup B', by the previous lemma, $B' \subset aBa^{-1}$. Since aBa^{-1} is solvable connected and B' maximal solvable connected, we must have $B' = aBa^{-1}$.

Let us prove now that G/B is projective for a suitable connected maximal solvable B. Since B contains the solvable radical of G, we can assume that G is semisimple. We do first the case $G = G_s(L)$, the simply connected group of a simple Lie algebra L. Consider for B the subgroup with Lie algebra $\mathfrak{b}^+$. We have proved in Theorem 6.2 that B is the stabilizer of a line ℓ generated by a highest weight vector in an irreducible representation M of $G_s(L)$ relative to a regular dominant weight. We claim that the orbit $G_s(L)/B \subset \mathbb{P}(M)$ is closed in this projective space, and hence $G_s(L)/B$ is projective. Otherwise, one could find in its closure a fixed point under B which is different from ℓ. This is impossible since it would correspond to a new highest weight vector in the irreducible representation M. Moreover, we also see that the center Z of $G_s(L)$ is contained in B. In fact, since on any irreducible representation the center of $G_s(L)$ acts as scalars, Z acts trivially on the projective space. In particular, it is contained in the stabilizer B of the line ℓ.

The general case now follows from 6.1. A semisimple group G is the quotient $G = \prod_i G_s(L_i)/Z$, of a product $\prod_i G_s(L_i)$ of simply connected groups with simple Lie algebras L_i by a finite group Z in the center. Taking the Borel subgroups B_i in $G_s(L_i)$ we have that $\prod_i B_i$ contains Z, $B := \prod_i B_i/Z$ is a Borel subgroup in G and $G/B = \prod_i G_s(L_i)/B_i$. □

Corollary of proof. *The center of G is contained in all Borel subgroups.*
All Borel subgroups are conjugate.
The normalizer of B is B.
A parabolic subgroup is conjugate to one and only one of the groups P_J, $J \subset \Delta$.

Proof. All the statements follow from the theorem and the previous lemmas. The only thing to clarify is why two groups P_J are not conjugate. Suppose $gP_Jg^{-1} = P_I$. Since gBg^{-1} is a Borel subgroup of P_I there is an $h \in P_I$ with $hgBg^{-1}h^{-1} = B$. Hence $hg \in B$ and $P_J = hgP_J(hg)^{-1} = hP_Ih^{-1} = P_I$. □

The variety G/B plays a fundamental role in the theory and, by analogy to the linear case, it is called the *(complete) flag variety.*

By the theory developed, G/B appears as the orbit of the line associated to a highest weight vector for an irreducible representation V_λ when the weight $\lambda \in \Lambda^{++}$ is strongly dominant, i.e., in the interior of the Weyl chamber C (cf. §2.4).

The other varieties G/P_J also play an important role and appear as the orbit of the line associated to a highest weight vector for an irreducible representation V_λ when the weight λ is in a given set of walls of C.

6.4 Bruhat Decomposition

Let L be a simple Lie algebra, $\mathfrak{t}$ a maximal toral subalgebra, Φ^+ a system of positive roots, $\mathfrak{b}^+X$ the corresponding Borel subalgebra, and $\mathfrak{u}^+$ its nilpotent radical. We need to make several computations with the Weyl group W and with its lift $\tilde{W}$ generated by the elements s_i in the groups $SL_i(2, \mathbb{C})$. In order to avoid unnecessary confusion let us denote by σ_i the simple reflections in W lifting to the elements $s_i \in \tilde{W}$.

For any algebraic group G with Lie algebra L we have that $G_s(L) \to G \to G_a(L)$ (G is between the simply connected and the adjoint groups). In G we have the subgroups T, B, U, i.e., the torus, Borel group, and its unipotent radical with Lie algebras $\mathfrak{t}, \mathfrak{b}^+, \mathfrak{u}^+$, respectively. Let W be the Weyl group.

Proposition 1. *The map $T \times U \to B,\ (t, u) \mapsto tu$ is an isomorphism.*

Proof. Take a faithful representation of G and a basis of eigenvectors for T, ordered such that U acts as strictly upper triangular matrices. Call D the diagonal and V the strictly upper triangular matrices in this basis. The triangular matrices form a product $D \times V$ inside which $T \times U$ is closed. Since clearly the image $TU \subset B \subset DV$ is also open in B, we must have $TU = B$ and the map is an isomorphism. □

First, a simple remark:

Definition-Proposition 2. *A maximal solvable subalgebra of L is called a Borel subalgebra. A Borel subalgebra $\mathfrak{b}$ is the Lie algebra of a Borel subgroup.*

Proof. Let B be the Lie subgroup associated to $\mathfrak{b}$. Thus B is solvable. By Proposition 3 of Chapter 7, §3.5, the Zariski closure of B is solvable and connected so B is algebraic and clearly a Borel subgroup. □

Consider the adjoint action of $G_a(L)$ on L.

Lemma 1. *The stabilizer of $\mathfrak{b}^+$ and $\mathfrak{u}^+$ under the adjoint action is B.*

Proof. Clearly B stabilizes $\mathfrak{b}^+, \mathfrak{u}^+$. If the stabilizer were larger, it would be one of the groups P_J, which is impossible. □

Remark. According to Chapter 7, we can embed G/B in a Grassmann variety. We let $N = \dim \mathfrak{u}^+$ (the number of positive roots) and consider the line $\bigwedge^N \mathfrak{u}^+ \subset \bigwedge^N L$. The orbit of this line is G/B. Now clearly a vector in $\bigwedge^N \mathfrak{u}^+$ is a highest weight vector of weight $\sum_{\alpha \in \Phi^+} \alpha = 2\rho$. It remains puzzling to understand if there is also a more geometric interpretation of the embedding of G/B associated to ρ. This in fact can be explained using a theory that we will develop in Chapter 11, §7. The adjoint group $G = G_a(L)$ preserves the Killing form and the subspace $\mathfrak{u}^+$ is totally isotropic. If we embed $\mathfrak{u}^+$ in a maximal totally isotropic subspace we can apply to it the spin formalism and associate to it a pure spinor. We leave to the reader to verify (using the theory of Chapter 7) that this spinor is a highest weight vector of weight ρ.

We want to develop some geometry of the varieties G/P. We will proceed in a geometric way which is a special case of a general theory of Bialynicki-Birula [BB].

Let us consider $\mathbb{C}^*$ acting linearly on a vector space V. Denote by $\rho : \mathbb{C}^* \to GL(V)$ the corresponding homomorphism. ρ is called a *1-parameter group*. Decompose V according to the weights $V = \sum_i V_{m_i}$, $V_{m_i} = \{v \in V \mid \rho(t)v = t^{m_i}v\}$. The action induces an action on the projective space of lines $\mathbb{P}(V)$. A point $p \in \mathbb{P}(V)$ is a line $\mathbb{C}v \subset V$, and it is a fixed point under the action of $\mathbb{C}^*$ if and only if v is an eigenvector.

Given a general point in $\mathbb{P}(V)$ corresponding to the line through some vector $v = \sum_i v_i$, $v_i \in V_{m_i}$, we have $\rho(t)v = \sum_i t^{m_i}v_i$. In projective space we can dehomogenize the coordinates and choose the index i among the ones for which $v_i \neq 0$ and for which the exponent m_i is minimum. Say that this happens for $i = 1$. Choose a basis of eigenvectors among which the first is v_1, and consider the open set of projective space in which the coordinate of v_1 is nonzero and hence can be normalized to 1. In this set, in the affine coordinates chosen, we have $\rho(t)v = v_1 + \sum_{i>1} t^{m_i - m_1}v_i$. Therefore we have $\lim_{t\to 0}\rho(t)v = v_1$. We have thus proved in particular:

Lemma 2. *For a point $p \in \mathbb{P}(V)$ the limit $\lim_{t\to 0}\rho(t)p$ exists and is a fixed point of the action.*

Remark. If $W \subset \mathbb{P}(V)$ is a T-stable projective subvariety and $p \in W$, we have clearly $\lim_{t\to 0}\rho(t)p \in W$. We will apply this lemma to G/B embedded in a G-equivariant way in the projective space of a linear representation V_λ.

We want to apply this to a *regular 1-parameter subgroup* of T. By this we mean a homomorphism $\rho : \mathbb{C}^* \to T$ with the property that if $\alpha \neq \beta$ are two roots, considered as characters of T, we have that $\alpha \circ \rho \neq \beta \circ \rho$. Then we have the following simple lemma.

Lemma 3. *A subspace of L is stable under $\rho(\mathbb{C}^*)$ if and only if it is stable under T.*

Proof. A subspace of L is stable under $\rho(\mathbb{C}^*)$ if and only if it is a sum of weight spaces; since $\rho(\mathbb{C}^*)$ is regular, its weight spaces coincide with the weight spaces of T. □

We now introduce some special Borel subalgebras. For any choice Ψ of the set of positive roots, we define $\mathfrak{b}_\Psi = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Psi} L_\alpha$. In particular, let $\Psi = w(\Phi^+)$ be a choice of positive roots.

We know in fact that such a Ψ corresponds to a Weyl chamber and by Theorem 2.3 3), the Weyl group acts in a simply transitive way on the chambers. Thus we have defined algebras indexed by the Weyl group and we set $\mathfrak{b}_w = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi^+} L_{w(\alpha)}$.

Lemma 4. *Let $A \subset \Phi$ be a set of roots satisfying the two properties:*

(S) $$\alpha, \beta \in A,\ \alpha + \beta \in \Phi \implies \alpha + \beta \in A.$$

(T) $$\alpha \in A, \implies -\alpha \notin A.$$

Then $A \subset w(\Phi^+)$ for some $w \in W$.

Proof. By the theory of chambers, it suffices to find a regular vector v such that $(\alpha, v) > 0,\ \forall \alpha \in A$. In fact, since the regular vectors are dense and the previous condition is open, it suffices to find any vector v such that $(\alpha, v) > 0,\ \forall \alpha \in A$. We proceed in three steps.

(1) We prove by induction on m that given a sequence $\alpha_1, \dots, \alpha_m$ of m elements in A, we have $\alpha_1 + \alpha_2 + \cdots + \alpha_m \neq 0$. For $m = 1$ it is clear. Assume it for $m - 1$. If $-\alpha_1 = \alpha_2 + \cdots + \alpha_m$, we have $(-\alpha_1, \alpha_2 \cdots + \alpha_m) > 0$; thus, for some $j \geq 2$, we have $(\alpha_1, \alpha_j) < 0$. By Lemma 2.2, $\alpha_1 + \alpha_j$ is a root, by assumption in A, so we can rewrite the sum as a shorter sum $(\alpha_1 + \alpha_j) + \sum_{i \neq 1, j} \alpha_i = 0$, a contradiction.

(2) We find a nonzero vector v with $(\alpha, v) \geq 0,\ \forall \alpha \in A$. In fact, assume by contradiction that such a vector does not exist. In particular this implies that given $\alpha \in A$, there is a $\beta \in A$ with $(\alpha, \beta) < 0$, and hence $\alpha + \beta \in A$. Starting from any root $\alpha_0 \in A$ we find inductively an infinite sequence $\beta_i \in A$, such that $\alpha_{i+1} := \beta_i + \alpha_i \in A$. By construction $\alpha_i = \alpha_0 + \beta_1 + \beta_2 + \cdots + \beta_{i-1},\ \forall i$. For two distinct indices $i < j$ we must have $\alpha_i = \alpha_j$, and hence $0 = \sum_{h=i+1}^{j} \beta_h$, contradicting 1.

(3) By induction on the root system induced on the hyperplane $H_v := \{x \mid (x, v) = 0\}$, we can find a vector w with $(\alpha, w) > 0,\ \forall \alpha \in A \cap H_v$. If we take w sufficiently close to 0, we can still have that $(\beta, v + w) > 0,\ \forall \beta \in A - H_v$. The vector $v + w$ solves our problem. □

Lemma 5. *Let $A \subset \Phi$ be a set of roots. $\mathfrak{h} := \mathfrak{t} \oplus \bigoplus_{\alpha \in A} L_\alpha$ is a Lie algebra if and only if A satisfies the property (S) of Lemma 4.*

Furthermore $\mathfrak{h}$ is solvable if and only if A satisfies the further property (T).

Proof. The first part follows from the formula $[L_\alpha, L_\beta] = L_{\alpha+\beta}$. For the second, if $\alpha, -\alpha \in A$, we have inside $\mathfrak{h}$ a copy of $sl(2, \mathbb{C})$ which is not solvable. Otherwise, we can apply the previous lemma and see that $\mathfrak{h} \subset \mathfrak{b}_w$ for some w. □

Proposition 3. *(i) A Borel subalgebra $\mathfrak{h}$ is stable under the adjoint action of T if and only if $\mathfrak{t} \subset \mathfrak{h}$.*

(ii) The Borel subalgebras containing $\mathfrak{t}$ are the algebras $\mathfrak{b}_w,\ w \in W$.

Proof. (i) If a subalgebra $\mathfrak{h}$ is stable under T, it is stable under the adjoint action of $\mathfrak{t}$. Hence $\mathfrak{h}' := \mathfrak{h} + \mathfrak{t}$ is a subalgebra and $\mathfrak{h}$ is an ideal in $\mathfrak{h}'$. So, if $\mathfrak{h}$ is maximal solvable, we have that $\mathfrak{t} \subset \mathfrak{h}$. The converse is clear.

(ii) Since $\mathfrak{h}$ is T-stable we must have $\mathfrak{h} = \mathfrak{t} \oplus \bigoplus_{\alpha \in A} L_\alpha$ satisfies the hypotheses of Lemma 5, and hence it is contained in a $\mathfrak{b}_w$. Since $\mathfrak{h}$ is maximal solvable, it must coincide with $\mathfrak{b}_w$. □

From Theorem 6.3 and all the results of this section we have:

Theorem 1. *The set of Borel subalgebras of L is identified under the adjoint action with the orbit of* $\mathfrak{b}$ *and with the projective homogeneous variety* G/B. *The fixed points under T of* G/B *are the algebras* $\mathfrak{b}_w$ *indexed by W.*

We want to decompose G/B according to the theory of 1-parameters subgroups now. We choose a regular 1-parameter subgroup of T with the further property that if α is a positive root, $\alpha(\rho(t)) = t^{m_\alpha}$, $m_\alpha < 0$. From Lemma 3, it follows that the fixed points of $\rho(\mathbb{C}^*)$ on G/B coincide with the T fixed points $\mathfrak{b}_w$. We thus define

(6.4.1)
$$C_w^- := \{p \in G/B \mid \lim_{t\to 0} \rho(t)p = \mathfrak{b}_w\}.$$

From Lemma 2 we deduce:

Proposition 4. G/B *is the disjoint union of the sets* C_w^-, $w \in W$.

We need to understand the nature of these sets C_w^-. We use Theorem 2 of §3.5 in Chapter 4. First, we study G/B in a neighborhood of $B = \mathfrak{b}$, taking as a model the variety of Borel subalgebras. We have that $\mathfrak{u}^-$, the Lie algebra of the unipotent group U^-, is a complement of $\mathfrak{b}^+$ in L, so from the cited theorem we have that the orbit map restricted to U^- gives a map $i : U^- \to G/B$, $u \mapsto \mathrm{Ad}(u)(\mathfrak{b})$ which is an open immersion at 1. Since U^- is a group and i is equivariant with respect to the actions of U^-, we must have that i is an open map with an invertible differential at each point. Moreover, i is an isomorphism onto its image, since otherwise an element of U^- would stabilize $\mathfrak{b}_+$, which is manifestly absurd. In fact, $U^- \cap B = 1$ since it is a subgroup of U^- with trivial Lie algebra, hence a finite group. In a unipotent group the only finite subgroup is the trivial group. We have thus found an open set isomorphic to U^- in G/B and we claim that this set is in fact C_1. To see this, notice that U^- is T-stable, so $G/B - U^-$ is closed and T-stable. Hence necessarily $C_1 \subset U^-$. To see that it coincides, notice that the T-action on U^- is isomorphic under the exponential map to the T-action on $\mathfrak{u}^-$. By the definition of the 1-parameter subgroup ρ, all the eigenvalues of the group on $\mathfrak{u}^-$ are strictly positive, so for every vector $u \in \mathfrak{u}^-$ we have $\lim_{t\to 0} \rho(t)u = 0$, as desired.

Let us look now instead at another fixed point $\mathfrak{b}_w$. Choose a reduced expression of w and correspondingly an element $s_w = s_{i_1}s_{i_2}\dots s_{i_k} \in G$ as in Section 3.1, $s_i \in SL_i(2,\mathbb{C})$ the matrix $\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$. We have that $\mathfrak{b}_w = s_w(\mathfrak{b})$ has as an open neighborhood the orbit of the group $s_w(U^-)s_w^{-1}$, which by the exponential map is isomorphic to the Lie algebra $\mathfrak{u}_w^- = \sum_{\alpha\in\Phi^-} L_{w(\alpha)}$. This neighborhood is thus isomorphic, in a T-equivariant way, to $\mathfrak{u}_w^-$ with the adjoint action of T. On this space the 1-parameter subgroup ρ has positive eigenvalues on the root spaces $L_{w(\alpha)}$ for the roots $\alpha \prec 0$ such that $w(\alpha) \prec 0$ and negative eigenvalues for the roots $\alpha \prec 0$ such that $w(\alpha) \succ 0$. Clearly the Lie algebra of the unipotent group $U_w^- := U^- \cap s_w(U^-)s_w^{-1}$ is the sum of the root spaces L_β, where $\beta \prec 0$, $w^{-1}(\beta) \prec 0$. We have:

Lemma 6. C_w^- *is the closed set* $U_w^-\mathfrak{b}_w$ *of the open set* $s_w(U^-)s_w^{-1}\mathfrak{b}_w = s_w(U^-\mathfrak{b})$.
The orbit map from U_w^- *to* C_w^- *is an isomorphism.*

We need one final lemma:

Lemma 7. *Decompose* $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$ *as the direct sum of two Lie subalgebras, say* $\mathfrak{u}_i = Lie(U_i)$. *The map* $i : U_1 \times U_2 \to U, i : (x, y) \mapsto xy$ *and the map* $j : \mathfrak{u}_1 \oplus \mathfrak{u}_2 \to U,\ j(a, b) := \exp(a)\exp(b)$ *are isomorphisms of varieties.*

Proof. Since U_i is isomorphic to $\mathfrak{u}_i$ under the exponential map, the two statements are equivalent. The Jacobian of j at 0 is the identity, so the same is true for i. i is equivariant with respect to the right and left actions of U_2, U_1, so the differential of i is an isomorphism at each point.[101] Moreover, i is injective since otherwise, if $x_1y_1 = x_2y_2$ we have $x_2^{-1}x_1 = y_2y_1^{-1} \in U_1 \cap U_2$. This group is a subgroup of a unipotent group with trivial Lie algebra, hence it is trivial. To conclude we need a basic fact from affine geometry. Both $U_1 \times U_2$ and U are isomorphic to some affine space $\mathbb{C}^m$. We have embedded, via i, $U_1 \times U_2$ into U. Suppose then that we have an open set A of $\mathbb{C}^m$ which is an affine variety isomorphic to $\mathbb{C}^m$. Then $A = \mathbb{C}^m$. To see this, observe that in a smooth affine variety the complement of a proper affine open set is a hypersurface, which in the case of $\mathbb{C}^m$ has an equation $f(x) = 0$. We would have then that the function $f(x)$ restricted to $A = \mathbb{C}^m$ is a nonconstant invertible function. Since on $\mathbb{C}^n$ the functions are the polynomials, this is impossible. □

Given a $w \in W$ we know by 2.3 that if $w = s_{i_1} \dots s_{i_k}$ is a reduced expression, the set $B_w := \{\beta \in \Phi^+ \mid w^{-1}(\beta) \prec 0\}$ of positive roots sent into negative roots by w^{-1} is the set of elements $\beta_h := s_{i_1}s_{i_2}\dots s_{i_{h-1}}(\alpha_{i_h}),\ h = 1, \dots, k$. Let us define the unipotent group U_w as having Lie algebra $\bigoplus_{\beta \in B_w} L_\beta$. For a root α let U_α be the additive group with Lie algebra L_α.

Corollary. *Let* $w = s_{i_1} \dots s_{i_k}$ *be a reduced expression. Then the group* U_w *is the product* $U_{\beta_1}U_{\beta_2}\dots U_{\beta_k} = U_{\beta_k}U_{\beta_{k-1}}\dots U_{\beta_1}$.

Proof. We apply induction and the fact which follows by the previous lemma that U_w is the product of $U_{ws_{i_k}}$ with U_{β_k}. □

In particular it is useful to write the unipotent group U as a product of the root subgroups for the positive roots, ordered by a convex ordering. We can then complete our analysis.

Theorem (Bruhat decomposition). *The sets* C_w^- *are the orbits of* B^- *acting on* G/B.

Each $C_w^- = U_w^- \mathfrak{b}_w$ *is a locally closed subset isomorphic to an affine space of dimension* $\ell(ww_0)$ *where* w_0 *is the longest element of the Weyl group.*

The stabilizer in B^- *of* $\mathfrak{b}_w$ *is* $B^- \cap s_w(B)s_w^{-1} = TU'_w$, *where* $U'_w = U^- \cap s_w(U)s_w^{-1}$ *has Lie algebra* $\bigoplus_{\alpha \in \Phi^-,\ w^{-1}(\alpha) \in \Phi^+} L_\alpha$.

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[101] It is not known if this is enough. There is a famous open problem, the Jacobian conjecture, stating that if we have a polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ with everywhere nonzero Jacobian, it is an isomorphism.

Proof. Most of the statements have been proved. It is clear that $s_w(B)s_w^{-1}$ is the stabilizer of $\mathfrak{b}_w = \mathrm{Ad}(s_w)(\mathfrak{b})$ in G. Hence $B^- \cap s_w(B)s_w^{-1}$ is the stabilizer in B^- of $\mathfrak{b}_w$. This is a subgroup with Lie algebra

$$\mathfrak{b}^- \cap \mathfrak{b}_w = \mathfrak{t} \oplus \bigoplus_{\substack{\alpha\in\Phi^- \\ w^{-1}(\alpha)\in\Phi^+}} L_\alpha .$$

We have a decomposition

$$\mathfrak{b}^- = \mathfrak{t} \oplus \bigoplus_{\substack{\alpha\in\Phi^- \\ w^{-1}(\alpha)\in\Phi^+}} L_\alpha \oplus \bigoplus_{\substack{\alpha\in\Phi^- \\ w^{-1}(\alpha)\in\Phi^-}} L_\alpha ,$$

which translates in groups as $B^- = TU_w^- U_w = U_w^- TU_w'$ (the products giving isomorphisms of varieties). Hence $B^-\mathfrak{b}_w = U_w^- TU_w'\mathfrak{b}_w = U_w^-\mathfrak{b}_w = C_w^-$. □

Some remarks are in order. In a similar way one can decompose G/B into B-orbits. Since $B = s_{w_0}B^- s_{w_0}^{-1}$, we have that

$$G/B = \bigcup_{w\in W} s_{w_0}C_w^- = \bigcup_{w\in W} Bs_{w_0}\mathfrak{b}_w = \bigcup_{w\in W} B\mathfrak{b}_{w_0 w}.$$

Set $U_w := s_{w_0}U_{w_0 w}^- s_{w_0}^{-1}$. The cell

$$C_w^+ = s_{w_0}C_{w_0 w}^- = s_{w_0}U_{w_0 w}^-\mathfrak{b}_{w_0 w} = U_w\mathfrak{b}_w ,$$

with center $\mathfrak{b}_w$ is an orbit under B and has dimension $\ell(w)$.

U_w is the unipotent group with Lie algebra $\bigoplus_{\alpha\in B_w} L_\alpha$ where

$$B_w := \{\beta \in \Phi^+ \mid w^{-1}(\beta) \prec 0\}.$$

Finally $C_w^+ = \{p \in G/B \mid \lim_{t\to\infty} \rho(t)p = \mathfrak{b}_w\}$.

We have a decomposition of the open set $U_w^- = C_w^- \times C_{w_0 w}^+$. The cells are called *Bruhat cells* and form two *opposite* cell decompositions.

Second, if we now choose G to be any algebraic group with Lie algebra L and $G_a(L) = G/Z$ where Z is the finite center of G, we have that:

Proposition 5. *The preimage of a Borel subgroup B_a of $G_a(L)$ is a Borel subgroup B of G. Moreover there is a canonical* 1-1 *correspondence between the B-orbits on G/B and the double cosets of B in G, hence the decomposition:*

$$G = \bigsqcup_{w\in W} Bs_wB = \bigsqcup_{w\in W} U_w s_w B, \qquad \textit{(Bruhat decomposition)}. \tag{6.4.2}$$

The obvious map $U_w \times B \to U_w s_w B$ is an isomorphism.

Proof. Essentially everything has already been proved. Observe only that G/B embeds in the projective space of an irreducible representation. On this space Z acts as scalars, hence Z acts trivially on projective space. It follows that $Z \subset B$ and that $G/B = G_a(L)/B_a$. □

The fact that the Bruhat cells are orbits (under B^- or B^+) implies immediately that:

Proposition 6. *The closure $S_w := \overline{C_w}$ of a cell C_w is a union of cells.*

Definition. S_w is called a *Schubert variety.*

The previous proposition has an important consequence. It defines a partial order in W given by $x \prec y \iff C_x^+ \subset \overline{C}_y^+$. This order is called the *Bruhat order.* It can be understood combinatorially as follows.

Theorem (on Bruhat order). *Given $x, y \in W$ we have $x \prec y$ in the Bruhat order if and only if there is a reduced expression $y = s_{i_1} s_{i_2} \dots s_{i+k}$ such that x can be written as a product of some of the s_{i_j} in the same order as they appear in y.*

We postpone the proof of this theorem to the next section.

6.5 Bruhat Order

Let us recall the definitions and results of 6.2. Given a subset J of the nodes of the Dynkin diagram, we have the corresponding root system Φ_J with simple roots as those corresponding to the nodes not in J. Its Weyl group W_J is the subgroup generated by the corresponding simple reflections s_i, $i \notin J$ (the stabilizer of a point in a suitable stratum of the closure of the Weyl chamber). We also have defined a parabolic subalgebra $\mathfrak{p}_J$ and a corresponding parabolic subgroup $P = P_J$.

Reasoning as in 6.4 we have:

Lemma. *P is the stabilizer of $\mathfrak{p}_J$ under the adjoint action.*

G/P can be identified with the set of parabolic subalgebras conjugate to $\mathfrak{p}_J$.

The parabolic subalgebras conjugated to $\mathfrak{p}_J$ and fixed by T are the ones containing $\mathfrak{t}$ are in 1-1 correspondence with the cosets W/W_J.

Proof. Let us show the last statement. If $\mathfrak{q}$ is a parabolic subalgebra containing $\mathfrak{t}$, a maximal solvable subalgebra of $\mathfrak{q}$ is a Borel subalgebra of L containing $\mathfrak{t}$, hence is equal to $\mathfrak{b}_w$. This implies that for some $w \in W$ we have that $\mathfrak{q} = s_w(\mathfrak{p}_J)$. The elements s_w are in the group $\tilde{W}$ such that $\tilde{W}/\tilde{W} \cap T = W$. One verifies immediately that the stabilizer in $\tilde{W}$ of $\mathfrak{p}_J$ is the preimage of W_J and the claim follows. □

Theorem. *(i) We have a decomposition $P = \bigsqcup_{w \in W_J} U_w^+ s_w B$.*

(ii) The variety G/P has also a cell decomposition. Its cells are indexed by elements of W/W_J, and in the fibration $\pi : G/N \to G/P$, we have that $\pi^{-1} C_{xW_J} = \bigsqcup_{w \in xW_J} C_w$.

(iii) The coset xW_J has a unique minimal element in the Bruhat order whose length is the dimension of the cell C_{xW_J}.

(iv) Finally, the fiber P/B over the point P of this fibration is the complete flag variety associated to the Levi factor L_J.

Proof. All the statements follow easily from the Levi decomposition. Let L be the Levi factor, B the standard Borel subgroup, and R the solvable radical of P. We have that $L \cap B$ is a Borel subgroup in L and we have a canonical identification $L/L \cap B = P/B$. Moreover, L differs from L_J by a torus part in B. Hence we have also $P/B = L_J/B_J$, with $B_J = B \cap L_J$ a Borel subgroup in L_J. The Weyl group of L_J is W_J so the Bruhat decomposition for L_J induces a Bruhat decomposition for P given by i).

By the previous lemma, the fixed points $(G/P)^W$ are in 1-1 correspondence with W/W_J. We thus have a decomposition into locally closed subsets C_a, $a \in W/W_J$ as for the full flag variety. □

A particularly important case of the previous analysis for us is that of a *minimal parabolic*. By this we mean a parabolic P associated to the set J of all nodes except one i. In this case the root system Φ_J is a system of type A_1 and thus the semisimple Levi factor is the group $SL(2, \mathbb{C})$. The Bruhat decomposition for P, which we denote $P(i)$, reduces to only two double cosets $P(i) = B \sqcup Bs_iB$ and $P(i)/B = \mathbb{P}^1(\mathbb{C})$ is the projective line, a sphere. The two cells are the affine line and the point at infinity.

We pass now to the crucial combinatorial lemma.

Bruhat lemma. *Let $w \in W$ and $s_w = s_{i_1}s_{i_2}\dots s_{i+k}$ be associated to a reduced expression of $w = \sigma_{i_1}\sigma_{i_2}\dots\sigma_{i+k}$. Let us consider the element s_i associated to a simple reflection σ_i. Then*

$$Bs_iBBs_wB = \begin{cases} Bs_is_wB & \text{if } \ell(\sigma_iw) = \ell(w) + 1 \\ Bs_is_wB \cup Bs_wB & \text{if } \ell(\sigma_iw) = \ell(w) - 1. \end{cases}$$

Proof. For the first part it is enough to see that $s_iBs_wB \subset Bs_is_wB$. Since $s_i^2 \in T$ this is equivalent to proving that $s_iBs_is_is_wB \subset Bs_is_wB$. We have (by the Corollary of the previous section), that $s_iBs_i \subset BU_{-\alpha_i}$ and

$$U_{-\alpha_i}s_is_w = s_is_w(s_is_w)^{-1}U_{-\alpha_i}s_is_w = U_{-w^{-1}\sigma_i(\alpha_i)}s_is_w = U_{w^{-1}(\alpha_i)}s_is_w.$$

Since $\ell(\sigma_iw) = \ell(w) + 1$ we have $w^{-1}(\alpha_i) \succ 0$, and so $U_{w^{-1}(\alpha_i)} \subset B$.

In the other case $w^{-1}(\alpha_i) \prec 0$. Set $w = \sigma_iu$. By the previous case $Bs_wB = Bs_iBs_uB$. Let us thus compute Bs_iBBs_iB. We claim that $Bs_iBBs_iB = Bs_iB \cup B$. This will prove the claim. Clearly this is true if $G = SL(2, \mathbb{C})$ and $s_i = s = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$, since in this case $BsB \cup B = SL(2, \mathbb{C})$, and $1 = (-1)s^2 \in BsBBsB$ and $s \in BsBBsB$ since

$$\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}\begin{vmatrix} -1 & -1 \\ 0 & -1 \end{vmatrix}\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}.$$

For the general case we notice that this is a computation in the minimal parabolic associated to i. We have that all the double cosets contain the solvable radical and thus we can perform the computation modulo the radical, reducing to the previous special case. □

Geometric lemma. *Let $w = \sigma_i u$, $\ell(w) = \ell(u) + 1$. Then $S_y = PS_u$ where $P \supset B$ is the minimal parabolic associated to the simple reflection s_i.*

Proof. Since $P = Bs_iB \cup B$, from Bruhat's lemma we have $PC_u = C_w \cup C_u$. Moreover, from the proof of the lemma, it follows that C_u is in the closure of C_w; thus by continuity $C_w \subset PS_u \subset S_w$. Thus it is sufficient to prove that PS_u is closed. By the Levi decomposition for P using $SL(2, \mathbb{C}) \subset P$, we have that $PS_u = SL(2, \mathbb{C})S_u$. In $SL(2, \mathbb{C})$ every element can be written as product ab with $a \in SU(2, \mathbb{C})$, and b upper triangular, hence in $B \cap SL(2, \mathbb{C})$. Thus $PS_u = SL(2, \mathbb{C})S_u = SU(2, \mathbb{C})P$. Finally since $SU(2, \mathbb{C})$ is compact, the action map $SU(2, \mathbb{C}) \times S_u \to G/B$ is proper and so its image is closed. □

Remark. With a little more algebraic geometry one can give a proof which is valid in all characteristics. One forms the algebraic variety $P \times_B S_u$ whose points are pairs (p, v), $p \in P$, $v \in S_u$, modulo the identification $(pb, v) \equiv (p, bv)$, $\forall v \in S_u$.

The action map factors through a map $P \times_B S_u \to S_w$ which one proves to be proper. This in fact is the beginning of an interesting construction, the Bott–Samelson resolution of singularities for S_w.

Proof (of the theorem on Bruhat order stated in §6.4*).* Let $y \in W$ and $T_y := \cup_{x \prec y} C_x$, where $\prec$ is the Bruhat order. We have to prove that $T_y = S_y$. We work by induction on the length of y. Let $y = s_{i_1}u$, $\ell(y) = \ell(u) + 1$. By induction S_u is the union of the cells C_x where x is obtained from the reduced expressions of u, deleting some factors. Given this we have by the Bruhat decomposition of P that $PS_u = (B \cup Bs_{i_1}B)S_u = S_u \cup Bs_{i_1}S_u$. Now if $x \prec u$ in the Bruhat order, we have that also $x, s_{i_1}x$ precede y. This shows that $S_w \subset T_w$. Conversely, let $x \prec y$. Then we have a reduced expression $y = s_{j_1}s_{j_2}\dots s_{j_k} = s_{j_1}w$ and x is obtained by dropping some of the factors; hence either $x \prec w$ or $x = s_{j_1}x'$ and $x' \prec w$. The same argument as before shows that $C_x \subset S_y$. □

Remark. The theory we have discussed holds in any characteristic, and in fact also over finite fields, where it gives the basic ingredients for the representation theory of the finite Chevalley groups. For instance, in the case of a finite field F with q elements, one takes the flag variety as basis of a permutation representation $\mathbb{C}[G/B]$. One applies next the discussion of Chapter 1, §3.2, where we showed that the endomorphism algebra of $\mathbb{C}[G/B]$ is the Hecke algebra of double cosets. The theory of Bruhat implies that this algebra has a basis T_w indexed by $w \in W$ and that $T_uT_v = T_{uv}$, if $\ell(uv) = \ell(u)+\ell(v)$, while $T_{s_i}^2 = (q-1)T_{s_i}+q$ for a simple reflection. This is the beginning of a rather deep theory.

Remark. Given a representation V_λ, $\lambda = \sum_i m_i\omega_i$ such that the stabilizer of the highest weight vector v_λ is P_J, $J = \{i \mid m_i \neq 0\}$, the set of T fixed points in the orbit $G\mathbb{C}v_\lambda = G/P_J \subset \mathbb{P}(V_\lambda)$ is the set of lines of the highest weight vectors for the various algebras $\mathfrak{b}_w$. These vectors are called the *extremal weight vectors* and their weights (the W orbit of λ) the *extremal weights.* There are (very few) representations which are particularly simple, and have the extremal weight vectors as basis; these are called *minuscule*. Among them we find the exterior powers $\bigwedge^k V$ for type A_n and, as we will see, the spin representations.

Examples. In classical groups we can represent the variety G/B in a more concrete way as a *flag variety*. We have already seen the definition of flags in Chapter 7, §4.1 where we described Borel subgroups of classical groups as stabilizers of totally isotropic flags.

Examples of fixed points Let us understand in this language which points are the T-fixed points. A flag of subspaces V_i of the defining representation is fixed under T if and only if each V_i is fixed, i.e., it is a sum of eigenspaces. For $SL(n, \mathbb{C})$, where T consists of the diagonal matrices, the standard basis $e_1, \dots, e_n$ is a basis of distinct eigenvalues, so a T-stable space has as basis a subset of the e_i. Thus a stable flag is constructed from a permutation σ as the sequence of subspaces $V_i := \langle e_{\sigma(1)}, \dots, e_{\sigma(i)} \rangle$. We see concretely how the fixed points are indexed by S_n.

Exercise. Prove directly the Bruhat decomposition for $SL(n, \mathbb{C})$, using the method of *putting a matrix into canonical row echelon form.*

For the other classical groups, the argument is similar. Consider for instance the symplectic group, with basis e_i, f_i of eigenvectors with distinct eigenvalues. Again a T-stable space has as basis a subset of the e_i, f_i. The condition for such a space to be totally isotropic is that, for each i, it should contain at most one and not both of the elements e_i, f_i. This information can be encoded with a permutation plus a sequence of ± 1, setting $+1$ if e_i appears, -1 if f_i appears. It is easily seen that we are again encoding the fixed flags by the Weyl group. The even orthogonal group requires a better analysis. The problem is that in this case, the set of complete isotropic flags is no longer an orbit under $SO(2n, \mathbb{C})$. This is explained by the familiar example of the *two rulings* of lines in a quadric in projective 3-space (which correspond to totally isotropic planes in 4 space). In group theoretical terms this means that the set of totally isotropic spaces of dimension n form two orbits under $SO(2n, \mathbb{C})$. This can be seen by induction as follows. One proves:

1. If $m = \dim V > 2$, the special orthogonal group acts transitively on the set of nonzero isotropic vectors.
2. If e is such a vector the orthogonal $e^\perp$ is an $m-1$-dimensional space on which the symmetric form is degenerate with kernel generated by e. Modulo e, we have an $m-1$-dimensional space $U := e^\perp/\mathbb{C}e$. The stabilizer of e in $SO(V)$ induces on U the full special orthogonal group.
3. By induction, two k-dimensional totally isotropic spaces are in the same orbit for $k < m/2$.
4. Finally $SO(2, \mathbb{C})$ is the subgroup of $SL(2, \mathbb{C})$ stabilizing the degenerate conic $\{xe+yf \mid xy = 0\}$. $SO(2, \mathbb{C})$ consists of diagonal matrices $e \mapsto te,\ f \mapsto t^{-1}f$. There are only two isotropic lines $x = 0,\ y = 0$. This analysis explains why the fixed flags correspond to the Weyl group which has $n!2^{n-1}$ elements.

Now let us understand the *partial flag varieties* G/P with P a parabolic subgroup. We leave the details to the reader (see also Chapter 13).

For type A_n, with group $SL(V)$, a parabolic subgroup is the stabilizer of a *partial flag* $V_1 \subset V_2 \cdots \subset V_k$ with $\dim V_i = h_i$. The dimensions h_i correspond to the positions, in the Dynkin diagram A_n, of the set J of nodes that we remove.

In particular, a maximal parabolic is the stabilizer of a single subspace, for instance, in a basis $e_1, \dots, e_m$, the k-dimensional subspace spanned by $e_1, \dots, e_k$. In fact, the stabilizer of this subspace is the stabilizer of the line through the vector $e_1 \wedge e_2 \wedge \cdots \wedge e_k \in \bigwedge^k V$. The irreducible representation $\bigwedge^k V$ is a fundamental representation with weight ω_k. The orbit of the highest weight vector $e_1 \wedge \cdots \wedge e_k$ is the set of decomposable exterior vectors. It corresponds in projective space to the set of k-dimensional subspaces, which is the *Grassmann variety*, isomorphic to $G/P_{\{k\}}$ where the parabolic subgroup is the group of block matrices $\begin{vmatrix} A & B \\ 0 & C \end{vmatrix}$ with A a $k \times k$ matrix. We plan to return to these ideas in Chapter 13, where we will take a more combinatorial approach which will free us from the characteristic 0 constraint.

Exercise. Show that the corresponding group $W_{\{k\}}$ is $S_k \times S_{n+1-k}$. The fixed points correspond to the decomposable vectors $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$, $i_1 < i_2 < \cdots < i_k$.

For the other classical groups one has to impose the restriction that the subspaces of the flags be totally isotropic. One can then develop a parallel theory of isotropic Grassmannians. We have again a more explicit geometric interpretation of the fundamental representations. These representations are discussed in the next chapter.

Exercise. Visualize, using the matrix description of classical groups, the parabolic subgroups in block matrix form.

For the symplectic and orthogonal group we will use the Theorems of Chapter 11.

For the symplectic group of a space V, the fundamental weight ω_k corresponds to the irreducible representation $\bigwedge_0^k(V) \subset \bigwedge^k(V)$ consisting of traceless tensors. Its highest weight vector is $e_1 \wedge e_2 \wedge \ldots \wedge e_k$ (cf. Chapter 11, §6.7).

In $\bigwedge^k V$, the orbit under $Sp(V)$ of the highest weight vector $e_1 \wedge \cdots \wedge e_k$ is the set of decomposable exterior vectors which are traceless. The condition to be traceless corresponds to the constraint, on the corresponding k-dimensional subspace, to be totally isotropic. This is the *isotropic Grassmann variety*. It is then not hard to see that to a set $J = \{j_1, j_2, \dots, j_k\}$ of nodes corresponds the variety of *partial isotropic flags* $V_{j_1} \subset V_{j_2} \subset \cdots \subset V_{j_k}$, $\dim V_{j_t} = j_t$.

Exercise. Prove that the intersection of the usual Grassmann variety, with the projective subspace $\mathbb{P}(\bigwedge_0^k(V))$ is the isotropic Grassmann variety.

For the orthogonal groups we have, besides the exterior powers $\bigwedge^k(V)$ which remain irreducible, also the spin representations.

On each of the exterior powers $\bigwedge^k(V)$ we have a quadratic form induced by the quadratic form on V.

Exercise. Prove that a decomposable vector $u := v_1 \wedge v_2 \wedge \ldots \wedge v_k$ corresponds to an isotropic subspace if and only if it is isotropic. The condition that the vector u

corresponds to a totally isotropic subspace is more complicated although it is given by quadratic equations. In fact one can prove that it is: $u \otimes u$ belongs to the irreducible representation in $\bigwedge^k V^{\otimes 2}$ of the orthogonal group, generated by the highest weight vector.

Finally for the maximal totally isotropic subspaces we should use, as fundamental representations, the spin representations, although from §6.2 we know we can also use twice the fundamental weights and work with exterior powers (cf. Chapter 11, §6.6). The analogues of the decomposable vectors are called in this case *pure spinors*.

6.6 Quadratic Equations

E. Cartan discovered that pure spinors, as well as the usual decomposable vectors in exterior powers, can be detected by a system of quadratic equations. This phenomenon is quite general, as we will see in this section. We have seen that the parabolic subgroups P give rise to compact homogeneous spaces G/P. Such a variety is the orbit, in some projective space $\mathbb{P}(V_\lambda)$, of the highest weight vector. In this section we want to prove a theorem due to Kostant, showing that, as a subvariety of $\mathbb{P}(V_\lambda)$, the ideal of functions vanishing on G/P is given by explicit quadratic equations.

Let v_λ be the highest weight vector of V_λ. We know by §5.2 Proposition 3 that $v_\lambda \otimes v_\lambda$ is a highest weight vector of $V_{2\lambda} \subset V_\lambda \otimes V_\lambda$ and $V_\lambda \otimes V_\lambda = V_{2\lambda} \bigoplus_{\mu \prec 2\lambda} V_\mu$.

If $v \in V_\lambda$ is in the orbit of v_λ we must have that $v \otimes v \in V_{2\lambda}$.

The theorem we have in mind is proved in two steps.

1. One proves that the Casimir element C has a scalar value on $V_{2\lambda}$ which is different from the values it takes on the other V_μ of the decomposition and interprets this as quadratic equations on G/P.
2. One proves that these equations generate the desired ideal.

As a first step we compute the value of C on an irreducible representation V_λ, or equivalently on v_λ. Take as basis for L the elements $(\alpha, \alpha)e_\alpha/2$, f_α for all positive roots, and an orthonormal basis k_i of $\mathfrak{h}$. From 1.8.1 (computing the dual basis f_α, $(\alpha, \alpha)e_\alpha/2$, k_i)

$$C = \sum_{\alpha \in \Phi^+} (\alpha, \alpha)(e_\alpha f_\alpha + f_\alpha e_\alpha)/2 + \sum_i k_i^2.$$

We have

$$f_\alpha e_\alpha v_\lambda = 0, \quad e_\alpha f_\alpha v_\lambda = [e_\alpha, f_\alpha]v_\lambda = h_\alpha v_\lambda = \lambda(h_\alpha)v_\lambda = \langle \lambda \mid \alpha \rangle v_\lambda,$$

(cf. 1.8.1). Finally, $\sum_i k_i^2 v_\lambda = \sum_i \lambda(k_i)^2 v_\lambda = (\lambda, \lambda)v_\lambda$ by duality and:

Lemma 1. *C acts on V_λ by the scalar*

$$C(\lambda) := \sum_{\alpha \in \Phi^+} (\lambda, \alpha) + (\lambda, \lambda) = (\lambda, 2\rho) + (\lambda, \lambda) = (\lambda + \rho, \lambda + \rho) - (\rho, \rho).$$

We can now complete step 2.

Lemma 2. *If $\mu \prec \lambda$ is a dominant weight, we have $C(\mu) < C(\lambda)$.*

Proof. $C(\lambda) - C(\mu) = (\lambda + \rho, \lambda + \rho) - (\mu + \rho, \mu + \rho)$. Write $\mu = \lambda - \gamma$ with γ a positive combination of positive roots. Then $(\lambda + \rho, \lambda + \rho) - (\mu + \rho, \mu + \rho) = (\lambda + \mu + 2\rho, \gamma)$. Since $\lambda + \mu + 2\rho$ is a regular dominant weight and γ a nonzero sum of positive roots, we have $(\lambda + \mu + 2\rho, \gamma) > 0$. □

Corollary. $V_{2\lambda} = \{a \in V_\lambda \otimes V_\lambda \mid Ca = C(2\lambda)a\}$.

Proof. The result follows from the lemma and the decomposition $V_\lambda \otimes V_\lambda = V_{2\lambda} \bigoplus_{\mu \prec 2\lambda} V_\mu$, which shows that the summands V_μ are eigenspaces for C with eigenvalue strictly less than the one obtained on $V_{2\lambda}$. □

Let us now establish some notations. Let R denote the polynomial ring on the vector space V_λ. It equals the symmetric algebra on the dual space which is V_μ, with $\mu = -w_0(\lambda)$, by Proposition 5.3. Notice that $-w_0(\rho) = \rho$ and $(\mu, \mu) = (\lambda, \lambda)$, so $C(\lambda) = C(\mu)$.

The space R_2 of polynomials of degree 2 consists (always, by §5.3) of $V_{2\mu}$ over which the Casimir element has value $C(2\mu)$ and lower terms. Let X denote the affine cone corresponding to $G/P \subset \mathbb{P}(V_\lambda)$. X consists of the vectors v which are in the G-orbit of the multiples of v_λ. Let A be the coordinate ring of X which is R/I, where I is the ideal of R vanishing on X. Notice that since X is stable under G, I is also stable under G, and $A = \bigoplus_k A_k$ is a graded representation. Since $v_\lambda \otimes v_\lambda \in V_{2\lambda}$, $x \otimes x \in V_{2\lambda}$ for each element $x \in X$. In particular let us look at the restriction to X of the homogeneous polynomials R_2 of degree 2 with image A_2. Let $Q \subset R_2$ be the kernel of the map $R_2 \to A_2$. Q is a set of quadratic equations for X. From the corollary it follows that Q equals the sum of all the irreducible representations different from $V_{2\mu}$, in the decomposition of R_2 into irreducibles. Since $V_{2\lambda}$ is irreducible, A_2 is dual to $V_{2\lambda}$, and so it is isomorphic to $V_{2\mu}$. The Casimir element $C = \sum_i a_i b_i$ acts as a second order differential operator on R and on A, where the elements of the Lie algebra a_i, b_i act as derivations.

Theorem (Kostant). *Let J be the ideal generated by the quadratic equations Q in R.*

(i) $R/J = A$, J is the ideal of the definition of X.
(ii) The coordinate ring of X, as a representation of G, is $\bigoplus_{k=0}^{\infty} V_{k\mu}$.

Proof. Let $R/J = B$. The Lie algebra L acts on R by derivations and, since J is generated by the subrepresentation Q, L preserves J and induces an action on B. The corresponding simply connected group acts as automorphisms. The Casimir element $C = \sum_i a_i b_i$ acts as a second order differential operator on R and B. Finally, $B_1 = V_\mu$, $B_2 = V_{2\mu}$.

Let $x, y \in B_1$. Then, $xy \in B_2 = V_{2\mu}$, hence $C(x) = C(\lambda)x$, $C(xy) = C(2\lambda)xy$. On the other hand, $C(xy) = C(x)y + xC(y) + \sum_i a_i(x)b_i(y) + b_i(x)a_i(y)$. We have hence $\sum_i a_i(x)b_i(y) + b_i(x)a_i(y) = [C(2\lambda) - 2C(\lambda)]xy = (2\lambda, 2\rho) + 4(\lambda, \lambda) - 2[(\lambda, 2\rho) + (\lambda, \lambda)] = 2(\lambda, \lambda)$. On an element of degree k we have

$$C(x_1x_2\dots x_k) = \sum_{i=1}^{k} x_1x_2\dots C(x_i)\dots x_k + \sum_{i<j}\sum_h x_1x_2\dots a_hx_i\dots b_hx_j\dots x_k$$
$$+ x_1x_2\dots b_hx_i\dots a_hx_j\dots x_k = [kC(\lambda)+2\binom{k}{2}(\lambda, \lambda)]x_1x_2\dots x_k.$$

Now $[kC(\lambda) + 2\binom{k}{2}(\lambda, \lambda)] = C(k\lambda)$. We now apply Lemma 2. B_k is a quotient of $V_\mu^{\otimes k}$, and on B_k we have that C acts with the unique eigenvalue $C(k\lambda)$; therefore we must have that $B_k = V_{k\mu}$ is irreducible. We can now finish. The map $\pi : B \to A$ is surjective by definition. If it were not injective, being L-equivariant, we would have that some B_k maps to 0. This is not possible since if on a variety all polynomials of degree k vanish, this variety must be 0. Thus J is the defining ideal of X and $B = \bigoplus_{k=0}^{\infty} V_{k\mu}$ is the coordinate ring of X. □

Corollary. *A vector $v \in V_\lambda$ is such that $v \otimes v \in V_{2\lambda}$ if and only if a scalar multiple of v is in the orbit of v_λ.*

Proof. In fact $v \otimes v \in V_{2\lambda}$ if and only if v satisfies the quadratic relations Q. □

Remark. On $V_\lambda \otimes V_\lambda$ the Casimir operator is $C_\lambda \otimes 1 + 1 \otimes C_\lambda + 2D$, where

$$D = \sum_{\alpha\in\Phi^+} (\alpha, \alpha)(e_\alpha \otimes f_\alpha + f_\alpha \otimes e_\alpha)/2 + \sum_i k_i \otimes k_i.$$

Thus a vector $a \in V_\lambda \otimes V_\lambda$ is in $V_{2\lambda}$ if and only if $Da = (\lambda, \lambda)a$.

The equation $D(v \otimes v) = (\lambda, \lambda)v \otimes v$ expands in a basis to a system of quadratic equations, defining in projective space the variety G/P.

Examples. We give here a few examples of fundamental weights for classical groups. More examples can be obtained from the theory to be developed in the next chapters.

1. The defining representation.

For the special linear group acting on $\mathbb{C}^n = V_{\omega_1}$ there is a unique orbit of nonzero vectors and so $X = \mathbb{C}^n$. $V_{2\omega_1} = S^2(\mathbb{C}^n)$, the symmetric tensors, and the condition $u \otimes u \in S^2(\mathbb{C}^n)$ is always satisfied.

For the symplectic group the analysis is the same.

For the special orthogonal group it is easily seen that X is the variety of isotropic vectors. In this case the quadratic equations reduce to $(u, u) = 0$, the canonical *quadric*.

2. For the other fundamental weights in $\bigwedge^k \mathbb{C}^n$, the variety X is the set of decomposable vectors, and its associated projective variety is the Grassmann variety. In this case the quadratic equations give rise to the theory of standard diagrams. We refer to Chapter 13 for a more detailed discussion.

For the other classical groups the Grassmann variety has to be replaced by the variety of totally isotropic subspaces. We discuss in Chapter 11, §6.9 the theory of maximal totally isotropic spaces in the orthogonal case using the theory of *pure spinors*.

6.7 The Weyl Group and Characters

We now deduce the internal description of the Weyl group and its consequences for characters.

Theorem 1. *The normalizer N_T of the maximal torus T is the union $\bigcup_{w \in W} s_w T$. We have an isomorphism $W = N_T/T$.*

Proof. Clearly $N_T \supset \bigcup_{w \in W} s_w T$. Conversely, let $a \in N_T$. Since a normalizes T it permutes its fixed points in G/B. In particular we must have that for some $w \in W$ the element $s_w^{-1}a$ fixes $\mathfrak{b}^+$. By Lemma 1 of 6.4, this implies that $s_w^{-1}a \in B$. If we have an element $tu \in B$, $t \in T$, $u \in U$ in the normalizer of T we also have $u \in N_T$. We claim that $N_T \cap U = 1$. This will prove the claim. Otherwise $N_T \cap U$ is a subgroup of U which is a unipotent group normalizing T. Recall that a unipotent group is necessarily connected. The same argument of Lemma 3.6 of Chapter 7 shows that $N_T \cap U$ must commute with T. This is not possible for a nontrivial subgroup of U, since the Lie algebra of U is a sum of nontrivial eigenspaces for T. □

We collect another result which is useful for the next section.

Proposition. *If G is a semisimple algebraic group, then the center of G is contained in all maximal tori.*

Proof. From the corollary of 6.3 we have $Z \subset B = TU$. If we had $z \in Z, z = tu$, we would have that u commutes with T. We have already remarked that the normalizer of T in U is trivial so $u = 1$ and $z \in T$. Since maximal tori are conjugate, Z is in every one of them. □

We can now complete the proof of Theorem 1, Chapter 8, §4.1 in a very precise form. Recall that an element $t \in T$ is *regular* if it is not in the kernel of any root character. We denote by T^{reg} this set of regular elements. Observe first that a generic element of G is a regular element in some maximal torus T.

Lemma. *The map $c : G \times T^{\text{reg}} \to G$, $c : (g, t) \mapsto gtg^{-1}$ has surjective differential at every point. Its image is a dense open set of G.*

Proof. Since the map c is G-equivariant, with respect to the left action on $G \times T^{\text{reg}}$ and conjugation in G, it is enough to compute it at some element $(1, t_0)$ where the tangent space is identified with $L \oplus \mathfrak{t}$. To compute the differential we can compose with $L_{t_0^{-1}}$ and consider separately the two maps $g \mapsto t_0^{-1}gt_0g^{-1}$ and $t \mapsto t$. The first map is the composition $g \mapsto (t_0^{-1}gt_0, g^{-1}) \mapsto t_0^{-1}gt_0g^{-1}$, and so it has differential $a \mapsto \text{Ad}(t_0^{-1})(a) - a$; the second is the identity of $\mathfrak{t}$. Thus we have to prove that the map $(a, u) \mapsto [\text{Ad}(t_0^{-1}) - 1]a + u$ is surjective. Since by hypothesis $\text{Ad}(t_0^{-1})$ does not possess any eigenvalue 1 on the root subspaces, the image of L under $\text{Ad}(t_0^{-1}) - 1$ is $\bigoplus_{\alpha \in \Phi} L_\alpha$. We can then conclude that the image of c is open; since it is algebraic it is also dense. □

Theorem 2. *Let G be a simply connected semisimple group.*

(i) The ring of regular functions, invariant under conjugation by G, is the polynomial ring $\mathbb{C}[\chi_{\omega_i}]$ in the characters χ_{ω_i} of the fundamental representations.
(ii) The restriction to a maximal torus T of the irreducible characters $\chi_\lambda, \lambda \in \Lambda^+$ forms an integral basis of the ring $\mathbb{Z}[\hat{T}]^W$ of W-invariant characters of T.
(iii) $\mathbb{Z}[\hat{T}]^W = \mathbb{Z}[\chi_{\omega_1}, \dots, \chi_{\omega_r}]$ is a polynomial ring over $\mathbb{Z}$ generated by the restrictions of the characters χ_{ω_i} of the fundamental representations.

Proof. From the previous theorem the restriction to T of a function on G, invariant under conjugation, is W-invariant. Since the union of all maximal tori in G is dense in G we have that this restriction is an injective map. The rings $\mathbb{Z}[\Lambda]$, $\mathbb{C}[\Lambda]$ are both permutation representations under W. From Theorem 2.4, every element of Λ is W-conjugate to a unique element $\lambda \in \Lambda^+$. Therefore if we set S_λ, $\lambda \in \Lambda^+$, to be the sum of all the conjugates under W of $\lambda \in \Lambda^+$, we have that

$$\mathbb{Z}[\Lambda] = \bigoplus\nolimits_{\lambda \in \Lambda^+} \mathbb{Z} S_\lambda. \tag{6.7.1}$$

From the highest weight theory it follows that the restriction to a maximal torus T of the irreducible character $\chi_\lambda, \lambda \in \Lambda^+$, which by abuse of notation we still denote by χ_λ, is of the form

$$\chi_\lambda = S_\lambda + \sum_{\mu \prec \lambda} c_{\mu,\lambda} S_\mu$$

for suitable positive integers $c_{\mu,\lambda}$ (which express the multiplicity of the space of weight μ in V_λ). In particular, we deduce that the irreducible characters $\chi_\lambda, \lambda \in \Lambda^+$, form an integral basis of the ring $\mathbb{Z}[\hat{T}]^W$ of W-invariant characters of T. Writing a dominant character $\lambda = \sum_{i=1}^r n_i \omega_i$, $n_i \in \mathbb{N}$ we see that χ_λ and $\prod_{i=1}^r \chi_{\omega_i}^{n_i}$ have the same leading term S_λ (in the dominance order) and thus $\mathbb{Z}[\hat{T}]^W$ is a polynomial ring over $\mathbb{Z}$ generated by the restrictions of the characters χ_{ω_i} of the fundamental representations.

The statement for regular functions over $\mathbb{C}$ follows from this more precise analysis. □

The reader will note the strong connection between this general theorem and various theorems on symmetric functions and conjugation invariant functions on matrices.

6.8 The Fundamental Group

We have constructed the group $G_s(L)$ with the same representations as a semisimple Lie algebra L. We do not yet know that $G_s(L)$ is simply connected. The difficulty comes from the fact that we cannot say a priori that the simply connected group associated to L is a linear group, and so it is obtained by integrating a finite-dimensional representation. The next theorem answers this question. In it we will use some basic facts of algebraic topology for which we refer to standard books, such as [Sp], [Ha]. We need to know that if G is a Lie group and H a closed subgroup, we have a locally trivial fibration $H \to G \to G/H$. To any such fibration one has an associated long exact sequence of homotopy groups. This will allow us to compute $\pi_1(G)$ for G an adjoint semisimple group. The fibration we consider is $B \to G \to G/B$. In order to compute the long exact sequence of this fibration we need to develop some topology of B and of G/B.

First, let us analyze B. Let T_c be the compact torus in T.

Proposition. *The inclusion of $T_c \subset T \subset B$ is a homotopy equivalence.* $\pi_1(T_c) = \hom_{\mathbb{Z}}(\hat{T}, \mathbb{Z})$.

Proof. We remark that $B = TU$ is homotopic to the maximal torus T since U is homeomorphic to a vector space. $T = (\mathbb{C}^*)^n$ is homotopic to $(S^1)^n = \mathbb{R}^n/\mathbb{Z}^n$. The homotopy groups are $\pi_i(\mathbb{R}^n/\mathbb{Z}^n) = 0, \forall i > 1,\ \pi_1(\mathbb{R}^n/\mathbb{Z}^n) = \mathbb{Z}^n$. The homotopy group of $(S^1)^n$ is the free abelian group generated by the canonical inclusions of S^1 in the n factors. In precise terms, in each homotopy class we have the loop induced by a 1-parameter subgroup μ:

$$\begin{array}{ccc} S^1 & \xrightarrow{\mu} & T_c \\ i\big\downarrow & & i\big\downarrow \\ \mathbb{C}^* & \xrightarrow{\mu} & T \end{array}.$$

More intrinsically $\pi_1(T)$ is identified with the group $\hom_{\mathbb{Z}}(\hat{T}, \mathbb{Z})$ of 1-parameter subgroups. □

By the Bruhat decomposition G/B is a union of even-dimensional cells. In order to apply the standard theory of CW-complexes we need more precise information. Let $(G/B)_h$ be the union of all the Schubert cells of complex dimension $\leq h$. We need to show that $(G/B)_h$ is the $2h$-dimensional skeleton of a CW complex and that every Schubert cell of complex dimension h is the interior of a ball of real dimension $2h$ with its boundary attached to $(G/B)_{h-1}$. If we can prove these statements, we will deduce by standard theory that $\pi_1(G/B) = 1,\ \pi_2(G/B) = H_2(G/B, \mathbb{Z})$ having as basis the orientation classes of the complex 1-dimensional Schubert varieties, which correspond to the simple reflections s_i and are each homeomorphic to a 2-dimensional sphere.

Let us first analyze the basic case of $SL(2, \mathbb{C})$. We have the action of $SL(2, \mathbb{C})$ on $\mathbb{P}^1$, and in homogeneous coordinates the two cells of $\mathbb{P}^1$ are $p_0 := \{(1, 0)\}$, $C := \{a, 1\},\ a \in \mathbb{C}$. The map

$$\begin{vmatrix} 1 & a \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \begin{vmatrix} a \\ 1 \end{vmatrix}$$

is the parametrization we have used for the open cell.

Consider the set D of unitary matrices:

$$D := \left\{ \begin{vmatrix} se^{i\theta} & r \\ -r & se^{-i\theta} \end{vmatrix} : r^2 + s^2 = 1,\ r, s \geq 0,\ \theta \in [0, 2\pi] \right\}.$$

Setting $se^{i\theta} = x + iy$ we see that this is in fact the 2-cell $x^2 + y^2 + r^2 = 1,\ r \geq 0$, with boundary the circle with $r = 0$. When we apply these matrices to p_0, we see that the boundary ∂D fixes p_0 and the interior $\overset{\circ}{D}$ of the cell maps isomorphically to the open cell of $\mathbb{P}^1$.

If B_0 denotes the subgroup of upper triangular matrices in $SL(2, \mathbb{C})$, we have, comparing the actions on $\mathbb{P}^1$, that

$$\overset{\circ}{D} B_0 = B_0 s B_0, \quad \partial D \subset B_0.$$

We can now use this attaching map to recursively define the attaching maps for the Bruhat cells. For each node i of the Dynkin diagram, we define D_i to be the copy of D contained in the corresponding group $SU_i(2, \mathbb{C})$.

Proposition. *Given $w = \sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_k} \in W$ a reduced expression, consider the $2k$-dimensional cell $D_w = D_{i_1} \times D_{i_2} \times \cdots \times D_{i_k}$. The multiplication map:*

$$D_{i_1} \times D_{i_2} \times \cdots \times D_{i_k} \to D_{i_1} D_{i_2} \dots D_{i_k} B$$

has image S_w. The interior of D_w maps homeomorphically to the Bruhat cell C_w, while the boundary maps to $S_w - C_w$.

Proof. Let $u = \sigma_{i_1} w$. By induction we can assume the statement for u. Then by induction $D_{i_1}(S_u - C_u) \subset (S_w - C_w)$ and $\partial D_{i_1} S_u \subset S_u$ since $\partial D_{i_1} \subset B$. It remains to prove only that we have a homeomorphism $\overset{\circ}{D} \times C_u \to C_w$. By the description of the cells, every element $x \in C_w$ has a unique expression as $x = a s_{i_1} c$ where a is in the root subgroup $U_{\alpha_{i_1}}$, and $c \in C_u$. We have that $a s_{i_1} = db$, for a unique element $d \in \overset{\circ}{D}$ and $b \in SL_{i_1}(2, \mathbb{C})$ upper triangular. The claim follows. □

We thus have:

Corollary. *G/B has the structure of a CW complex, with only even-dimensional cells D_w of dimension $2\ell(w)$ indexed by the elements of W.*

Each Schubert cell is a subcomplex.

If $w = s_i u$, $\ell(w) = \ell(u) + 1$, then S_w is obtained from S_u by attaching the cell D_w.

$\pi_1(G/B) = 0$, $H_i(G/B, \mathbb{Z}) = 0$ if i is odd, while $H_{2k}(G/B, \mathbb{Z}) = \bigoplus_{\ell(w)=k} \mathbb{Z}[D_w]$, where D_w is the homology class induced by the cell D_w.

$\pi_2(G/B) = H_2(G/B, \mathbb{Z})$.

Proof. These statements are all standard consequences of the CW complex structure. The main remark is that in the cellular complex which computes homology, the odd terms are 0, and thus the even terms of the complex coincide with the homology.

Given that $\pi_1(G/B) = 0$, $\pi_2(G/B) = H_2(G/B, \mathbb{Z})$ is the Hurewicz isomorphism. □

For us, the two important facts are that $\pi_1(G/B) = 0$ and $\pi_2(G/B, \mathbb{Z}) = \bigoplus_{s_i} \mathbb{Z}[D_{s_i}]$, a sum on the simple reflections.

Theorem 1. *Given a root system Φ, if $G_s(L)$ is as in §6.1 then $G_s(L)$ is simply connected.*

Moreover, its center Z is isomorphic to $\hom_{\mathbb{Z}}(\Lambda/\Lambda_r, \mathbb{Q}/\mathbb{Z})$ *where Λ is the weight lattice and Λ_r is the root lattice.*

Finally, $Z = \pi_1(G)$, where $G := G_a(L)$ is the associated adjoint group.

Proof. Given any dominant weight λ and the corresponding irreducible representation V_λ, by Schur's lemma, Z acts as some scalars which are elements ϕ_λ of $\hat{Z} = \hom(Z, \mathbb{C}^*)$. Since Z is a finite group, any such homomorphism takes values in the roots of 1, which can be identified with $\mathbb{Q}/\mathbb{Z}$. By §5.2 Proposition 3, we have that $\phi_{\lambda+\mu} = \phi_\lambda \phi_\mu$. Hence we get a map from Λ to $\hat{Z}$. If this mapping were not surjective, we would have an element $a \in Z$, $a \neq 1$ in the kernel of all the ϕ_λ. This is impossible since by definition $G_s(L)$ has a faithful representation which is the sum of the V_{ω_i}. Since $Z \subset T$ and Z acts trivially on the adjoint representation, we have that the homomorphism factors to a homomorphism of Λ/Λ_r to $\hat{Z}$.

Apply the previous results to $G = G_a(L)$. We obtain that the long exact sequence of homotopy groups of the fibration $B \to G \to G/B$ gives the exact sequence:

(6.8.1) $$0 \to \pi_2(G) \to H_2(G/B, \mathbb{Z}) \to \hom_{\mathbb{Z}}(\hat{T}, \mathbb{Z}) \to \pi_1(G) \to 0.$$

It is thus necessary to understand the mapping $H_2(G/B, \mathbb{Z}) \to \hom_{\mathbb{Z}}(\hat{T}, \mathbb{Z})$.

Next we treat $SL(2, \mathbb{C})$. In the diagram

$$\begin{array}{ccccc} U(1,\mathbb{C}) & \longrightarrow & SU(2,\mathbb{C}) & \longrightarrow & \mathbb{P}^1 \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & SL(2,\mathbb{C}) & \longrightarrow & \mathbb{P}^1 \end{array}$$

the vertical arrows are homotopy equivalences. We can thus replace $SL(2, \mathbb{C})$ by $SU(2, \mathbb{C})$.

We have the homeomorphisms $SU(2, \mathbb{C}) = S^3$, (Chapter 5, §5.1) $U(1, \mathbb{C}) = S^1$, $\mathbb{P}^1 = S^2$. The fibration $S^1 \to S^3 \to S^2$ is called the *Hopf fibration.*

Since $\pi_1(S^3) = \pi_2(S^3) = 0$ we get the isomorphism $\pi_1(S^1) = H_2(S^2, \mathbb{Z})$. A more precise analysis shows that this isomorphism preserves the standard orientations of S^1, S^2.

The way to achieve the general case, for each node i of the Dynkin diagram we embed $SU(2, \mathbb{C})$ in $SL_i(2, \mathbb{C}) \subset G_s(L) \to G = G_a(L)$ and we have a diagram

$$\begin{array}{ccccc} U(1,\mathbb{C}) & \longrightarrow & SU(2,\mathbb{C}) & \longrightarrow & \mathbb{P}^1 \\ {\scriptstyle i}\downarrow & & \downarrow & & {\scriptstyle j}\downarrow \\ B & \longrightarrow & G & \longrightarrow & G/B \end{array}$$

The mapping i of $S^1 = U(1, \mathbb{C})$ into the maximal torus T of G is given by $e^{\phi\sqrt{-1}} \mapsto e^{\phi h_i \sqrt{-1}}$. As a homotopy class in $\hom(\hat{T}, \mathbb{Z}) = \hom(\Lambda_r, \mathbb{Z})$ it is the element which is the evaluation of $\beta \in \Lambda_r$ at h_i. From 1.8.1 this value is $\langle \beta \mid \alpha_i \rangle$.

Next j maps $\mathbb{P}^1$ to the cell D_{s_i}.

We see that the homology class $[D_{s_i}]$ maps to the linear function $\tau_i \in \hom(\Lambda_r, \mathbb{Z})$, $\tau_i : \beta \mapsto \langle \beta \mid \alpha_i \rangle$. By 2.4.2 these linear functions are indeed a basis of the dual of the weight lattice and this completes the proof that

$$\text{(6.8.2)} \qquad \hom_{\mathbb{Z}}(\Lambda/\Lambda_r, \mathbb{Q}/\mathbb{Z}) = \hom_{\mathbb{Z}}(\Lambda_r, \mathbb{Z})/\hom_{\mathbb{Z}}(\Lambda, \mathbb{Z}) = \pi_1(G).$$

Now we have a mapping on the universal covering group $\tilde{G}$ of G to $G_s(L)$, which maps surjectively the center of $\tilde{G}$, identified to $\hom_{\mathbb{Z}}(\Lambda/\Lambda_r, \mathbb{Q}/\mathbb{Z})$, to the center Z of $G_s(L)$. Since we have seen that Z has a surjective homomorphism to $\hom_{\mathbb{Z}}(\Lambda/\Lambda_r, \mathbb{Q}/\mathbb{Z})$ and these are all finite groups, the map from $\tilde{G}$ to $G_s(L)$ is an isomorphism. □

By inspecting the Cartan matrices and computing the determinants we have the following table for Λ/Λ_r:

A_n : $\Lambda/\Lambda_r = \mathbb{Z}/(n+1)$. In fact the determinant of the Cartan matrix is $n+1$ but $SL(n+1, \mathbb{C})$ has as center the group of $(n+1)^{\text{th}}$ roots of 1.

For G_2, F_4, E_8 the determinant is 1. Hence $\Lambda/\Lambda_r = 0$, and the adjoint groups are simply connected.

For E_7, D_n, B_n we have $\Lambda/\Lambda_r = \mathbb{Z}/(2)$. For E_6, $\Lambda/\Lambda_r = \mathbb{Z}/(3)$, by the computation of the determinant.

For type D_n the determinant is 4. There are two groups of order 4, $\mathbb{Z}/(4)$ and $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$. A closer inspection of the elementary divisors of the Cartan matrix shows that we have $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$ when n is even and $\mathbb{Z}/(4)$ when n is odd.

6.9 Reductive Groups

We have seen the definition of reductive groups in Chapter 7 where we proved that a reductive group is linearly reductive, modulo the same theorem for semisimple groups. We have now proved this from the representation theory of semisimple Lie algebras. From all the work done, we have now proved that if an algebraic group is semisimple, that is, if its Lie algebra $\mathfrak{g}$ is semisimple, then it is the quotient of the simply connected semisimple group of $\mathfrak{g}$ modulo a finite subgroup of its center. The simply connected semisimple group of Lie algebra $\mathfrak{g}$ is the product of the simply connected groups of the simple Lie algebras $\mathfrak{g}_i$ which decompose $\mathfrak{g}$.

Lemma. *Let G be a simply connected semisimple algebraic group with Lie algebra $\mathfrak{g}$ and H any algebraic group with Lie algebra $\mathfrak{h}$. If $\phi : \mathfrak{g} \to \mathfrak{h}$ is a complex linear homomorphism of Lie algebras, ϕ integrates to an algebraic homomorphism of algebraic groups.*

Proof. Consider a faithful linear representation of $H \subset GL(n, \mathbb{C})$. When we integrate the homomorphism ϕ, we are in fact integrating a linear representation of $\mathfrak{g}$. We know that these representations integrate to rational representations of G. □

Given a connected reductive group G, let Z be the connected component of its center. We know that Z is a torus. Decompose the Lie algebra of G as $\oplus \mathfrak{g}_i \oplus \mathfrak{z}$ where $\mathfrak{z}$ is the Lie algebra of Z and the algebras $\mathfrak{g}_i$ are simple. Let G_i be the simply connected algebraic group with Lie algebra $\mathfrak{g}_i$. The previous lemma implies that for each i, there is an algebraic homomorphism $\phi_i : G_i \to G$ inducing the inclusion of the Lie algebra. Thus we deduce a map $\phi : \prod_i G_i \times Z \to G$ which is the identity on the Lie algebras. This is thus a surjective algebraic homomorphism with finite kernel contained in the product $\prod_i Z_i \times Z$, where Z_i is the finite center of G_i. Conversely:

Theorem. *Given simply connected algebraic groups G_i with simple Lie algebras and centers Z_i, a torus Z and a finite subgroup $A \subset \prod_i Z_i \times Z$ with $A \cap Z = 1$, the group $\prod_i G_i \times Z/A$ is reductive with Z as its connected center.*

In this way all reductive groups are obtained and classified.

The irreducible representations of $\prod_i G_i \times Z/A$ are the tensor products $\otimes_i V_{\lambda_i} \otimes \chi$ with χ a character of Z with the restriction that A acts trivially.

6.10 Automorphisms

The construction of Serre (cf. §3.1) allows us to also determine the entire group of automorphisms of a simple Lie algebra L. Recall that since all derivations are inner, the adjoint group is the connected component of the automorphism group. Now let $\phi : L \to L$ be any automorphism. We use the usual notations $\mathfrak{t}$, Φ, Φ^+ for a maximal toral subalgebra, roots and positive roots. Since maximal toral subalgebras are conjugate under the adjoint group, there is an element g of the adjoint group such that $g(\mathfrak{t}) = \phi(\mathfrak{t})$. Thus setting $\psi := g^{-1}\phi$, we have $\psi(\mathfrak{t}) = \mathfrak{t}$. From Proposition 3 of §5.4 we have that $\psi(\mathfrak{b}^+) = \mathfrak{b}_w$ for some $w \in W$. Hence $s_w^{-1}\psi(\mathfrak{b}^+) = \mathfrak{b}^+$. The outcome of this discussion is that we can restrict our study to those automorphisms ϕ for which $\phi(\mathfrak{t}^+) = \mathfrak{t}^+$ and $\phi(\mathfrak{b}^+) = \mathfrak{b}^+$.

One such automorphism permutes the roots preserving the positive roots, and hence it induces a permutation of the simple roots, hence a symmetry of the Dynkin diagram. On the other hand, we see immediately that the group of symmetries of the Dynkin diagram is $\mathbb{Z}/(2)$ for type $A_n, n > 1$ (reversing the orientation), $D_n, n > 4$ (exchanging the two last nodes $n-1, n$), E_6. It is the identity in cases $B_n, C_n, G_2, F_4, E_7, E_8$. Finally, D_4 has as a symmetry group the symmetric group S_3 (see the triality in the next Chapter 7.3). Given a permutation σ of the nodes of the Dynkin diagram we have that we can define an automorphism ϕ_σ of the Lie algebra by $\phi_\sigma(h_i) = h_{\sigma(i)}$, $\phi_\sigma(e_i) = e_{\sigma(i)}$, $\phi_\sigma(f_i) = f_{\sigma(i)}$. This is well defined since the Serre relations are preserved. We finally have to understand the nature of an automorphism fixing the roots. Thus $\phi(h_i) = h_i$, $\phi(e_i) = \alpha_i e_i$, for some numbers α_i. It follows that $\phi(f_i) = \alpha_i^{-1} f_i$ and that ϕ is conjugation by an element of the maximal torus, of coordinates α_i.

Theorem. *The full group* Aut(L) *of automorphisms of the Lie algebra L is the semidirect product of the adjoint group and the group of symmetries of the Dynkin diagram.*

Proof. We have seen that we can explicitly realize the group of symmetries of the Dynkin diagram as a group S of automorphisms of L and that every element of $\mathrm{Aut}(L)$ is a product of an inner automorphism in $G_a(L)$ and an element of S. It suffices to see that $S \cap G_a(L) = 1$. For this, notice that an element of $S \cap G_a(L)$ normalizes the Borel subgroup. But we have proved that in $G_a(L)$ the normalizer of B is B. It is clear that $B \cap S = 1$. □

Examples. In A_n, as an outer automorphism we can take $x \mapsto (x^{-1})^t$.

In D_n, as an outer automorphism we can take conjugation by any improper orthogonal transformation.

7 Compact Lie Groups

7.1 Compact Lie Groups

At this point we can complete the classification of compact Lie groups. Let K be a compact Lie group and $\mathfrak{k}$ its Lie algebra. By complete reducibility we can decompose $\mathfrak{k}$ as a direct sum of irreducible modules, hence simple Lie algebras. Among simple Lie algebras we distinguish between the 1-dimensional ones, which are abelian, and the nonabelian. The abelian summands of $\mathfrak{k}$ add to the center $\mathfrak{z}$ of $\mathfrak{k}$.

The adjoint group is a compact group with Lie algebra the sum of the nonabelian simple summands of $\mathfrak{k}$. First, we study the case $\mathfrak{z} = 0$ and K is adjoint. On $\mathfrak{k}$ there is a K-invariant (positive real) scalar product for which the elements $\mathrm{ad}(a)$ are skew symmetric. For a skew-symmetric real matrix A we see that A^2 is a negative semidefinite matrix, since $(A^2v, v) = -(Av, Av) \leq 0$. For a negative semidefinite nonzero matrix, the trace is negative and we deduce

Proposition 1. *The Killing form for the Lie algebra of a compact group is negative semidefinite with kernel the Lie algebra of the center.*

Definition. A real Lie algebra with negative definite Killing form is called a *compact* Lie algebra.

Before we continue, let L be a real simple Lie algebra; complexify L to $L \otimes \mathbb{C}$. By Chapter 6, §3.2 applied to L as a module on the algebra generated by the elements $\mathrm{ad}(a)$, $a \in L$, we may have that either $L \otimes \mathbb{C}$ remains simple or it decomposes as the sum of two irreducible modules.

Lemma. *Let L be a compact simple real Lie algebra. $L \otimes \mathbb{C}$ is still simple.*

Proof. Otherwise, in the same chapter the elements of $\mathrm{ad}(L)$ can be thought of as complex or quaternionic matrices (hence also complex).

If a real Lie algebra has also a complex structure we can compute the Killing form in two ways, taking either the real or the complex trace. A complex $n \times n$ matrix A is a real $2n \times 2n$ matrix. The real trace $\mathrm{tr}_{\mathbb{R}}(A)$ is obtained from the complex trace as $2\,\mathrm{Re}(\mathrm{tr}_{\mathbb{C}} A)$ twice its real part. Given a complex quadratic form, in some basis it is a sum of squares $\sum_h (x_h + iy_h)^2$, its real part is $\sum_h x_h^2 - y_h^2$. This is indefinite, contradicting the hypotheses made on L. □

Proposition 2. *Conversely, if for a group G the Killing form on the Lie algebra is negative definite, the adjoint group is a product of compact simple Lie groups.*

Proof. If the Killing form (a, a) is negative definite, the Lie algebra, endowed with $-(a, a)$ is a Euclidean space. The adjoint group G acts as a group of orthogonal transformations. We can therefore decompose $L = \bigoplus_i L_i$ as a direct sum of orthogonal irreducible subspaces. These are necessarily ideals and simple Lie algebras. Since the center of L is trivial, each L_i is noncommutative and, by the previous proposition, $L_i \otimes \mathbb{C}$ is a complex simple Lie algebra. We claim that G is a closed subgroup of the orthogonal group. Otherwise its closure $\overline{G}$ has a Lie algebra bigger than $\mathrm{ad}(L)$. Since clearly $\overline{G}$ acts as automorphisms of the Lie algebra L, this implies that there is a derivation D of L which is not inner. Since $L \otimes \mathbb{C}$ is a complex semisimple Lie algebra, D is inner in $L \otimes \mathbb{C}$, and being real, it is indeed in L. Therefore the group G is closed and the product of the adjoint groups of the simple Lie algebras L_i. Each G_i is a simple group. We can quickly prove at least that G_i is simple as a Lie group, although a finer analysis shows that it is also simple as an abstract group. Since G_i is adjoint, it has no center, hence no discrete normal subgroups. A proper connected normal subgroup would correspond to a proper two-sided ideal of L_i. This is not possible, since L_i is simple. □

To complete the first step in the classification we have to see, given a complex simple Lie algebra L, of which compact Lie algebras it is the complexification. We use the theory of Chapter 8, §6.2 and §7.1. For this we look first to the Cartan involution.

7.2 The Compact Form

We prove that the semisimple groups which we found are complexifications of compact groups. For this we need to define a suitable adjunction on the semisimple Lie algebras. This is achieved by the *Cartan involution*, which can be defined using the Serre relations. Let L be presented as in §3.1 from a root system.

Proposition 1. *There is an antilinear involution ω, called the* ***Cartan involution****, on a semisimple Lie algebra which, on the Chevalley generators, acts as*

$$\omega(e_i) = f_i, \ \omega(h_i) = h_i. \tag{7.2.1}$$

Proof. To define ω means to define a homomorphism to the conjugate opposite algebra. Since all relations are defined over $\mathbb{Q}$ the only thing to check is that the relations are preserved. This is immediate. For instance $\delta_{ij} h_i = \omega([e_i, f_j]) = [\omega(f_j), \omega(e_i)] = [e_j, f_i]$. The fact that ω is involutory is clear since it is so on the generators. □

We now need to show that:

Theorem 1. *The real subalgebra $\mathfrak{k} := \{a \in L \mid \omega(a) = -a\}$ gives a compact form for L.*

Clearly for each $a \in L$ we have $a = (a + \omega(a))/2 + (a - \omega(a))/2$, $(a - \omega(a))/2 \in \mathfrak{k}$, $(a + \omega(a))/2 \in \sqrt{-1}\mathfrak{k}$. Since ω is an antilinear involution, we can easily verify that for the Killing form we have $(\omega(a), \omega(b)) = \overline{(a, b)}$. This gives the Hermitian form $\langle a, b\rangle := (a, \omega(b))$. We claim it is positive definite. Let us compute $\langle a, a\rangle = (a, \omega(a))$ using the orthogonal decomposition for the Killing form $L = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi^+}(L_\alpha \oplus L_{-\alpha})$. On $\mathfrak{t} = E_{\mathbb{C}}$, with E the real space generated by the elements h_i, for $a \in E, \alpha \in \mathbb{C}$ we have $(a \otimes \alpha, a \otimes \alpha) = (a, a)|\alpha|^2 > 0$.

For $L_\alpha \oplus L_{-\alpha}$ one should first remark that the elements s_i which lift the simple reflections preserve the Hermitian form. Next, one can restrict to $\mathbb{C}e_i \oplus \mathbb{C}f_i$ and compute

$$(ae_i + bf_i, \omega(ae_i + bf_i)) = (ae_i + bf_i, \bar{a}f_i + \bar{b}e_i)$$
$$= (a\bar{a} + b\bar{b})(e_i, f_i) = 2(a\bar{a} + b\bar{b})/(\alpha_i, \alpha_i) > 0.$$

In conclusion we have a self-adjoint group and a compact form:

Proposition 2. *$\langle a, b\rangle$ is a Hilbert scalar product for which the adjoint of* $\mathrm{ad}(x)$, $x \in L$ *is given by* $\mathrm{ad}(\omega(x))$.

$\mathfrak{k}$ is the Lie algebra of the unitary elements in the adjoint group of L.

Proof. We have just checked positivity. For the second statement, notice that since $[x, \omega(b)] = -\omega[\omega(x), b]$, we have

$$\langle \mathrm{ad}(x)(a), b\rangle = (a, -\mathrm{ad}(x)(\omega(b))) = (a, \omega(\mathrm{ad}(\omega(x))(b)))$$
$$= \langle a, \mathrm{ad}(\omega(x))(b)\rangle. \tag{7.2.2}$$

The last statement follows from the previous ones. □

We have at this point proved that the adjoint group of a semisimple algebraic group is self-adjoint for the Hilbert structure given by the Cartan involution. In particular, it has a Cartan decomposition $G = KP$ with K a maximal compact subgroup. If the Lie algebra of G is simple, G is a simple algebraic group and K a simple compact group. Let us pass now to the simply connected cover $G_s(L)$. Let K_s be the preimage of K in $G_s(L)$.

Proposition 3. *K_s is connected maximal compact and is the universal cover of K. K_s is Zariski dense in $G_s(L)$.*

Proof. Since the map $\pi : G_s(L) \to G$ is a finite covering, the map $K_s \to K$ is also a finite covering. The inclusion of K in G is a homotopy equivalence. In particular, it induces an isomorphism of fundamental groups. Thus K_s is connected compact and the universal cover of K. If it were not maximal compact, we would have a larger compact group with image a compact group strictly larger than K. The first claim follows.

The Zariski closure of K_s is an algebraic subgroup H containing K_s. Its image in G contains K, so it must coincide with G. Since the kernel of π is in H we must have $G_s(L) = H$. □

From Proposition 2, Chapter 8, §6.1 we have:

Theorem 2. *Given any rational representation M of $G_s(L)$ choose a Hilbert space structure on M invariant under K_s. Then $G_s(L)$ is self-adjoint and K_s is the subgroup of unitary elements.*

In the correspondence between a compact K and a self-adjoint algebraic group G, we have seen that the algebraic group is topologically $G = K \times V$ with V affine space. Thus G is simply connected if and only if K is simply connected.

Remark. At this point we can complete the analysis by establishing the full classification of compact connected Lie groups and their algebraic analogues, the linearly reductive groups. Summarizing all our work we have proved:

Theorem 3. *There is a correspondence between connected compact Lie groups and reductive algebraic groups, which to a compact group K associates its algebraic envelope defined in Chapter 8, §7.2.*

Conversely, to a reductive group G we associate a maximal compact subgroup K unique up to conjugacy.

In any linear representation of G, a Hilbert metric invariant under K makes G self-adjoint.

G has a Cartan decomposition relative to K.

Then Theorem 6.9 becomes the classification theorem for connected compact Lie groups:

Theorem 4. *Given simply connected compact groups K_i with simple Lie algebras and centers Z_i, a compact torus T and a finite subgroup $A \subset \prod_i Z_i \times T$ with $A \cap T = 1$, the group $\prod_i K_i \times T/A$ is compact with Z as its connected center.*

In this way all connected compact groups are obtained and classified.

Proof. The compact group $\prod_i K_i \times T/A$ is the one associated to the reductive group $\prod_i G_i \times Z/A$, where G_i is the complexification of K_i and Z the complexification of T. □

From these theorems we can also deduce the classification of irreducible representations of reductive or compact Lie groups. For $G = (\prod_i G_i \times Z)/A$, we must give for each i an irreducible representation V_{λ_i} of G_i and also a character χ of Z. The representations $\otimes_i V_{\lambda_i} \otimes \chi$ are the list of irreducible representations of $\prod_i G_i \times Z$. Such a representation factors through G if and only if A acts trivially on it. For each i, λ_i induces a character on Z_i which we still call λ_i. Thus the condition is that the character $\prod_i \lambda_i \chi$ should be trivial on A.

7.3 Final Comparisons

We have now established several correspondences. One is between reductive groups and compact groups, the other between Lie algebras and groups. In particular we have associated to a complex simple Lie algebra two canonical algebraic groups, the adjoint group and the simply connected group, their compact forms and the compact Lie algebra. Several other auxiliary objects have appeared in the classification, and we should compare them all.

First, let us look at tori. Let L be a simple Lie algebra, $\mathfrak{t}$ a Cartan subalgebra, $G_a(L), G_s(L)$ the adjoint and simply connected groups. $G_a(L) = G_s(L)/Z$, where Z is the center of $G_s(L)$. Consider the maximal tori T_a, T_s associated to $\mathfrak{t}$ in $G_a(L), G_s(L)$, respectively. From §7.8, it follows that $Z \subset T_s$. Since T_s and T_a have the same Lie algebra it follows that $T_a = T_s/Z$. Since the exponential from the nilpotent elements to the unipotents is an isomorphism of varieties, the unipotent elements of $G_s(L)$ are mapped isomorphically to those of $G_a(L)$ under the quotient map. For the Borel subgroup associated to positive roots we thus have T_aU^+ in $G_a(L)$ and T_sU^+ in $G_s(L)$; for the Bruhat decomposition we have

$$G_a(L) = \bigsqcup_{w \in W} U_w^+ s_w T_a U^+, \qquad G_s(L) = \bigsqcup_{w \in W} U_w^+ s_w T_s U^+.$$

A similar argument shows that the normalizer of T_s in $G_s(L)$ is $N_{T_s} = \bigsqcup_{w\in W} s_w T_s$ and $N_{T_s}/Z = N_{T_a}$. In particular $N_{T_s}/T_s = N_{T_a}/T_a = W$. Another simple argument, which we leave to the reader, shows that there is a 1-1 correspondence between maximal tori in any group with Lie algebra L and maximal toral subalgebras of L. In particular maximal tori of G are all conjugate (Theorem 3.2).

More interesting is the comparison with compact groups. In this case, the second main tool, besides the Cartan decomposition, is the *Iwasawa decomposition*. We explain a special case of this theorem. Let us start with a very simple remark. The Cartan involution, by definition, maps $\mathfrak{u}^+$ to $\mathfrak{u}^-$ and $\mathfrak{t}$ into itself.

Let $\mathfrak{k}$ be the compact form associated to the Cartan involution. Let us look first at the Cartan involution on $\mathfrak{t}$. From formulas 7.2.1 we see that $\mathfrak{t}_c := \mathfrak{t} \cap \mathfrak{k}$ is the real space with basis $i\, h_j$. It is clearly the Lie algebra of the maximal compact torus T_c in T, and T has a Cartan decomposition.

Proposition 1. *(i)* $\mathfrak{k} = \mathfrak{t}_c \oplus \mathfrak{m}$ *where* $\mathfrak{m} := \{a - \omega(a),\ a \in \mathfrak{u}^-\}$.
(ii) $L = \mathfrak{k} + \mathfrak{b}^+,\ \mathfrak{t}_c = \mathfrak{k} \cap \mathfrak{b}^+$.
(iii) If K *is the compact group* $B \cap K = T_c$.

Proof. (i) The first statement follows directly from the formula 7.2.1 defining ω which shows in particular that $\omega(\mathfrak{u}^-) = \mathfrak{u}^+, \omega(\mathfrak{t}) = \mathfrak{t}$. It follows that every element of $\mathfrak{k}$ is of the form $-\omega(a) + t + a, t \in \mathfrak{t}_c,\ a \in \mathfrak{u}^-$.

(ii) Since $\mathfrak{b}^+ = \mathfrak{t} \oplus \mathfrak{u}^+$ any element $x = a + t + b, a \in \mathfrak{u}^+, t \in \mathfrak{t}, b \in \mathfrak{u}^-$ in L equals $a + \omega(b) + t + b - \omega(b)$, $a + \omega(b) + t \in \mathfrak{b}^+$, $b - \omega(b) \in \mathfrak{k}$ showing that $L = \mathfrak{k} + \mathfrak{b}^+$.

Consider now an element $-\omega(a) + t + a \in \mathfrak{k}$ with $t \in \mathfrak{t}_c, a \in \mathfrak{u}^-$. If we have $t + (a - \omega(a)) \in \mathfrak{b}^+$ since $t \in \mathfrak{b}^+$ we have $a - \omega(a) \in \mathfrak{b}^+$. This clearly implies that $a = 0$.

For the second part, we have from the first part that $B \cap K$ is a compact Lie group with Lie algebra $\mathfrak{t}_c$. Clearly $B \cap K \supset T_c$. Thus it suffices to remark that T_c is maximal compact in B. Let $H \supset T_c$ be maximal compact. Since unitary elements are semisimple we have that $H \cap U^+ = 1$. Hence in the quotient, H maps injectively into the maximal compact group of T. This is T_c. Hence $H = T_c$. □

The previous simple proposition has a very important geometric implication. Let K be the associated compact group. We can assume we are working in the adjoint case. As the reader will see, the other cases follow. Restrict the orbit map of G to G/B to the compact group K. The stabilizer of $[B] \in G/B$ in K is then, by Lemma 1 of 6.4 and the previous proposition, $B \cap K = T_c$. The tangent space of G/B in B is $L/\mathfrak{b}^+$. Hence from the same proposition the Lie algebra of $\mathfrak{k}$ maps surjectively to this tangent space. By the implicit function theorem this means that the image of K under the orbit map contains an open neighborhood of B. By equivariance, the image of K is open. Since K is compact the image of K is also closed. It follows that $K[B] = G/B$ and:

Theorem 1. *$KB = G$ and the homogeneous space G/B can also be described in compact form as K/T_c.*

It is interesting to see concretely what this means at least in one classical group. For $SL(n, \mathbb{C})$, the flag variety is the set of flags $V_1 \subset V_2 \subset \cdots \subset V_n = V$. Fixing a maximal compact subgroup is like fixing a Hilbert structure on V. When we do this, each V_i has an orthogonal complement L_i in V_{i+1}. The flag is equivalent to the sequence $L_1, L_2, \dots, L_n$ of mutually orthogonal lines. The group $SU(n, \mathbb{C})$ acts transitively on this set and the stabilizer of the set of lines generated by the standard orthonormal basis e_i in $SU(n, \mathbb{C})$ is the compact torus of special unitary diagonal matrices.

Consider next the normalizer N_{T_c} of the compact torus in K. First, let us recall that for each i in the Dynkin diagram, the elements s_i inducing the simple reflections belong to the corresponding $SU_i(2, \mathbb{C}) \subset SL_i(2, \mathbb{C})$. In particular all the elements s_i belong to K. We have.

Proposition 2. *$N_{T_c} = K \cap N_T$. Moreover $N_{T_c}/T_c = N_T/T = W$.*

Proof. If $a \in N_{T_c}$ since T_c is Zariski dense in T we have $a \in N_T$, hence the first statement. Since the classes of the elements $s_i \in N_{T_c}$ generate $W = N_T/T$, the second statement follows. □

We have thus proved that the Weyl group can also be recovered from the compact group. When we are dealing with compact Lie groups the notion of maximal torus is obviously that of a maximal compact abelian connected subgroup (Chapter 4, §7.1). Let us see then:

Theorem 2. *In a compact connected Lie group K all maximal tori are conjugate. Every element of K is contained in a maximal torus.*

Proof. Let Z be the connected part of the center of K. If A is a torus, it is clear that AZ is also a compact connected abelian group. It follows that all maximal tori contain Z. Hence we can pass to K/Z and assume that K is semisimple. Then K is a maximal compact subgroup of a semisimple algebraic group G. We use now the identification $G/B = K/T_c$, where T_c is the compact part of a maximal torus in G. Let A be any torus in K. Since A is abelian connected, it is contained in a maximal connected solvable subgroup P of G, that is a Borel subgroup of G. From the theory developed, we have that P has a fixed point in G/B, and hence A has a fixed point in K/T_c. By the fixed point principle A is conjugate to a subgroup of T_c. If A is a maximal torus, we must have A conjugate to T_c.

For the second part, every element of G is contained in a Borel subgroup, but a Borel subgroup intersects K in a maximal torus. □

One should also see [A] for a more direct proof based on the notion of degree of a map between manifolds.

Remark. In the algebraic case it is not true that every semisimple element of G is contained in a maximal torus!

In the description K/T_c we lose the information about the B-action and the algebraic structure, but we gain a very interesting topological insight.

Proposition 3. *Let $n \in N_{T_c}$. Given a coset $kT_c, k \in K$ the coset $kn^{-1}T_c$ is well defined and depends only on the class of n in W. In this way we define an action of W on K/T_c.*

Proof. If $t \in T_c$ we must show that $kn^{-1}T_c = ktn^{-1}T_c$. Now $ktn^{-1}T_c = kn^{-1}ntn^{-1}T_c = kn^{-1}T_c$ since $ntn^{-1} \in T_c$. It is clear that the formula, since it is well defined, defines an action of N_{T_c} on K/T_c. We have to verify that T_c acts trivially, but this is clear. □

Example. In the case of $SU(n, \mathbb{C})$ where K/T_c is the set of sequence $L_1, L_2, \dots, L_n$ of mutually orthogonal lines and W is the symmetric group, the action of a permutation σ on a sequence is just the sequence $L_{\sigma(1)}, L_{\sigma(2)}, \dots, L_{\sigma(n)}$.

Exercise. Calculate explicitly the action of $S_2 = \mathbb{Z}/(2)$ on the flag variety of $SL(2, \mathbb{C})$ which is just the projective line. Verify that it is not algebraic.

The Bruhat decomposition and the topological action of W on the flag variety are the beginning of a very deep theory which links geometry and representations but goes beyond the limits of this book.