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Tableaux

1 The Robinson–Schensted Correspondence

We start by explaining some combinatorial aspects of representation theory by giving a beautiful combinatorial analogue of the decomposition of tensor space $V^{\otimes n} = \bigoplus_{\lambda \vdash n} M_\lambda \otimes V_\lambda$.

Recall that in Chapter 9, §10.3 we have shown that this decomposition can be refined to a basis of weight vectors with indices pairs of tableaux. In this chapter tableaux are taken as the main objects of study, and from a careful combinatorial study we will get the applications to symmetric functions and the Littlewood–Richardson rule.

1.1 Insertion

We thus start from a totally ordered set A which in combinatorics is called an *alphabet.* We consider A as an ordered basis of a vector space V.

Consider the set of *words* of length n in this alphabet, i.e., sequences $a_1 a_2 \dots a_n$, $a_i \in A$. If $|A| = m$, this is a set with m^n elements in correspondence with the induced basis of $V^{\otimes n}$.

Next we shall construct from A certain combinatorial objects called *column and row tableaux.* Let us use pictorial language, and illustrate with examples. We shall use as an alphabet either the usual alphabet or the integers.

A *standard column* of length k consists in placing k distinct elements of A in a column (i.e., one on top of the other) so that they decrease from top to bottom:

Example.

$$\begin{array}{ccc} & s & \\ t & & 10 \\ & p & \\ g & & 9 \\ & g & \\ e & & 6 \\ & e & \\ b & & 5 \\ & d & \\ a & & 1 \\ & c & \end{array}$$

A sequence of columns of non-increasing length, placed one next to the other, identify a tableau and its *rows*. If the columns are standard and the elements in the rows going from left to right are weakly increasing (i.e., they can be also equal) the tableau is *semistandard*:

Definition 1. A semistandard tableau is a filling of a diagram with the letters from the alphabet so that the columns are strictly increasing from bottom to top while the rows are weakly increasing from left to right (cf. Chapter 9, §10.1).[117]

Example.

$$\begin{array}[b]{llll} t &&& \\ g & g & j & \\ e & f & f & \\ b & c & d & u \\ a & b & b & f \end{array} \qquad \begin{array}[b]{lll} s && \\ p && \\ g & h & p \\ e & f & g \\ d & e & f \\ c & d & e \end{array} \qquad \begin{array}[b]{lllll} 10 &&&& \\ 9 &&&& \\ 6 &&&& \\ 5 & 5 & 5 && \\ 1 & 1 & 1 & 1 & 1 \end{array}$$

The main algorithm which we need is that of *inserting a letter in a standard column.*

Assume we have a letter x and a column c; we begin by placing x on top of the column. If the resulting column is standard, this is the result of inserting x in c. Otherwise we start going down the column attempting to replace the entry that we encounter with x and we stop at the first step in which this produces a standard column. We thus replace the corresponding letter y with x, obtaining a new column c' and an expelled letter y.

Example.

$$\text{Inserting} \quad h \quad \text{in} \quad \begin{array}{c} t \\ g \\ e, \\ b \\ a \end{array} \quad \text{we obtain} \quad \begin{array}{c} t \\ h \\ e, \\ b \\ a \end{array} \quad \text{expelling} \quad g.$$

It is possible that the entering and exiting letters are the same. For instance, in the previous case if we wanted to insert g we would also extract g. A special case is when c is empty, and then inserting x just creates a column consisting of only x.

The first remark is that, from the new column c' and, if present, the expelled letter y one can reconstruct c and x. In fact we try backwards to insert y in c' from bottom upwards, stopping at the first position that makes the new column standard and expelling the relative entry. This is the reconstruction of c, x.

The second point is that we can now insert a letter x in a semistandard tableau T as follows. T is a sequence of columns $c_1, c_2, \ldots, c_i$. We first insert x in c_1; if we get an expelled element x_1 we insert it in c_2; if we get an expelled element x_2 we insert it in c_3, etc.

[117] Notice that we changed the display of a tableau! We are using the French notation.

Example.

1) Insert d in

$$\begin{array}{llll} t & & & \\ g & g & j & \\ e & f & f & \\ b & c & d & u \\ a & b & b & f \end{array}$$

get

$$\begin{array}{lllll} t & & & & \\ g & g & j & & \\ d & e & f & & \\ b & c & d & u & \\ a & b & b & f & f. \end{array}$$

2) Insert d in

$$\begin{array}{lll} s & & \\ p & & \\ g & h & p \\ e & f & g \\ d & e & f \\ c & d & e \end{array}$$

and get

$$\begin{array}{llll} s & & & \\ p & & & \\ g & h & p & \\ e & f & g & \\ d & e & f & \\ c & d & d & e. \end{array}$$

In any case, inserting a letter in a semistandard tableau, we always get a new semistandard tableau (verify it) with one extra box occupied by some letter. By the previous remark, the knowledge of this box allows us to recursively reconstruct the original tableau and the inserted letter.

All this can be made into a recursive construction. Starting from a word $w = a_1a_2\dots a_k = a_1w_1$ of length k, we construct two tableau $T(w),\ D(w)$ of the same shape, with k entries. The first, called *the insertion tableau*, is obtained recursively from the empty tableau by inserting a_1 in the tableau $T(w_1)$ (constructed by recursion). This tableau is semistandard and contains as entries exactly the letters appearing in w.

The tableau $D(w)$ is *the recording tableau*, and is constructed as follows. At the i^{th} step of the insertion a new box is produced, and in this box we insert i. Thus $D(w)$ records the way in which the tableau $T(w)$ has been recursively constructed. It is filled with all the numbers from 1 to k. Each number appears only once and we will refer to this property as standard.[118]

$D(w)$ is constructed from $D(w_1)$, which by inductive hypothesis has the same shape as $T(w_1)$, by placing in the position of the new box (occupied by the procedure of inserting a_1), the number k.

An example should illustrate the construction. We take the word *standard* and construct the sequence of associated insertion tableaux $T(w_i)$, inserting its letters starting from the right. We get:

$$
\begin{array}[b]{l} d \end{array} \quad
\begin{array}[b]{l} r \\ d \end{array} \quad
\begin{array}[b]{ll} r & \\ a & d \end{array} \quad
\begin{array}[b]{ll} d & r \\ a & d \end{array} \quad
\begin{array}[b]{ll} n & \\ d & r \\ a & d \end{array} \quad
\begin{array}[b]{lll} n & & \\ d & r & \\ a & a & d \end{array} \quad
\begin{array}[b]{lll} t & & \\ n & & \\ d & r & \\ a & a & d \end{array} \quad
\begin{array}[b]{lll} s & & \\ n & t & \\ d & r & \\ a & a & d; \end{array}
$$

the sequence of recording tableaux is

$$
\begin{array}[b]{l} 1 \end{array} \quad
\begin{array}[b]{l} 2 \\ 1 \end{array} \quad
\begin{array}[b]{ll} 2 & \\ 1 & 3 \end{array} \quad
\begin{array}[b]{ll} 2 & 4 \\ 1 & 3 \end{array} \quad
\begin{array}[b]{ll} 5 & \\ 2 & 4 \\ 1 & 3 \end{array} \quad
\begin{array}[b]{lll} 5 & & \\ 2 & 4 & \\ 1 & 3 & 6 \end{array} \quad
\begin{array}[b]{lll} 7 & & \\ 5 & & \\ 2 & 4 & \\ 1 & 3 & 6 \end{array} \quad
\begin{array}[b]{lll} 7 & & \\ 5 & 8 & \\ 2 & 4 & \\ 1 & 3 & 6. \end{array}
$$

Theorem (Robinson–Schensted correspondence). *The map $w \to (D(w), T(w))$ is a bijection between the set of words of length k and pairs of tableaux of the same shape of which $D(w)$ is standard and $T(w)$ semistandard.*

Proof. The proof follows from the sequence of previous remarks about the reversibility of the operation of inserting a letter. The diagram $D(w)$ allows one to determine which box has been filled at each step and thus to reconstruct the insertion procedure and the original word. $\square$

Definition 2. We call the *shape* of a word w the common shape of the two tableaux $(D(w), T(w))$.

Given a semistandard tableau T we call its *content* the set of elements appearing in it with the respective multiplicity. Similarly, we speak of the content of a given

[118] Some authors prefer doubly standard for this restricted type, and standard in place of semistandard.

word, denoted by $c(w)$. It is convenient to think of the content as the commutative monomial associated to the word, e.g., $w = abacbaa$ gives $c(w) = a^4b^2c$. The Robinson–Schensted correspondence preserves contents.

There is a special case to be observed. Assume that the recording tableau $D(w)$ is such that if we read it starting from left to right and then from the bottom to the top, we find the numbers $1, 2, \dots, k$ in increasing order, e.g.,

$$\begin{matrix} 9 & & & \\ 7 & 8 & & \\ 5 & 6 & & \\ 1 & 2 & 3 & 4 \end{matrix}$$

Then the word w can be very quickly read off from $T(w)$. It is obtained by reading $T(w)$ from top to bottom and from left to right (as in the English language), e.g.,[119]

$$spuntatu \implies \begin{matrix} s & & \\ p & u & \\ n & t & \\ a & t & u \end{matrix} \qquad \begin{matrix} 8 & & \\ 6 & 7 & \\ 4 & 5 & \\ 1 & 2 & 3 \end{matrix}$$

Such a word will be called a *semistandard word.*

1.2 Knuth Equivalence

A natural construction from the R-S correspondence is the *Knuth equivalence*.

Let us ask the question of when two words w_1, w_2 have the same insertion tableau, i.e., when $T(w_1) = T(w_2)$. As the starting point let us see the case of words of length 3.

For the 6 words w in a, b, c with the 3 letters appearing we have the simple table of corresponding tableaux $T(w)$:

$$abc \to \begin{matrix} a & b & c \end{matrix};\ acb \to \begin{matrix} c & \\ a & b \end{matrix};\ bac \to \begin{matrix} b & \\ a & c \end{matrix};\ bca \to \begin{matrix} b & \\ a & c \end{matrix};\ cab \to \begin{matrix} c & \\ a & b \end{matrix};\ cba \to \begin{matrix} c \\ b \\ a \end{matrix}.$$

We write $acb \stackrel{K}{\equiv} cab$, $bac \stackrel{K}{\equiv} bca$ to recall that they have the same insertion tableau. For words of length 3 with repetition of letters we further have $aba \stackrel{K}{\equiv} baa$ and $bab \stackrel{K}{\equiv} bba$.

At this point we have two possible approaches to Knuth equivalence that we will prove are equivalent. One would be to declare equivalent two words if they have the same insertion tableau. The other, which we take as definition, is:

[119] I tried to find a real word; this is maybe dialect.

Definition. Knuth equivalence on words is the minimal equivalence generated by the previous equivalence on all possible subwords of length 3. We will write $w_1 \overset{K}{\cong} w_2$.

In other words we pass from one word to another in the same Knuth equivalence class, by substituting a string of 3 consecutive letters with an equivalent one according to the previous table.

Proposition. *Knuth equivalence is compatible with multiplication of words. It is the minimal compatible equivalence generated by the previous equivalence on words of length 3.*

Proof. This is clear by definition. □

The main result on Knuth equivalence is the following:

Theorem. *Two words w_1, w_2 are Knuth equivalent if and only if they have the same insertion tableau, i.e., when $T(w_1) = T(w_2)$.*

Proof. We start by proving that if w_1, w_2 are Knuth equivalent, then $T(w_1) = T(w_2)$. The reverse implication will be proved after we develop, in the next section, the *jeu de taquin*.

By the construction of the insertion tableau it is clear that we only need to show that if w_1, w_2 are Knuth equivalent words of length 3, and z is a word, then $T(w_1 z) = T(w_2 z)$. In other words, when we insert the word w_1 in the tableau $T(z)$ we obtain the same result as when we insert w_2.

The proof is done by case analysis. For instance, let us do the case $w_1 = wuv \overset{K}{\cong} uwv = w_2$ for 3 arbitrary letters $u < v < w$. We have to show that inserting these letters in the 2 given orderings in a semistandard tableau T produces the same result.

Let c be the first column of T, and T' the tableau obtained from T by removing the first column.

Suppose first that inserting in succession uwv in c, we place these letters in 3 distinct boxes, expelling successively some letters f, g, e. From the analysis of the positions in which these letters were, it is easily seen that $e < f < g$ and that, inserting wuv, we expel f, e, g. Thus in both cases the first column is obtained from c replacing e, f, g with u, v, w.

The tableau T' now is modified by inserting the word egf or gef, which are elementary Knuth equivalent. Thus we are in a case similar to the one in which we started for a smaller tableau and induction applies.

Some other cases are possible and are similarly analyzed.

If the top element of c is $< u$, the result of insertion is, in both cases, to place u, w on top of c and insert v in T'. The analysis is similar if w or u expels v. □

The set of words modulo Knuth equivalence is thus a monoid under multiplication, called by Schützenberger *le monoide plactique*. We will see how to use it.

There is a very powerful method to understand these operations which was invented by Schützenberger and it is called *jeu de taquin*[120] [Sch]. In order to explain it we must first of all discuss skew diagrams and tableaux.

2 Jeu de Taquin

2.1 Slides

In our convention (the French convention) we draw our tableaux in the *quadrant* $Q = \{(i, j) \mid i, j \in \mathbb{N}^+\}$. We refer to the elements of Q as *boxes*. In Q we have the partial order $(i, j) \leq (h, k) \iff i \leq h,$ and $j \leq k$. Given $c \leq d,\ c, d \in Q$ the set $R(c, d) := \{a \in Q \mid c \leq a \leq d\}$ is a *rectangle*, the set $R_c := \{a \in Q \mid a \leq c\}$ is the rectangle $R((1, 1), c)$ while the set $Q_c := \{a \in Q \mid c \leq a\}$ is itself a quadrant.

Definition 1. A finite set $S \subset Q$ is called:

1) A diagram, if for every $c \in S$ we have $R_c \subset S$.
2) A skew diagram if given $c \leq d$, both in S, we have $R(c, d) \subset S$.
3) A box c is called an outer (resp. inner) box of S if $S \cap Q_c = \emptyset$ (resp. $S \cap R_c = \emptyset$).

Observe that, in our definition, it is possible that a box is at the same time inner and outer.

We will usually denote diagrams with greek letters. If $\mu \subset \lambda$ are diagrams we see that $\lambda - \mu$ is a skew diagram, indicated by λ/μ. Each skew diagram can be expressed in this way although not uniquely. A skew diagram is also called a *shape* and one refers to a diagram as a *normal shape*.

Definition 2. A standard (resp. semistandard) skew tableau T is a filling of a skew diagram λ/μ satisfying the restrictions as for diagrams (strictly increasing upwards on columns and nondecreasing from left to right on rows). We call $\lambda/\mu = sh(T)$ the shape of the tableau.

If $\mu = \emptyset$ and $sh(T) = \lambda$ we say that the tableau is in normal shape.

Example. Let $\lambda = 4, 4, 3, 2,\ \mu = 2, 2, 2$. We show λ/μ and a semistandard diagram of this shape:

$$\begin{array}{cccc} \cdot & \cdot & & \\ & & \cdot & \\ & & \cdot & \cdot \\ & & \cdot & \cdot \end{array} \qquad\qquad \begin{array}{cccc} a & a & & \\ & & f & \\ & & d & u \\ & & b & f \end{array}$$

Given a semistandard skew tableau, its *row word* or *reading word* is the word one obtains by successively reading the rows of the tableau, starting from the top one, and proceeding downwards.

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[120] *Il gioco del 15.* Sam Loyd (1841–1911) was the creator of famous mathematical puzzles and recreations. One of his most famous was the "15 Puzzle," which consisted of a 4×4 square of tiles numbered 1 to 15, with one empty space. The challenge was, starting from an arbitrary ordering and sliding tiles one by one to an adjacent empty space, to rearrange them in numerical order.

In the previous example, the row word is $aafdubf$.

Conversely, given any word w there is a unique way of decomposing it as a product of maximal standard rows $w = w_1 w_2 \dots w_r$. For instance,[121]

$$pr \,|\, e \,|\, cip \,|\, it \,|\, ev \,|\, o \,|\, liss \,|\, im \,|\, ev \,|\, o \,|\, lm \,|\, ent \,|\, e$$

we can immediately use this decomposition to present the word as a semistandard skew tableau, stringing the standard rows together in the obvious trivial way:

$$\begin{array}{cccccccccccccccccccccccc}
p & r \\
 & & e \\
 & & & c & i & p \\
 & & & & & & i & t \\
 & & & & & & & & e & v \\
 & & & & & & & & & & o \\
 & & & & & & & & & & & l \\
 & & & & & & & & & & & & i & s & s \\
 & & & & & & & & & & & & & & & i & m \\
 & & & & & & & & & & & & & & & & & e & v \\
 & & & & & & & & & & & & & & & & & & & o \\
 & & & & & & & & & & & & & & & & & & & & l & m \\
 & & & & & & & & & & & & & & & & & & & & & & e & n & t \\
 & & & & & & & & & & & & & & & & & & & & & & & & & e
\end{array}$$

There is also another trivial way that consists of sliding each row as far to the left as is possible, maintaining the structure of semistandard tableau:

$$\begin{array}{cccccccccc}
p & r \\
 & e \\
 & c & i & p \\
 & & & i & t \\
 & & & & e & v \\
 & & & & & o \\
 & & & & & l \\
 & & & & & i & s & s \\
 & & & & & & i & m \\
 & & & & & & & e & v \\
 & & & & & & & & o \\
 & & & & & & & & l & m \\
 & & & & & & & & & e & n & t \\
 & & & & & & & & & & & e
\end{array}$$

[121] This is the longest word in Italian.

The word we started from is a standard word if and only if, after this trivial slide, it gives a tableau (not a skew one). In general this trivial slide is in fact part of a more subtle slide game, the *jeu de taquin*, which eventually leads to the standard insertion tableau of the word. The game is applied in general to a semistandard skew tableau with one *empty box*, which we visualize by a dot, as in

$$\begin{array}{cccc} g & g & & \\ e & . & f & \\ & b & d & u \\ & & b & f \end{array}$$

When we have such a tableau we can perform either a forward or a backward slide. For a backward slide we move one of the adjacent letters into the empty box from right to left or from top to bottom and empty the corresponding box, provided we maintain the standard structure of the tableau. One uses the opposite procedure for a forward slide:

$$\begin{array}{cccc} g & g & & \\ e & . & f & \\ & b & d & u \\ & & b & f \end{array} \quad \Longrightarrow \quad \begin{array}{cccc} g & g & & \\ e & f & . & \\ & b & d & u \\ & & b & f \end{array} \qquad \text{backward.}$$

It is clear that at each step we can perform one and only one backward slide until the empty box is expelled from the tableau. Similarly for forward slides.

A typical sequence of slides comes from Knuth equivalence. Consider for instance the equivalences $bca \overset{K}{\cong} bac$ and $acb \overset{K}{\cong} cab$. We obtain them through the backward slides:

$$\begin{array}{cc} b & c \\ . & a \end{array} \Longrightarrow \begin{array}{cc} b & c \\ a & . \end{array} \Longrightarrow \begin{array}{cc} b & . \\ a & c \end{array} \;;\; \begin{array}{cc} a & c \\ . & b \end{array} \Longrightarrow \begin{array}{cc} . & c \\ a & b \end{array} \Longrightarrow \begin{array}{cc} c & . \\ a & b \end{array}$$

Definition 3. An inner corner (resp. an outer corner) of a skew diagram S is an inner (resp. outer) box c such that $S \cup \{c\}$ is still a diagram.

A typical step of *jeu de taquin* consists in picking one of the *inner corners* of a skew diagram and perform the backward slides until the empty box (the hole) exits the diagram and becomes an outer corner. We will call this a *complete backward slide*, similarly for *complete forward slides*.

Example. We have 2 inner corners and show sequences of complete backward slides:

$$\begin{array}{ccc} i & t & \\ \bullet & e & v \\ . & . & o \end{array} \Rightarrow \begin{array}{ccc} i & t & \\ e & \bullet & v \\ . & . & o \end{array} \Rightarrow \begin{array}{ccc} i & \bullet & \\ e & t & v \\ . & . & o \end{array} \Rightarrow \begin{array}{ccc} i & & \\ e & t & v \\ . & o & \bullet \end{array} \Rightarrow \begin{array}{ccc} i & & \\ e & t & \\ \bullet & o & v \end{array} \Rightarrow \begin{array}{ccc} i & & \\ \bullet & t & \\ e & o & v \end{array} \Rightarrow \begin{array}{ccc} & & \\ i & t & \\ e & o & v \end{array}$$

We could have started from the other inner corner:

$$\begin{array}{ccc} i & t & \\ . & e & v \\ . & . & o \end{array} \Rightarrow \begin{array}{ccc} i & t & \\ . & e & v \\ . & o & . \end{array} \Rightarrow \begin{array}{ccc} i & t & \\ . & e & \\ . & o & v \end{array} \Rightarrow \begin{array}{ccc} i & t & \\ e & . & \\ . & o & v \end{array} \Rightarrow \begin{array}{ccc} i & & \\ e & t & \\ . & o & v \end{array} \Rightarrow \begin{array}{ccc} i & & \\ . & t & \\ e & o & v \end{array} \Rightarrow \begin{array}{ccc} & & \\ i & t & \\ e & o & v \end{array}$$

In this example we discover that, at the end, we always obtain the same tableau.

One should also take care of the trivial case in which an inner corner is also outer and nothing happens:

$$\begin{array}{ccc} i & & \\ . & \bullet & \\ . & . & o \end{array} \qquad \begin{array}{ccc} i & & \\ . & . & \\ . & . & o \end{array}$$

We want to introduce a notation for this procedure. Starting from an inner corner c of a semistandard tableau T, proceed with the corresponding backward slides on T. As the final step we vacate a cell d of T obtaining a new semistandard tableau T'. We set

$$T' := j_c(T), \quad d := v_c(T).$$

Then d is an outer corner of T' and if we make the forward slides starting from it, we just invert the previous sequence, restore T and vacate c. We set

$$T := j^d(T'), \quad c := v^d(T').$$

We thus have the identity $T = j^d j_c(T)$ if $d = v_c(T)$. We have a similar identity if we start from forward slides.

We see in this example the fundamental results of Schützenberger:

Theorem 1. *Two skew semistandard tableaux of row words w_1, w_2 can be transformed one into the other, performing in some order complete backward slides starting from inner corners or complete forward slides from outer corners, if and only if the two row words are Knuth equivalent.*

Corollary. *Starting from a skew semistandard tableau of row word w, if we perform in any order, complete backward slides starting from inner corners, at the end we always arrive at the semistandard insertion tableau of w.*

In order to prove this theorem we first need a:

Lemma. *Let $a_1a_2 \dots a_n$ be a standard row (i.e., $a_i \leq a_{i+1}, \forall i$).*

(i) If $y \leq a_1$ we have $a_1a_2 \dots a_n y \overset{K}{\cong} a_1 y a_2 \dots a_n$.
(ii) If $y \geq a_n$ we have $y a_1 a_2 \dots a_n \overset{K}{\cong} a_1 a_2 \dots a_{n-1} y a_n$.

Proof. By induction on n. If $n = 2$ we are just using the definition of Knuth equivalence on words of length 3. Otherwise, in the first case if $y < a_1$ (or $y = a_1$) by induction $a_1a_2 \dots a_n y \stackrel{K}{\cong} a_1a_2y \dots a_n$. Next we use the equivalence $a_1ya_2 \stackrel{K}{\cong} a_1a_2y$.

In the second case $y > a_n$ (or $y = a_n$). By induction $ya_1a_2 \dots a_n \stackrel{K}{\cong} a_1a_2 \dots ya_{n-1}a_n$ and then we use the equivalence $ya_{n-1}a_n \stackrel{K}{\cong} a_{n-1}ya_n$. □

The reader should remark that this inductive proof corresponds in fact to a sequence of backward, respectively forward, slides for the tableaux:

$$y < a_1 : \quad \begin{matrix} a_1 & a_2 & a_3 & \dots & a_n & \\ . & . & . & \dots & . & y \end{matrix} \quad \Longrightarrow \quad \begin{matrix} a_1 & . & . & \dots & . \\ y & a_2 & a_3 & \dots & a_n \end{matrix}$$

$$y > a_n : \quad \begin{matrix} & y & . & . & \dots & \\ a_1 & a_2 & a_3 & \dots & a_n & \end{matrix} \quad \Longrightarrow \quad \begin{matrix} a_1 & a_2 & a_3 & \dots & y \\ . & . & . & \dots & a_n \end{matrix}$$

Proof of the Theorem. We must show that the row words we obtain through the slides are always Knuth equivalent. The slides along a row do not change the row word, so we only need to analyze slides from top to bottom or conversely. Let us look at one such move:

$$\begin{matrix} \dots u & c_1 & c_2 & \dots & c_n & x & d_1 & d_2 & \dots & d_m & \\ & a_1 & a_2 & \dots & a_n & . & b_1 & b_2 & \dots & b_m & \dots v \end{matrix}$$

to

$$\begin{matrix} \dots u & c_1 & c_2 & \dots & c_n & . & d_1 & d_2 & \dots & d_m & \\ & a_1 & a_2 & \dots & a_n & x & b_1 & b_2 & \dots & b_m & \dots v \end{matrix}$$

Since Knuth equivalence is compatible with multiplication, we are reduced to analyze:

$$\begin{matrix} c_1 & c_2 & \dots & c_n & x & d_1 & d_2 & \dots & d_m \\ a_1 & a_2 & \dots & a_n & . & b_1 & b_2 & \dots & b_m \end{matrix}$$

to

$$\begin{matrix} c_1 & c_2 & \dots & c_n & . & d_1 & d_2 & \dots & d_m \\ a_1 & a_2 & \dots & a_n & x & b_1 & b_2 & \dots & b_m \end{matrix}$$

Here $a_n < c_n \leq x \leq b_1$. If $n = m = 0$ there is nothing to prove. Let $n > 0$. Perform the slide which preserves the row word:

$$\begin{matrix} c_1 & & & & & & & & \\ . & c_2 & \dots & c_n & x & d_1 & d_2 & \dots & d_m \\ a_1 & a_2 & \dots & a_n & . & b_1 & b_2 & \dots & b_m \end{matrix}$$

Next the slide:

$$\begin{array}{cccccccccc}
c_1 &&&&&&&&& \\
a_1 & c_2 & \dots & c_n & x & d_1 & d_2 & \dots & d_m \\
. & a_2 & \dots & a_n & . & b_1 & b_2 & \dots & b_m
\end{array}$$

is compatible with Knuth equivalence by the previous Lemma (i) applied to $y = a_1 < c_1$ and the standard row $c_1c_2\dots c_nxd_1d_2\dots d_m$. Now we can apply induction and get Knuth equivalence for the slide:

$$\begin{array}{ccccccccc}
c_1 &&&&&&&& \\
a_1 & c_2 & \dots & c_n & . & d_1 & d_2 & \dots & d_m \\
. & a_2 & \dots & a_n & x & b_1 & b_2 & \dots & b_m
\end{array}$$

Next we slide

$$\begin{array}{ccccccccc}
c_1 &&&&&&&& \\
. & c_2 & \dots & c_n & . & d_1 & d_2 & \dots & d_m \\
a_1 & a_2 & \dots & a_n & x & b_1 & b_2 & \dots & b_m
\end{array}
\qquad
\begin{array}{ccccccccc}
. &&&&&&&& \\
c_1 & c_2 & \dots & c_n & . & d_1 & d_2 & \dots & d_m \\
a_1 & a_2 & \dots & a_n & x & b_1 & b_2 & \dots & b_m
\end{array}$$

To justify that the first slide preserves Knuth equivalence, notice that we can slide to the left:

$$\begin{array}{ccccccccc}
. & c_2 & \dots & c_n & d_1 & d_2 & \dots & d_m & \\
a_1 & a_2 & \dots & a_n & x & b_1 & b_2 & \dots & b_m
\end{array}$$

The same argument as before allows us to take away b_m and analyze the same type of slide as before but in fewer elements, and apply induction.

In the induction we performed we were reducing the number n, thus we still have to justify the slide in the case $n = 0,\ m > 0$. In this case we have to use the second part of the lemma. Slide:

$$\begin{array}{ccccc}
x & d_1 & d_2 & \dots & d_m \\
. & b_1 & b_2 & \dots & b_m
\end{array}
\Rightarrow
\begin{array}{cccccc}
x & d_1 & d_2 & \dots & d_{m-1} & d_m \\
. & b_1 & b_2 & \dots & b_{m-1} & . \\
&&&&& b_m
\end{array}
\Rightarrow
\begin{array}{cccccc}
x & d_1 & d_2 & \dots & d_{m-1} & . \\
. & b_1 & b_2 & \dots & b_{m-1} & d_m \\
&&&&& b_m
\end{array}$$

The last slide preserves Knuth equivalence by the previous case. Now by induction we pass to the $\overset{K}{\cong}$ equivalent:

$$\begin{array}{cccccc}
. & d_1 & d_2 & \dots & d_{m-1} & . \\
x & b_1 & b_2 & \dots & b_{m-1} & d_m \\
&&&&& b_m
\end{array}$$

Apply next Lemma (ii) to $y = d_m > b_m$ and the standard row $xb_1b_2\dots b_{m-1}b_m$, obtaining

$$\begin{array}{cccccc}
. & d_1 & d_2 & \dots & d_{m-1} & d_m \\
x & b_1 & b_2 & \dots & b_{m-1} & . \\
&&&&& b_m
\end{array}
\Rightarrow
\begin{array}{cccccc}
. & d_1 & d_2 & \dots & d_{m-1} & d_m \\
x & b_1 & b_2 & \dots & b_{m-1} & b_m
\end{array}$$

completing the required slide and preserving Knuth equivalence. □

We can now complete the proof of:

Theorem 2. *Two words w_1, w_2 are Knuth equivalent if and only if $T(w_1) = T(w_2)$.*

Proof. We have already proved one part of this statement. We are left to show that if $T(w_1) = T(w_2)$, then $w_1 \overset{K}{\cong} w_2$. The algorithm given by *jeu de taquin* shows that, starting from w_1, we can construct first a semistandard tableau of row word w_1 then, by a sequence of slides, a standard tableau P, of semistandard word s_1, which is Knuth equivalent to w_1. Then $P = T(s_1) = T(w_1)$. Similarly for w_2, we construct a semistandard word s_2. Since $T(s_1) = T(w_1) = T(w_2) = T(s_2)$ we must have $s_1 = s_2$. Hence $w_1 \overset{K}{\cong} w_2$ by transitivity of equivalence. □

We can finally draw the important consequences for the monoid plactique. We start by:

Remark. In each Knuth equivalence class of words we can choose as canonical representative the unique semistandard word (defined in 1.1).

This means that we can formally identify the monoid plactique with the set of semistandard words. Of course the product by juxtaposition of two semistandard words is not in general semistandard, so one has to perform the Robinson–Schensted algorithm to transform it into its equivalent semistandard word and compute in this way the multiplication of the monoide plactique.

2.2 Vacating a Box

Suppose we apply a sequence of complete backward slides to a given skew tableau T and in this way n boxes of T are vacated. We can mark in succession with decreasing integers, starting from n, the boxes of T which are emptied by the procedure. We call the resulting standard skew tableau the *vacated tableau.*

We can proceed similarly for forward slides, but with increasing markings.

One explicit way of specifying a sequence of backward (resp. forward) slides for a semistandard skew tableau T of shape λ/μ is to construct a standard skew tableau U of shape μ/ν (resp. ν/λ). The sequence is determined in the following way. Start from the box of U occupied by the maximum element, say n. It is clearly an inner corner of T, so we can perform a complete backward slide from this corner. The result is a new tableau T' which occupies the same boxes of T, except that it has a new box which previously was occupied by n in U. At the same time it has vacated a box in the rim, which becomes the first box to build the vacated tableau. Then proceed with U_1, which is U once we have removed n. Let us give an example-exercise in which we write the elements of U with numbers and of T with letters. In order not to cause confusion, we draw a square around the boxes, the boxes vacated at each step.

$$\begin{array}{llll} g & g & & \\ 4 & 6 & f & \\ 2 & 5 & d & u \\ 1 & 3 & b & f \end{array}, \quad \begin{array}{llll} g & g & & \\ 4 & f & \boxed{6} & \\ 2 & 5 & d & u \\ 1 & 3 & b & f \end{array}, \quad \begin{array}{llll} g & g & & \\ 4 & f & \boxed{6} & \\ 2 & d & u & \boxed{5} \\ 1 & 3 & b & f \end{array}, \quad \begin{array}{llll} g & \boxed{4} & & \\ f & g & \boxed{6} & \\ 2 & d & u & \boxed{5} \\ 1 & 3 & b & f \end{array}$$

$$\begin{array}{llll} g & \boxed{4} & & \\ f & g & \boxed{6} & \\ 2 & d & u & \boxed{5} \\ 1 & b & f & \boxed{3} \end{array}, \quad \begin{array}{llll} g & \boxed{4} & & \\ f & \boxed{2} & \boxed{6} & \\ d & g & u & \boxed{5} \\ 1 & b & f & \boxed{3} \end{array}, \quad \begin{array}{llll} g & \boxed{4} & & \\ f & \boxed{2} & \boxed{6} & \\ d & g & \boxed{1} & \boxed{5} \\ b & f & u & \boxed{3} \end{array}$$

Let us give some formal definitions. Let T be a semistandard skew tableau of shape λ/μ, and U, V standard of shapes μ/ν, ν/λ. We set

$$j_U(T),\ v_U(T);\quad j^V(T),\ v^V(T)$$

to be the tableaux we obtain by performing backward slides with U or forward slides with V on T. If $Q := v_U(T)$; $R := v^V(T)$ we have seen:

Proposition. $T = j^Q j_U(T),\ T = j_R j^V(T)$

From the previous theorem it follows, in particular, that a skew tableau T can be transformed by *jeu de taquin* into a tableau of normal shape, and this normal shape is uniquely determined. We may sometimes refer to it as *the normal shape of* T.

3 Dual Knuth Equivalence

3.1 Dual equivalence

A fundamental discovery of Schützenberger has been the following (cf. [Sch2]).

Consider a permutation σ of $1, 2, \dots, n$, as a word $\sigma(1), \sigma(2), \dots, \sigma(n)$. We associate to it, by the Robinson–Schensted correspondence, a pair of standard tableaux P, Q.

Theorem 1. *If a permutation σ corresponds, by R-S, to the pair of tableaux A, B, then σ^{-1} corresponds, by R-S, to the pair of tableaux B, A.*

We do not prove this here since we will not need it (see [Sa]); what we shall use instead are the ideas on duality that are introduced in this proof .

Let us start by defining and studying dual Knuth equivalence for words.

Definition 1. We say that two words are *dually Knuth equivalent* if they have the same recording tableau.

For our purposes it will be more useful to study a general dual equivalence of semistandard tableaux. Let us consider two semistandard skew tableaux T, T' of shape λ/μ, having the same content.

Definition 2. A sequence of slides for a tableau T is a sequence of boxes $c_1, c_2, \dots, c_m$, satisfying the following recursive property:

c_1 is either an inner or an outer corner of T. Set $T_1 := j_{c_1}(T)$ if c_1 is inner or $T_1 := j^{c_1}(T)$ if c_1 is outer. We then have that $c_2, \dots, c_m$ is a sequence of slides for T_1.

The definition is so organized that it defines a corresponding sequence of tableaux T_i with $T_{i+1} = j_{c_{i+1}} T_i$ or $T_{i+1} = j^{c_{i+1}} T_i$.

Now we give the notion of dual equivalence for semistandard tableaux.

Definition 3. Two tableaux T, T' are said to be *dually equivalent,* denoted $T \overset{*}{\cong} T'$ if and only if any given sequence of slide operations that can be done on T can also be performed on T', thus producing tableaux of the same shape.[122]

We analyze dual equivalence, following closely Haiman (see [Hai]).

Let us take a sequence of diagrams $\lambda \supset \mu \supset \nu$. Let P be a standard tableau of shape λ/ν. We say that P decomposes as $P_1 \cup P_2$ with P_1 of shape μ/ν and P_2 of shape λ/μ if the entries of P_2 are all larger than the entries of P_1.

Let us observe how decompositions arise. If $n = |P|$ is the number of boxes of P and $k = |P_2|$, then P_2 is formed by the boxes in which the numbers $1, \dots, k$ appear. Conversely, any number k with $1 \leq k < n$ determines such a decomposition.

We have the converse, starting from diagrams $\nu \subset \mu \subset \lambda$. Consider two standard tableaux P_1, P_2 with shapes λ/μ (a diagram with h boxes) and μ/ν (a diagram with k boxes). By definition, P_1, P_2 have respectively entries $1, 2, \dots, h$ and $1, 2, \dots, k$. We then can form the tableau $P_1 \cup P_2$ by placing the tableau P_2 in μ/ν and the tableau P_1 in λ/μ, but with each entry shifted by k.

Remark. Since all the numbers appearing in P_1 are strictly larger than those appearing in P_2, one easily verifies the following: when we perform a slide, for instance a complete backward slide from some cell c on $P_1 \cup P_2$, we first have to apply the complete slide to P_2 which leave some cell d of P_2 vacant, and then we apply the slide determined by d on P_1. In other words

$$j_c(P_1 \cup P_2) = j_d(P_1) \cup j_c(P_2).$$

For forward slides we have with similar notation the following:

$$j^c(P_1 \cup P_2) = j^c(P_1) \cup j^d(P_2).$$

122 In fact, requiring that any sequence performed on T can also be performed on T' implies that the two shapes must be the same.

Lemma 1. *Consider two standard skew tableaux of the same shape and decomposed as $P = X \cup S \cup Y,\ Q = X \cup T \cup Y$.*

If $S \stackrel{*}{\cong} T$, then also $P \stackrel{*}{\cong} Q$.

Proof. This follows from the *jeu de taquin* and the previous remark. Start from a slide, for instance a complete backward slide from some cell c. In both cases (of standard skew tableaux) we first have to apply the slide to Y, which leaves some cell d of Y vacant. Then, in the first case we have to apply the backward slide from d to S, and in the second case we have to apply it to T.

By hypothesis of equivalence, this will leave the same cell vacant in both cases. Next we have to apply the slide to X. So we see that under this slide, the two tableaux are transformed into tableaux $P' = X' \cup S' \cup Y',\ Q' = X' \cup T' \cup Y'$, with $S' \stackrel{*}{\cong} T'$. We can thus repeat the argument for any number of backward slides; for forward slides the proof is similar. □

We want to start the analysis with the first special case.

We say that a shape λ/μ is *miniature* if it has exactly three boxes. So the first result to prove is:

Proposition 1. *Two miniature tableaux of the same shape are dual equivalent if and only if they produce an insertion tableau of the same shape and with the same recording tableau.*

Proof. Assume the two tableaux dual equivalent: If we apply the *jeu de taquin* to the first tableau, in order to construct the insertion tableau, we have to apply a sequence of backward slides, which, by assumption, can also be applied to the second tableau, which then results in an insertion tableau of the same shape and with the same recording tableau.

As for the converse, we need first a reduction to some basic cases by applying translations; then we must do a case analysis on the reading word. We leave the details to the reader.

For the basic example for anti-chains we have the following equivalence:

$$\begin{array}{ccc}1&&\\&3&\\&&2\end{array} \quad \stackrel{*}{\cong} \quad \begin{array}{ccc}2&&\\&3&\\&&1\end{array} \tag{3.0.1}$$

with recording tableau

$$\begin{array}{cc}2&\\1&3\end{array}$$

$$\begin{array}{ccc}2&&\\&1&\\&&3\end{array} \quad \stackrel{*}{\cong} \quad \begin{array}{ccc}3&&\\&1&\\&&2\end{array} \tag{3.0.2}$$

with recording tableau

$$\begin{array}{cc}3&\\1&2\end{array}$$

□

The other two form single dual equivalence classes

$$\begin{array}{ccc} 1 & & \\ & 2 & \\ & & 3 \end{array}\ , \qquad \begin{array}{ccc} 3 & & \\ & 2 & \\ & & 1 \end{array}$$

with recording tableaux

$$\begin{array}{ccc} 1 & 2 & 3 \end{array}, \qquad \begin{array}{c} 3 \\ 2 \\ 1 \end{array}$$

For the main reduction we use:

Definition 4. An *elementary dual equivalence* is one of the form $X \cup S \cup Y \stackrel{*}{\cong} X \cup T \cup Y$ in which S and T are miniature.

Lemma 2. *Let $U \stackrel{*}{\cong} V$ be an elementary dual equivalence. Applying any slide to U and V respectively yields U' and V' so that $U' \stackrel{*}{\cong} V'$ is elementary.*

Proof. This follows from the description of the slide given in Lemma 1. □

Proposition 2. *If S, T are two standard tableaux having the same normal shape λ, then S and T are connected by a chain of elementary dual equivalences.*

Proof. The proof is by induction on $n = |\lambda|$. We may assume $n \geq 3$, otherwise the two tableaux would be identical and there would be nothing to prove.

Consider the box in which n is placed. If it is the same in both tableaux, we remove it, thus obtaining tableaux of the same shape to which we now can apply induction. Otherwise, n appears in two distinct boxes, c and c': they are both corners of λ, and we may assume that c lies in a row higher than c'. We can then find another box c'' in the next row which is lower than the one where c is and as far to the right as possible. We now place $n-2$ in c'' and $n, n-1$ in c, c' in two possible ways. We fill the remaining boxes with the numbers $1, 2, \ldots, n-3$ so as to make the tableaux standard. We obtain two standard tableaux of shape λ, say S' and T', which are elementary dual equivalent by construction. The first S' has n in box c, and thus by induction is connected by a chain of elementary dual equivalences to S; the second, by the same reasoning is connected to T, and the claim follows. □

Corollary. *Two tableaux of the same normal shape are dual equivalent.*

The main theorem which we will need about dual equivalence is the following.

Theorem 2. *Two standard tableaux S, T of the same shape are dual equivalent if and only if they are connected by a chain of elementary dual equivalences.*

Proof. In one direction the theorem is obvious. Let us prove the converse. Let λ/μ be the shape of S, T, and choose a standard tableau U of normal shape μ which we will use to define the sequence of slides to put both S and T in normal shape.

By the dual equivalence of S, T we obtain two standard tableaux of the same normal shape $S' = j_U(S)$, $T' = j_U(T)$, and in so doing we obtain the same tableau that was vacated, i.e., $V = v_U(T) = v_U(S)$. We know that S', T', which can be connected by a chain of elementary dual equivalences, yields $S' \overset{*}{\cong} R_1 \overset{*}{\cong} R_2 \overset{*}{\cong} \dots \overset{*}{\cong} R_k \overset{*}{\cong} T'$. Now we can apply Lemma 2; using the forward slides j^V, we have $T = j^V(T') \cdot S = j^V(S')$, and thus

$$S = j^V(S') \overset{*}{\cong} j^V(R_1) \overset{*}{\cong} j^V(R_2) \overset{*}{\cong} \dots \overset{*}{\cong} j^V(R_k) \overset{*}{\cong} j^V(T') = T$$

is the required chain of elementary equivalences. □

Let us now apply the theory to permutations.

First, write a permutation as a word, and then as a skew tableau as a *diagonal* as in

$$(3,4,1,5,2) = \begin{matrix} 3 & & & & \\ & 4 & & & \\ & & 1 & & \\ & & & 5 & \\ & & & & 2 \end{matrix}$$

We call such a standard tableau a *permutation tableau.*

Theorem 3. *Two permutation tableaux S, T are dual equivalent, if and only if the recording tableaux of their words are the same.*

Proof. If the two tableaux are dual equivalent, when we apply the Robinson–Schensted algorithm to both we follow the same sequence of slides and thus produce the same recording tableaux. Conversely, let U be a standard tableau, which we add to S or T to obtain a triangular tableau, which we use to define a sequence of slides of *jeu de taquin* that will produce in both cases the corresponding insertion tableaux $j_X(S)$, $j_X(T)$. Let λ be the shape of $j_X(S)$.

Consider the tableau $Y = v_X(S)$, which was vacated. We know that j_X and j^Y establish a 1–1 correspondence between dual equivalent permutation tableaux and tableaux of normal shape λ. By the Schensted correspondence, j_X establishes a 1–1 correspondence between tableaux of shape λ and the set $R(S)$ of permutation tableaux whose recording tableau is the same as that of S.

Now we claim that by the first part, $R(S)$ contains the class of permutation tableaux dually equivalent to S. Since by the previous remark j^Y produces as many dually equivalent tableaux as the number of standard tableaux of shape λ, we must have that $R(S)$ coincides with the dual equivalence class of S. □

But now let us understand directly the elementary dual equivalences on words; we see that the two elementary equivalences given by formulas 3.0.1, 3.0.2 are

$$\dots k \dots k+2 \dots k+1 \dots \overset{*}{\cong} \dots k+1 \dots k+2 \dots k \dots$$

$$\dots k+1 \dots k \dots k+2 \dots \overset{*}{\cong} \dots k+2 \dots k \dots k+1 \dots$$

Now we have the remarkable fact that the two previous basic elementary dual Knuth relations for a permutation σ correspond to the usual elementary Knuth relation for σ^{-1}. We deduce that:

Theorem 4. *Two permutations σ, τ are* ***dually Knuth equivalent if and only if*** *σ^{-1}, τ^{-1}* ***are Knuth equivalent.***

We have up to now been restricted to standard tableaux or words without repeated letters or permutations. We need to extend the theory to semistandard tableaux or words with repeated letters. Fortunately there is a simple reduction to the previous case.

Definition 5. Given a standard tableau T with entries $1, 2, \dots, n$, call i a *descent* if $i + 1$ is in a row higher than i.

We say that a word $a_1 \leq a_2 \leq \cdots \leq a_n$ is *compatible with the descent set of T* if it is such that $a_i < a_{i+1}$ if and only if i is a descent of T.

Proposition 3. *Replacing the entries of T with $a_1, \dots, a_n$ gives a semistandard tableau. This is a bijection between semistandard tableaux S and pairs (standard tableau T, word a compatible with the descent set of T).*

Proof. In one direction, given a standard tableau filled with $1, 2, \dots, n$ we replace each i with a_i. In the reverse direction we read the semistandard tableau as follows.

We start by finding the positions of a_1, then of a_2, and so on, for each letter a_j reading from left to right. At the i^{th} step of this procedure, we place i in the corresponding case. For instance, for

$$\begin{array}{cccccc} 5 & 5 & & & & \\ 4 & 4 & 4 & & & \\ 3 & 3 & 3 & 5 & & \\ 2 & 2 & 2 & 2 & 4 & \\ 1 & 1 & 1 & 1 & 2 & 3 \end{array} \quad \Longrightarrow \quad T = \begin{array}{cccccc} 18 & 19 & & & & \\ 14 & 15 & 16 & & & \\ 10 & 11 & 12 & 20 & & \\ 5 & 6 & 7 & 8 & 17 & \\ 1 & 2 & 3 & 4 & 9 & 13 \end{array} .$$

The a sequence is of course in this case

$$1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5.$$

The descent set of T is in fact 4, 9, 13, 17. □

One checks that the *jeu-de-taquin* operations and elementary dual equivalences do not change the descent set. Now define *jeu-de-taquin* operations, dual equivalence, etc., on semistandard tableaux S to operate on T while keeping a fixed.

It follows immediately that the semistandard tableaux T and U of the same normal shape are dual equivalent by the corresponding result for standard tableaux. Also, their reading words are dual Knuth equivalent, because elementary dual equivalences on the underlying standard tableaux induce dual Knuth relations on the reading words of the corresponding semistandard tableaux.

Example: If we switch $a+1, a, a+2 < - > a+1, a+2, a$ in a standard tableau, then both before and after the switch, a is a descent and $a + 1$ is not. So this translates into either $bac \overset{K}{\cong} bca$ (with $a < b < c$ consecutive) or $yxy \overset{K}{\cong} yyx$ (with $x < y$ consecutive) in the semistandard tableau.

4 Formal Schur Functions

4.1 Schur Functions

First, let us better understand the map which associates to a skew semistandard tableau T of shape λ/μ its associated semistandard tableau (insertion tableau of the row word w of T). To study this take a fixed standard tableau P of shape μ. The *jeu de taquin* shows that the map we are studying is $T \to U := j_P(T)$. Let $\nu = sh(U)$ be the normal shape of T. If $Q := v_P(T)$ we also have that $sh(Q) = \lambda/\nu$.

The set $S_{\lambda/\mu}$ of tableaux of shape λ/μ decomposes as the union:

$$S_{\lambda/\mu} = \bigcup_{\nu \subset \lambda} S^\nu_{\lambda/\mu}, \quad S^\nu_{\lambda/\mu} := \{T \in S_{\lambda/\mu},\ sh(j_P(T)) = \nu\}.$$

So let now U be a fixed semistandard tableau of shape ν and consider the set

(4.1.3) $$S_{\lambda/\mu}(U) := \{T \in S^\nu_{\lambda/\mu},\ j_P(T) = U\}.$$

We have not put the symbol P in the definition since P is just auxiliary. The result is independent of P by the basic theorem on *jeu de taquin*.

For $T \in S_{\lambda/\mu}(U)$ consider the vacated tableau $Q = v_P(T)$ (of shape λ/ν). Given another semistandard tableau U' with the same shape as U, consider the tableau $T' := j^Q(U')$. We claim that:

Lemma. *(i)* $sh(T') = \lambda/\mu$.

(ii) The map $\rho_P^{U'} : T \to T' := j^{v_P(T)}(U')$ *is a bijection between* $S_{\lambda/\mu}(U)$ *and* $S_{\lambda/\mu}(U')$.

Proof. (i) Since U, U' are semistandard tableaux of the same normal shape they are dually equivalent. Hence $T = j^Q(U)$ and $T' = j^Q(U')$ have the same shape. Moreover, the dual equivalence implies that the shapes are the same at each step of the operations leading to $j^Q(U), j^Q(U')$. Hence $v^Q(U) = v^Q(U')$.

(ii) We shall show now that the inverse of the map $\rho_P^{U'} : S_{\lambda/\mu}(U) \to S_{\lambda/\mu}(U')$ is the map $\rho_P^U : T' \to j^{v_P(T')}(U)$.

Since $T = j^{v_P(T)}(U)$, $T' = j^{v_P(T)}(U')$ we have that $T \overset{*}{\cong} T'$ (§3.1 Corollary to Theorem 2). Thus we must have $Q = v_P(T) = v_P(T')$ since these tableaux record the changes of shape under the operations j_P. So

(4.1.4) $$\rho_P^U \rho_P^{U'}(T) = \rho_P^U(T') = j^{v_P(T')}(U) = j^Q(U) = T,$$

and we have inverse correspondences between $S_{\lambda/\mu}(U)$ and $S_{\lambda/\mu}(U')$:

$$T \to T' := j^{v_P(T)}(U'), \quad T' \to T := j^{v_P(T')}(U). \qquad \square$$

Let us then define $d^\nu_{\lambda,\mu} := |S_{\lambda/\mu}(U)|$ for any semistandard tableau U of shape ν.

We arrive now at the construction of Schützenberger. Consider the monoide plactique $\mathcal{M}$ in an alphabet and define a *formal Schur function* $\mathcal{S}_\lambda \in \mathbb{Z}[\mathcal{M}]$ defined by $\mathcal{S}_\lambda = \sum w$ where w runs over all semistandard words of shape λ. Similarly define $\mathcal{S}_{\lambda/\mu}$ to be the sum of all row words which correspond to all semistandard skew tableaux of shape λ/μ. We have in the algebra of the monoide plactique:

Theorem.

$$S_{\lambda/\mu} = \sum_{\nu \subset \lambda} d^{\nu}_{\lambda,\mu} S_\nu, \quad S_\lambda S_\mu = \sum_\nu c^{\nu}_{\lambda,\mu} S_\nu. \tag{4.1.5}$$

for some nonnegative integers $d^{\nu}_{\lambda,\mu}$, $c^{\nu}_{\lambda,\mu}$.

Proof. The first statement is just a consequence of the previous lemma. As for the second it is enough to remark that $S_\lambda S_\mu = S_{\gamma/\rho}$ where ρ is a rectangular diagram and γ is obtained from ρ by placing λ on its top and μ on its right, as in

$$\lambda = \begin{array}{cc} \cdot & \\ \cdot & \cdot \end{array}, \qquad \mu = \begin{array}{cc} \cdot & \\ \cdot & \cdot \\ \cdot & \cdot \end{array}, \qquad \gamma/\rho = \begin{array}{cccc} \cdot & & & \\ \cdot & \cdot & & \\ & & \cdot & \\ & & \cdot & \cdot \\ & & \cdot & \cdot \end{array}$$

□

Now assume that we are using as an alphabet $1, 2, \ldots, m$. Then we consider the content of a word as a monomial in the variables x_i. Since $c(ab) = c(a)c(b)$ and content is compatible with Knuth equivalence, we get a morphism: $c : \mathbb{Z}[\mathcal{M}] \to \mathbb{Z}[x_1, \ldots, x_m]$.

Proposition. $c(S_\lambda) = S_\lambda(x_1, x_2, \ldots, x_m)$.

Proof. This is a consequence of Chapter 9, Theorem 10.3.1, stating that

$$S_\lambda(x_1, x_2, \ldots, x_m) = \sum_T x^T,$$

where the sum is indexed by semistandard tableaux T of shape λ, filled with $1, \ldots, m$. x^T is, in monomial form, the content $c(T)$. □

It follows that $S_\lambda(x) S_\mu(x) = \sum_\nu c^{\nu}_{\lambda,\mu} S_\nu(x)$. The interpretation of the symmetric function associated to $S_{\lambda/\mu}$ will be given in the next section.

5 The Littlewood–Richardson Rule

5.1 Skew Schur Functions

The Littlewood–Richardson rule describes in a combinatorial way the multiplicities of the irreducible representations of $GL(V)$ that decompose a tensor product $S_\lambda(V) \otimes S_\mu(V) = \bigoplus_\nu c^{\nu}_{\lambda,\mu} S_\nu(V)$. Using characters, this is equivalent to finding the multiplication between symmetric Schur functions.[123]

[123] These types of formulas are usually called Clebsch–Gordan formulas, since for $SL(2, \mathbb{C})$ they are really the ones discussed in Chapter 3.

(5.1.1)
$$S_\lambda(x)S_\mu(x) = \sum_\nu c^\nu_{\lambda,\mu} S_\nu(x).$$

We revert first to symmetric functions. Let us consider $\lambda \vdash n$ and the Schur function $S_\lambda(x_1, x_2, \dots, x_n, z_1, z_2, \dots, z_n)$ in a double set of variables. Since this is also symmetric separately in the x and z, we can expand it as a sum in the $S_\mu(x)$:

$$S_\lambda(x_1, x_2, \dots, x_n, z_1, z_2, \dots, z_n) = \sum_\mu S_\mu(x_1, x_2, \dots, x_n) S_\lambda/\mu(z_1, z_2, \dots, z_n).$$

$S_{\lambda/\mu}(z)$ is defined by this formula and it is symmetric in the z's. Now take Cauchy's formula for the variables $x_1, x_2, \dots, x_n, z_1, z_2, \dots, z_n$ and $y_1, \dots, y_n$:

$$\prod_{i,j} \frac{1}{1 - x_i y_i} \prod_{i,j} \frac{1}{1 - z_i y_i} = \sum_\lambda S_\lambda(x, z) S_\lambda(y)$$

$$= \Big(\sum_\mu S_\mu(x) S_\mu(y)\Big)\Big(\sum_\nu S_\nu(z) S_\nu(y)\Big).$$

Expand $S_\lambda(x, z) = \sum_\mu S_\mu(x) S_{\lambda/\mu}(z)$ to get

$$\sum_\lambda \sum_\mu S_\mu(x) S_{\lambda/\mu}(z) S_\lambda(y) = \sum_{\nu,\mu} S_\mu(x) S_\nu(z) \sum_\lambda c^\lambda_{\nu,\mu} S_\lambda(y),$$

hence

(5.1.2)
$$S_{\lambda/\mu}(z) = \sum_\nu c^\lambda_{\nu,\mu} S_\nu(z).$$

5.2 Clebsch–Gordan Coefficients

The last formula allows one to develop a different approach to compute the numbers $c^\lambda_{\nu,\mu}$, relating these numbers to semistandard skew tableaux.

In fact, if we consider the variables x_i as less than the z_j in the given order, $S_\lambda(x, z)$ is the sum of the contents of all the semistandard tableaux of shape λ filled with the x_i and z_j (or indices corresponding to them). In each such semistandard tableau the letters x must fill a subdiagram μ and the remaining letters fill a skew diagram of shape λ/μ. We can thus separate the sum according to the μ. Since, given μ, we can fill independently with x's the diagram μ and with z's the skew diagram λ/μ, we have that this contribution to the sum $S_\lambda(x, z)$ is the product of $S_\mu(x, z)$ with a function $S'_{\lambda/\mu}(z)$ sum of the contents of all the skew tableaux filled with the z's of shape λ/μ. We deduce finally that the skew Schur function $S_{\lambda/\mu}(z)$ equals to the content $c(\mathcal{S}_{\lambda/\mu})$ of the skew formal Schur function $\mathcal{S}_{\lambda/\mu}$ defined in 4.1, and so

(5.2.1)
$$S_{\lambda/\mu}(z) = c(\mathcal{S}_{\lambda/\mu}), \quad \sum_\nu c^\lambda_{\nu,\mu} S_\nu(z) = \sum_{\nu\subset\lambda} d^\nu_{\lambda,\mu} S_\nu(z), \quad c^\lambda_{\nu,\mu} = d^\nu_{\lambda,\mu}.$$

We deduce:

Proposition. $c^\lambda_{\nu,\mu} = d^\nu_{\lambda,\mu} := |S_{\lambda/\mu}(U)|$ *for any semistandard tableau U of shape ν.*

It is convenient to choose as U the semistandard tableau with i on the i^{th}row as

$$\begin{array}{ccccc} 4 & 4 & & & \\ 3 & 3 & & & \\ 2 & 2 & 2 & 2 & \\ 1 & 1 & 1 & 1 & 1 \end{array}$$

There is a unique such tableau for each shape λ; this tableau is called *the supercanonical tableau of shape* λ and denoted by C_λ. It allows us to interpret the combinatorics in terms of *lattice permutations.*

Definition. A word w in the numbers $1, \dots, r$ is called a lattice permutation if, for each initial subword (prefix) a, i.e., such that $w = ab$, setting k_i to be the number of occurrences of i in a, we have $k_1 \geq k_2 \geq \cdots \geq k_r$. A reverse lattice permutation is a word w such that the opposite word[124] w^o is a lattice permutation.

Of course the word of C_λ has this property (e.g., for $w = 4433222211111$, we have $w^o = 1111122223344$). Conversely:

Lemma. *The row word of a semistandard tableau U of shape λ is a reverse lattice permutation if and only if $U = C_\lambda$ is supercanonical.*

Proof. When we read in reverse the first row, we have by definition of semistandardness a decreasing sequence. By the lattice permutation condition this sequence must start with 1 and so it is the constant sequence 1. Now for the next row we must start with some $i > 1$ by standardness, but then $i = 2$ by the lattice permutation property, and so on. □

5.3 Reverse Lattice Permutations

Notice that for a supercanonical tableau the shape is determined by the content. To use these facts we must prove:

Proposition. *The property for the row word w of a tableau T to be a reverse lattice permutation is invariant under* jeu de taquin.

Proof. We must prove it for just one elementary vertical slide.

Let us look at one such move (the other is similar) from the word w of:

$$\begin{array}{cccccccccc} \dots u & c_1 & c_2 & \dots & c_n & x & d_1 & d_2 & \dots & d_m & \\ & a_1 & a_2 & \dots & a_n & . & b_1 & b_2 & \dots & b_m & \dots v \end{array}$$

with $a_n < c_n \leq x \leq b_1 < d_1$, to:

[124] The opposite word is just the word read from right to left.

$$\begin{array}{cccccccccccc} \dots u & c_1 & c_2 & \dots & c_n & . & d_1 & d_2 & \dots & d_m & & \\ & a_1 & a_2 & \dots & a_n & x & b_1 & b_2 & \dots & b_m & \dots v \end{array}$$

The opposite of the row word is read from right to left and from bottom to top. The only changes in the contents of the prefix words of w^o (or suffixes of w) can occur in the prefixes ending with one of the a's or d's. Here the number of x's has increased by 1, so we must only show that it is still $\leq$ than the number of $x - 1$. Let us see where these elements may occur. By semistandardness, no $x - 1$ occurs among the d's and no x among the a's or the d's. Clearly if $x - 1$ appears in the a's, there is a minimum j such that $x - 1 = a_j = a_{j+1} = \dots = a_n$. By semistandardness $x \geq c_j > a_j = x - 1$ implies that $x = c_j = c_{j+1} = \dots = c_n$. Let $A \geq B$ (resp $A' \geq B'$) be the number of occurrences of $x - 1, x$ in the suffix of w starting with b_1 (resp with c_j). We have $A' = A + n - j + 1$ while $B' = B + n - j + 2$. Hence $A > B$. This is sufficient to conclude since in the new word for the changed suffixes the number of occurrences of x is always $B + 1 \leq A$. □

Let us say that a word has content ν if it has the same content as C_ν. We can now conclude (cf. 4.1.1):

Theorem (The Littlewood–Richardson rule).

(i) $S_{\lambda/\mu}(C_\nu)$ *equals the set of skew semistandard tableaux of shape* λ/μ *and content* ν *whose associated row word is a reverse lattice permutation.*
(ii) The multiplicity $c^\lambda_{\nu,\mu}$ *of the representation* $S_\lambda(V)$ *in the tensor product* $S_\mu(V) \otimes S_\nu(V)$*, equals the number of skew semistandard tableaux of shape* λ/μ *and content* ν *whose associated row word is a reverse lattice permutation.*

Proof. (i) $\implies$ (ii) By Proposition 5.2, $c^\lambda_{\nu,\mu} = d^\nu_{\lambda,\mu} := |S_{\lambda/\mu}(U)|$ for any tableau U of shape ν. If we choose $U = C_\nu$ supercanonical, i.e., of content ν, clearly we get the claim.

(i) By the previous proposition $S_{\lambda/\mu}(C_\nu)$ is formed by skew semistandard tableaux of shape λ/μ and content ν whose associated row word is a reverse lattice permutation. Conversely, given such a tableau T, from the previous proposition and Lemma 5.2, its associated semistandard tableau is supercanonical of content ν. Hence we have $T \in S_{\lambda/\mu}(C_\nu)$. □